Written Examination Special Relativity F8066

Academic Year 2001–2002: 03 September 2002, 2-4 PM

Please read the following INSTRUCTIONS

A. Answer at most TWO questions. You may answer in english or in italian. A pass is obtained for one complete answer, and full marks for two complete answers.

B. You may not use notes or textbooks, but the course notes are available for consultation at the front desk.

C. On your answer paper, please rewrite and sign the pledge

"I swear on my honour that I have neither given nor received help during this examination."

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1. Derive the Einstein formula $E = \gamma mc^2$ from the variation of the action $S = \int -mc^2 d\tau$ for a free particle of mass m, where τ is its proper time. Show that for velocities $v << c$ it differs from the non-relativistic kinetic energy by a constant.

2. A rocket propels itself rectilinearly through empty space by emitting pure radiation (photons) in the direction opposite to its motion. If V is its final velocity relative to its initial rest - frame, show that the ratio of the initial to the final rest mass of the rocket is given by

$$\frac{M_i}{M_f} = \left(\frac{c+V}{c-V}\right)^{\frac{1}{2}}$$

3. Consider the components of a four–vector V^α as the matrix (with $i^2 = -1$)

$$\mathcal{V} = \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}$$

i) Show that V^α satisfies $V^\alpha V_\alpha = Det\ \mathcal{V}$

ii) Show that the transformation $\mathcal{V} \to \mathcal{V}' = \mathcal{A}\mathcal{V}\mathcal{A}$ with

(continued)

$$\mathcal{A} = \begin{pmatrix} \omega^{-\frac{1}{2}} & 0 \\ 0 & \omega^{\frac{1}{2}} \end{pmatrix}$$

and ω constant, corresponds to a Lorentz boost along the z axis. What is ω in terms of the rapidity parameter θ and the boost velocity v?
Ans. $\omega = \exp\theta = \sqrt{\frac{c+v}{c-v}}$.

4. Two particles have velocities $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$, not necessarily parallel or orthogonal. Show that their relative velocity $\mathbf{v} = |\tilde{\mathbf{v}}|$ is given by (**with $\mathbf{c = 1}$**)

$$\mathbf{v}^2 = \frac{(\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2)^2 - (\tilde{\mathbf{v}}_1 \wedge \tilde{\mathbf{v}}_2)^2}{(1 - \tilde{\mathbf{v}}_1.\tilde{\mathbf{v}}_2)^2}$$

(Hint: consider the Lorentz invariance of the product of their **four**-velocities ${v_1}^\alpha v_{2\alpha}$)