Chapter 1

ROOT SYSTEMS AND THEIR CLASSIFICATION

1.1 Cartan Subalgebras

We consider a semisimple Lie algebra $\mathbb{G}$ and we introduce the fundamental concept of Cartan subalgebra that will be the primary instrument to set up the reduction of the Lie algebra to a canonical form and its identification in terms of a **root system**.

Definition 1.1.1. A Cartan subalgebra $\mathcal{H} \subset \mathbb{G}$ is a subalgebra that satisfies the following two defining properties:

i) H is a **maximal abelian subalgebra**
ii) $\forall H \in \mathcal{H}$ the map ad(H) is a **semisimple endomorphism**

First we prove that every semisimple Lie algebra $\mathbb{G}$ has a Cartan subalgebra (frequently abbreviated as CSA). Then we show that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two CSA.s then they are isomorphic.

Let $H \in \mathbb{G}$ be an element of the semisimple Lie algebra and let $\lambda_0, \lambda_1, \ldots \lambda_r$ be the eigenvalues of ad(H): define

$$g(\mathrm{H}, \lambda_i) = \{X \in \mathbb{G}/\mathrm{ad}(\mathrm{H})X = \lambda_i X\} \tag{1.1.1}$$

the subspace of $\mathbb{G}$ pertaining to the eigenvalue λ_i. We have:

$$\mathbb{G} = \bigoplus_{i=0}^{r} g(\mathrm{H}, \lambda_i) \tag{1.1.2}$$

Definition 1.1.2. An element $H_0 \in \mathbb{G}$ is named **regular** if

$$\dim g\,(H_0, 0) = \underbrace{\min}_{X \in \mathbb{G}} (\dim g\,(X, 0)) \tag{1.1.3}$$

We have the

Theorem 1.1.1. If H_0 is a regular element then $g\,(H_0, 0)$ is a Cartan subalgebra

Proof 1.1.1.1. We have to show that:

a) $g\,(H_0, 0)$ is a subalgebra
b) $g\,(H_0, 0)$ is a maximal abelian subalgebra

c) if $H \in g\,(H_0, 0)$ then $\mathrm{ad}(H)$ is semisimple as an endomorphism.

We begin by observing that

$$[g\,(Z, \lambda)\ ,\ g\,(Z, \mu)] \subset g\,(Z, \lambda + \mu) \tag{1.1.4}$$

which immediately follows from Jacobi identities. This implies that $\mathcal{H} = g\,(H_0, 0)$ is a subalgebra. Next we prove that $\mathcal{H}$ is abelian. To this effect let us denote by $0 = \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_r$ the different eigenvalues of $\mathrm{ad}(H_0)$ and set:

$$\mathbb{G}' = \bigoplus_{i=1}^{r} g(\mathrm{H}, \lambda_i) \tag{1.1.5}$$

¿From eq. (1.1.4) it follows that $[\mathcal{H}\,,\,\mathbb{G}'] \subset \mathbb{G}'$. $\forall H \in \mathcal{H}$ let us denote $\mathrm{ad}'(H)$ the restriction of $\mathrm{ad}(H)$ to the subspace $\mathbb{G}'$ and name $d(H) = \det\left[\mathrm{ad}'(H)\right]$ the determinant of such an endomorphism. By definition $d(H)$ is a polynomial function on the finite dimensional vector space (algebra) $\mathcal{H}$, furthermore, by definition of the subspace $\mathbb{G}'$, the map $\mathrm{ad}'(H_0)$ has only non vanishing eigenvalues, so that $d(H_0) \neq 0$. If a polynomial function vanishes on an open set then it is identically zero. Since $d(H_0) \neq 0$ it follows that $d(H)$ is not identically zero and that its zeros are isolated. Calling S the set of elements of $\mathcal{H}$ for which $d(H) \neq 0$ we conclude that S is dense in $\mathcal{H}$. Let $H \in S \subset \mathcal{H}$: since $\det\left[\mathrm{ad}'(h)\right] \neq 0$ it follows that all the null eigenvectors of $\mathrm{ad}(H)$, if any, are contained in $\mathcal{H}$. Hence we have shown:

$$\forall H \in S \quad : \quad g\,(H, 0) \subset \mathcal{H} \tag{1.1.6}$$

Since the element H_0 is by hypothesis regular we conclude that $g\,(H, 0) = \mathcal{H}$. Hence it is proved that

$$\forall H \in S\,,\, \forall H_1 \in \mathcal{H} \quad : \quad \mathrm{ad}(H)(H_1) = 0 \tag{1.1.7}$$

Hence the restriction of $\mathrm{ad}(H)$ to the subalgebra $\mathcal{H}$ is nilpotent since it vanishes. Since S is dense in $\mathcal{H}$, by continuity it follows that

$$\forall H \in \mathcal{H} \quad : \quad \mathrm{ad}_{\mathcal{H}}\,(H) = 0 \tag{1.1.8}$$

namely that

$$\forall H_1, H_2 \in \mathcal{H} \quad : \quad [H_1\,,\,H_2] = 0 \tag{1.1.9}$$

This concludes the proof that $\mathcal{H} \equiv g\,(H_0, 0)$ is an abelian subalgebra. By definition it is also maximal. Indeed if there existed an element $X \notin g(H_0, 0)$ such that $[X, \mathcal{H}] = 0$ we would have a contradiction since, in particular $[X, H_0] = 0$ which implies $X \in g\,(H_0, 0)$.

Let us now show that if λ is a non vanishing eigenvalue of $\mathrm{ad}(H_0)$, then every endomorphism $ad(H)$ with $H \in \mathcal{H}$ maps the subspace $g\,(H_0, \lambda)$ into itself. Hence, denoting by $\mathrm{ad}_\lambda(H)$ the restriction of $\mathrm{ad}(H)$ to this subspace we have that $\mathrm{ad}_\lambda(H)$ is a *representation* of $\mathcal{H}$ on $g\,(H_0, \lambda)$. Since $\mathrm{ad}_\lambda(H)$ is a family of commuting endomorphisms (solvable algebra, in particular) we can put all of them simultaneously in a triangular form, by choosing some appropriate basis $\vec{e}_1, \ldots, \vec{e}_s$ of $g\,(H_0, \lambda)$.In this basis the semisimple part of $\mathrm{ad}_\lambda(H)$ will be the diagonal part:

$$\mathrm{ad}_\lambda(H) = \begin{pmatrix} \alpha_1(H) & 0 & \ldots & \ldots & 0 \\ 0 & \alpha_2(H) & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \alpha_{s-1}(H) & 0 \\ 0 & \ldots & \ldots & 0 & \alpha_s(H) \end{pmatrix} + \text{nilpotent matrix} \tag{1.1.10}$$

The diagonal elements $\alpha_i(H)$ are linear functions on $\mathcal{H}$ with the property that $\alpha_1(H_0) = \alpha_2(H_0) = \ldots = \alpha_s(H_0) = \lambda$. Let $\beta(H)$ be any linear function on $\mathcal{H}$ that takes the value $\beta(H_0) = \lambda$ at H_0. Let V_β be the subspace of $g(H_0, \lambda)$ spanned by the those basis vectors $\overrightarrow{e}_i$ such that $\alpha_i(H) = \beta(H)$ $(\forall H \in \mathcal{H})$. By definition it follows that:

$$\forall X \in V_\beta \quad \Rightarrow \quad \exists k \in \mathbb{N} \quad / \quad (\mathrm{ad}(H) - \beta(H)\mathbf{1})^k \; X = 0 \tag{1.1.11}$$

Indeed, once we have subtracted the diagonal part, what remains is nilpotent. In general we have:

$$\mathbb{G} = \sum_i V_{\beta_i} \qquad \text{for suitable } \beta_i \tag{1.1.12}$$

hence if $\kappa\,(\;,\;)$ is the Killing form we can write:

$$\forall\, H, H' \, \in \, \mathcal{H} \quad : \quad \kappa\,(H \,,\, H') = \sum_i \beta_i(H)\,\beta_i(H')\dim V_{\beta_i} \tag{1.1.13}$$

We decomposeá la Jordan:

$$\mathrm{ad}(H) = \underbrace{S(H)}_{\text{semisimple}} + \underbrace{N(H)}_{\text{nilpotent}} \tag{1.1.14}$$

and we recall that $S(H)$ is polynomial in $\mathrm{ad}(H)$. By construction the endomorphism $S(H)$ leaves each V_beta subspace invariant and:

$$S(H)X = \beta(H)X \quad ; \quad \forall X \, \in \, V_\beta \tag{1.1.15}$$

Furthermore since $[V_\alpha \,,\, V_\beta] \subset V_{\alpha+\beta}$ it follows that S is a derivation of the algebra. But for a semisimple Lie algebra every derivation is internal, hence $\exists Z \in \mathbb{G}$ such that $\mathrm{ad}(Z) = S(H)$. Since $S(H)$ commutes with all elements of $\mathcal{H}$ it follows that $Z \in H$. In other words $Z = H$ and this shows that $\mathrm{ad}(H)$ coincides with its semisimple part.

1.2 Root systems

Let $\mathcal{H} \subset \mathbb{G}$ be a Cartan subalgebra of the semisimple Lie algebra $\mathbb{G}$. Consider an element $\alpha \in \mathcal{H}^\star$ namely a linear functional:

$$\begin{array}{c} \alpha : \quad \mathcal{H} \rightarrow \mathbb{C} \\ \forall\, H_1, H_2 \, \in \, \mathcal{H} \quad ; \quad \forall\, \lambda, \mu \, \in \, \mathbb{C} : \alpha\,(\lambda H_1 + \mu H_2) = \lambda\,\alpha\,(H_1) + \mu\,\alpha\,(H_2) \end{array} \tag{1.2.16}$$

Let us define the linear subspace $\mathbb{G}^\alpha \, \subset \, \mathbb{G}$:

$$\mathbb{G}^\alpha \; : \; \{X \in \mathbb{G} \setminus [H \,,\, X] = \alpha(H)\,X \,,\, \forall H \in \mathcal{H}\} \tag{1.2.17}$$

If $\mathbb{G}^\alpha \neq \varnothing$ is not empty then we say that $\alpha \in \mathcal{H}^\star$ is a **root** and $\mathbb{G}^\alpha$ is named the **corresponding subspace** of root α. Since, by definition, $\mathcal{H}$ is maximal abelian, then we have $\mathbb{G}^0 = \mathcal{H}$. On the other hand from Jacobi identity we immediately obtain:

$$\left[\mathbb{G}^\alpha \,,\, \mathbb{G}^\beta\right] \, \subset \, \mathbb{G}^{\alpha+\beta} \quad \forall \alpha, \beta \, \in \mathcal{H}^\star \tag{1.2.18}$$

Let us next denote by Φ the set of all non–vanishing roots and with $\kappa\,(\;,\;)$ the Killing form. We have the

Theorem 1.2.1. The following statements are true:

i) $\mathbb{G} = \mathcal{H} \bigoplus \sum_{\alpha\in\Phi} \mathbb{G}^\alpha$ (direct sum)

ii) $\dim \mathbb{G}^\alpha = 1, \quad \forall \alpha \in \Phi$

iii) Let $\alpha, \beta \in \Phi$ be two roots such that $\alpha + \beta \neq 0$, then the corresponding subspaces $\mathbb{G}^\alpha$ and $\mathbb{G}^\beta$ are mutually orthogonal with respect to the Killing form $\kappa\,(\,,\,)$.

iv) The restriction of the Killing form $\kappa\,(\,,\,)$ to $\mathcal{H} \bigotimes \mathcal{H}$ is non degenerate and for each root $\forall \alpha \in \Phi$ there exists an element $\exists H_\alpha \in \mathcal{H}$ of the Cartan subalgebra such that

$$\kappa\,(H\,,\,H) = \alpha(H) \quad \forall H \in \mathcal{H} \tag{1.2.19}$$

v) If $\alpha \in \Phi$ is a root then also its negative is a root:$-\alpha \in \Phi$. Furthermore we have:

$$\begin{aligned} \left[\mathbb{G}^\alpha\,,\,\mathbb{G}^{-\alpha}\right] &= \text{const}\, H_\alpha \\ \alpha(H_\alpha) &\neq 0 \end{aligned} \tag{1.2.20}$$

Proof 1.2.1.1. We begin with point **i)** in the above list and we show first that the sum is direct. If it were not this would mean that there exists a linear relation:

$$H^\star + \sum_i X_{\alpha_i} = 0 \tag{1.2.21}$$

where $H^\star \in \mathcal{H}$ and $X_{\alpha_i} \in \mathbb{G}^{\alpha_i}$. We can choose an element $H \in \mathcal{H}$ such that $\alpha_i(H) \neq 0$ for all the roots α_i. Indeed the subset $N \subset \mathcal{H}$ on which all the roots α_i are different and non vanishing is the complement of the union of a finite number of hyperplanes ($\alpha(H) = 0 \Leftrightarrow \textit{hyperplane} \ni H$). Hence H with the required properties exists and, as a consequence, $H^\star$ and X_{α_i} belong to different eigenspaces of $\mathrm{ad}(H)$. As such they are linearly independent which contradicts the assumption of eq.(1.2.21). This shows that the sum of subspaces in statement **i)** is direct. On the other hand since $\mathrm{ad}_{\mathbb{G}}(\mathcal{H})$ is a set of semisimple endomorphisms it follows that $\mathbb{G}$ can be decomposed into eigenspaces and the relation advocated in statement **i)** follows. Furthermore if $\alpha(H_0) = 0$ for all roots $\forall \alpha \in \Phi$ then $H_0 = 0$. Indeed by hypothesis we have $[H_0\,,\,X] = 0$, $\forall X \in \mathbb{G}$ and since the Lie algebra $\mathbb{G}$ is semisimple this implies $H_0 = 0$.

Let us next proof the statement **iii)**. To this effect we choose $X \in \mathbb{G}^\alpha$ and $Y \in \mathbb{G}^\beta$. With this choice the endomorphism $\mathrm{ad}(x).\mathrm{ad}(Y)$ maps the space $\mathbb{G}^\gamma$ into $\mathbb{G}^{\alpha+\beta+\gamma}$ and since $\alpha + \beta \neq 0$ we get:

$$\mathbb{G}^\gamma \bigcap \mathbb{G}^{\alpha+\beta+\gamma} = 0 \tag{1.2.22}$$

Therefore if we use a basis where every basis vector lies in some root subspace $\mathbb{G}^\gamma$ we immediately see that:

$$\kappa\,(X\,,\,Y) \equiv \mathrm{Tr}\,(\mathrm{ad}(X).\mathrm{ad}(Y)) = 0 \tag{1.2.23}$$

which is what we wanted to show

Next let us prove the statement **iv)**. If $H_0 \in \mathcal{H}$ satisfies the condition:

$$\kappa\,(H_0\,,\,H) = 0 \qquad \forall H \in \mathcal{H} \tag{1.2.24}$$

then as a consequence of statement **iii)** that we have already proved it follows that:

$$\kappa\,(H_0\,,\,X) = 0 \qquad \forall X \in \mathbb{G} \tag{1.2.25}$$

This would imply that that the Killing $\kappa\left(\, , \, \right)$ is degenerate in contradiction with Cartan's criterion for semisimple Lie algebras. Hence there are no vectors in $\mathcal{H}$ which are orthogonal to all vectors of $\mathcal{H}$, which proves the first part of statement **iv)**. Next choose a basis $\{H_i\}$ $(i = 1, \dots \mathrm{rank}\mathbb{G})$ of the Cartan subalgebra $\mathcal{H}$ and set:

$$\kappa_{ij} = \kappa\left(H \, , \, H_j\right) \tag{1.2.26}$$

Writing $\forall H \in \mathcal{H} \quad H = h^i H_i$ we obtain $\alpha(H) = h^i \alpha_i$ where $\alpha_i \equiv \alpha(H_i)$. Statement **iv)** advocates that we should be able to find an element $H_\alpha = \alpha^i H_i$ such that $\kappa\left(H \, , \, H_\alpha\right) = \alpha(H)$. In the chosen basis this means $H^i \alpha^j \kappa_{ij} = h^i \alpha_i$ which implies:

$$\alpha^j \kappa_{ij} = \alpha_i \tag{1.2.27}$$

Since κ_{ij} is a non degenerate matrix we can always find its inverse and set

$$\alpha^j = \left(\kappa^{-1}\right)^{ji} \alpha_i \tag{1.2.28}$$

which concludes the proof of statement **iv)**.

Let us come to the proof of statement **v)**. Let us assume that $-\alpha \notin \Phi$. This would imply that $\mathbb{G}^{-\alpha} = \emptyset$. In this case an element $X \in \mathbb{G}^\alpha$ being orthogonal to all the other subspaces $\mathbb{G}^\beta$ would imply that

$$\kappa\left(X_\alpha \, , \, Y\right) = 0 \qquad \forall Y \in \mathbb{G} \tag{1.2.29}$$

In this case the Killing form would be degenerate which is impossible for a semisimple Lie algebra by Cartan criterion. So $-\alpha \in \Phi$. Let now $H, X_\alpha, X_{-\alpha}$ be arbitrary elements respectively in $\mathcal{H}, \mathbb{G}^\alpha$ and $\mathbb{G}^{-\alpha}$. Then by properties of the Killing form we have:

$$\begin{aligned}
\kappa\left(\left[X_\alpha \, , \, X_{-\alpha}\right] \, , \, H\right) &= \kappa\left(X_{-\alpha} \, , \, \left[H \, , \, X_\alpha\right] H\right) \\
&= \kappa\left(X_{-\alpha} \, , \, X_\alpha\right) \, \alpha(H) \\
&= \kappa\left(X_{-\alpha} \, , \, X_\alpha\right) \, \kappa\left(H_\alpha \, , \, H\right)
\end{aligned} \tag{1.2.30}$$

so that we are forced to identify:

$$\left[X_\alpha \, , \, X_{-\alpha}\right] = \kappa\left(X_{-\alpha} \, , \, X_\alpha\right) \, H_\alpha = \kappa\left(X_{-\alpha} \, , \, X_\alpha\right) \, \alpha^i H_i \tag{1.2.31}$$

which concludes the proof of statement **v)**.

Finally let us prove statement **ii)**. Let us assume that $\dim \mathbb{G}^\alpha > 1$. In this case let us choose $X_\alpha \in \mathbb{G}^\alpha$ and $X_{-\alpha} \in \mathbb{G}^{-\alpha}$ such that:

$$\kappa\left(X_{-\alpha} \, , \, X_\alpha\right) = 1 \tag{1.2.32}$$

If $\dim \mathbb{G}^\alpha > 1$ it follows that there exists $\exists D_\alpha \in \mathbb{G}^\alpha$ such that:

$$\kappa\left(D_\alpha \, , \, X_{-\alpha}\right) = 0 \tag{1.2.33}$$

Set $D_n = \left(\mathrm{ad}\left(X_\alpha\right)\right)^n \, D_\alpha$ for $n = 0, 1, 2, \dots$. We have $D_n \in \mathbb{G}^{(n+1)\alpha}$ and hence

$$\left[H_\alpha \, , \, D_n\right] = \alpha(H) \, (n+1) \, D_n \tag{1.2.34}$$

Furthermore by induction we can show that:

$$\left[X_{-\alpha} \, , \, D_n\right] = -n \frac{(n+1)}{2} \alpha(H_\alpha) \, D_{n-1} \tag{1.2.35}$$

For $n = 0$ we have $[X_{-\alpha}\,,\,D_\alpha] = \kappa\,(D_\alpha\,,\,X_{-\alpha})\,H_\alpha = 0$. On the other hand if eq.(1.2.35) is true for n it follows that it is also true for $n+1$. Indeed:

$$\begin{aligned}
[X_{-\alpha}\,,\,D_{n+1}] &= [X_{-\alpha}\,,\,[X_\alpha\,,\,D_n]] \\
&= -\,[X_\alpha\,,\,[D_n\,,\,X_{-\alpha}]] - [D_n\,,\,[X_{-\alpha}\,,\,X_\alpha]] \\
&= -\,n\frac{n+1}{2}\alpha(H_\alpha)\,D_n + (n+1)\alpha(H_\alpha)D_n \\
&= -\,(n+1)\frac{n+2}{2}\,\alpha(H_\alpha)D_n
\end{aligned} \tag{1.2.36}$$

which shows what we claimed. Therefore if $D_0 = D_\alpha$ exists also all the other D_n do exist and are non vanishing. This implies that there are infinite roots $(n+1)\alpha$ and correspondingly infinite orthogonal subspaces $\mathbb{G}^{(n+1)\alpha}$ this is manifestly absurd since the dimension of the semisimple Lie algebra $\mathbb{G}$ is finite. Hence D_α cannot exist and the dimension of the subspace $\mathbb{G}^\alpha = 1$ as claimed.

This concludes the proof of the theorem

1.2.1 Final form of the semisimple Lie algebra

Using the result provided by theorem 1.2.1 we can now write a final general form of a semisimple Lie algebra in terms of Cartan generators H_i and step operators E^α associated with the roots α. To this effect we normalize the Cartan subalgebra (CSA) generators in the following way:

$$\begin{aligned}
\kappa\,(H_i\,,\,H_j) &= \delta_{ij} \quad \Rightarrow \quad H_\alpha = \alpha_i\,H_i \\
\kappa\left(E^\alpha\,,\,E^{-\alpha}\right) &= 1 \\
\kappa\,(H_i\,,\,E^\alpha) &= 0
\end{aligned} \tag{1.2.37}$$

With this normalization the commutation relations of the *complex semisimple Lie algebra* take the following general form:

$$\begin{aligned}
[H_i\,,\,H_j] &= 0 \\
[H_i\,,\,E^\alpha] &= \alpha_i\,E^\alpha \\
\left[E^\alpha\,,\,E^{-\alpha}\right] &= \alpha^i\,H_i \\
\left[E^\alpha\,,\,E^\beta\right] &= N(\alpha,\beta)\,E^{\alpha+\beta} \qquad \text{if} \qquad \alpha+\beta \in \Phi \\
\left[E^\alpha\,,\,E^\beta\right] &= 0 \qquad \text{if} \qquad \alpha+\beta \notin \Phi
\end{aligned} \tag{1.2.38}$$

where $N(\alpha,\beta)$ is a coefficient that has to be determined using Jacobi identities.

1.2.2 Properties of Root systems

Let us now consider the properties of a root system associated with a semisimple Lie algebra. We have the

Theorem 1.2.2. If $\alpha, \beta \in \Phi$ are two roots, then the following two statements are true:

(i) $2\,\frac{(\alpha\,,\beta)}{(\alpha\,,\alpha)} \in \mathbb{Z}$

(ii) $\beta - 2\,\alpha\,\frac{(\alpha\,,\beta)}{(\alpha\,,\alpha)} \in \Phi$ is also a root.

Proof 1.2.2.1. Let $\alpha, \beta \in \Phi$ be two roots, and let us define the non negative integer $j \in \mathbb{N}$ by means of the following conditions

$$\begin{aligned} \gamma \equiv \beta + j\,\alpha \in \Phi \\ \gamma + \alpha \notin \Phi \end{aligned} \tag{1.2.39}$$

In other words j is the maximal integer n for which $\beta + n\alpha$ is a root.

We know that $-\alpha$ is a root and hence we can conclude that:

$$\begin{aligned} \left[E^{-\alpha}\,,\, E^{\gamma}\right] &= \widehat{E}^{\gamma-\alpha} \\ \left[E^{-\alpha}\,,\, \widehat{E}^{\gamma-\alpha}\right] &= \widehat{E}^{\gamma-2\alpha} \\ &\ldots\ldots\ldots \end{aligned} \tag{1.2.40}$$

where $\widehat{E}^{\gamma-n\alpha}$ denotes some element in the one–dimensional subspace pertaining to the root $\gamma - n\alpha$. Since the number of roots is necessarily finite it follows that there exists some positive integer $g \in \mathbb{N}$ such that:

$$\left[E^{-\alpha}\,,\, \widehat{E}^{\gamma-g\alpha}\right] = \widehat{E}^{\gamma-(g+1)\alpha} = 0 \tag{1.2.41}$$

In general, due to the one–dimensionality of the each root space we can set:

$$\left[E^{\alpha}\,,\, \widehat{E}^{\gamma-(n+1)\alpha}\right] = \mu_{n+1}\,\widehat{E}^{\gamma-n\alpha} \tag{1.2.42}$$

where μ_{n+1} is some normalization factor. From Jacobi identities we immediately obtain a recursion relation satisfied by these normalization factors. Indeed:

$$\begin{aligned} \left[E^{\alpha}\left[E^{-\alpha}\,,\, \widehat{E}^{\gamma-n\alpha}\right]\right] &= -\left[E^{-\alpha}\left[\widehat{E}^{\gamma-n\alpha}\,,\, E^{\alpha}\right]\right] - \left[E^{\gamma-n\alpha}\left[E^{\alpha}\,,\, E^{-\alpha}\right]\right] \\ &= \mu_n\,\widehat{E}^{\gamma-n\alpha} + \alpha^i\left[H_i\,,\, \widehat{E}^{\gamma-n\alpha}\right] \\ &= \left(\mu_n + (\gamma\,,\,\alpha) - n\,(\alpha\,,\,\alpha)\right)\,\widehat{E}^{\gamma-n\alpha} \end{aligned} \tag{1.2.43}$$

which implies the recursion relation:

$$\mu_{n+1} = \mu_n + (\gamma\,,\,\alpha) - n\,(\alpha\,,\,\alpha) \tag{1.2.44}$$

Since by hypothesis $\gamma + \alpha$ is not a root we have

$$\left[E^{\alpha}\,,\, E^{\gamma}\right] = \mu_0\,E^{\gamma+\alpha} = 0 \quad \text{namely} \quad \mu_0 = 0 \tag{1.2.45}$$

This allows to solve the recursion relation explicitly obtaining:

$$\mu_n = n\,(\alpha\,,\,\gamma) - \tfrac{n(n-1)}{2}\,(\alpha\,,\,\alpha) \tag{1.2.46}$$

Since, at the other end of the chain, we have assumed that $\gamma - (g+1)\alpha$ is not a root, we conclude that $\mu_{g+1} = 0$ and hence:

$$(g+1)\,\left\{(\alpha\,,\,\gamma) - \tfrac{g}{2}\,(\alpha\,,\,\alpha)\right\} = 0 \tag{1.2.47}$$

This implies that $2\frac{(\gamma\,,\,\alpha)}{(\alpha\,,\,\alpha)} = g \in \mathbb{N}$ Hence for each pair of roots $\alpha\,\beta$ there exists a non negative integer $j \geq 0$ such that $\gamma = \beta + j\alpha$ is a root and

$$(\alpha\,,\,\beta) = (\alpha\,,\,\gamma) - j\,(\alpha\,,\,\alpha) = \left(\tfrac{g}{2} - j\right)\,(\alpha\,,\,\alpha) \tag{1.2.48}$$

namely:

$$2\,\frac{(\alpha\,,\beta)}{(\alpha\,,\alpha)} = g - 2j \in \mathbb{Z} \qquad \text{(positive or negative)} \tag{1.2.49}$$

This concludes the first part of our proof. Let us now consider the string or roots that we have constructed to make the above argument:

$$\begin{aligned}
\beta_0 &= \gamma = \beta + j\alpha \\
\beta_1 &= \gamma - \alpha = \beta + (j-1)\alpha \\
\beta_2 &= \gamma - 2\alpha = \beta + (j-2)\alpha \\
\ldots\ldots\ldots & \qquad (1.2.50) \\
\beta_g &= \gamma - g\alpha = \beta + (j-g)\alpha \qquad (1.2.51)
\end{aligned}$$

Since $2\frac{(\gamma\,,\alpha)}{(\alpha\,,\alpha)} = g$, it is evident by means of the replacement:

$$\beta \mapsto \beta - 2\,\alpha\,\frac{(\beta\,,\,\alpha)}{(\alpha\,,\,\alpha)} \tag{1.2.52}$$

the string (1.2.51) is simply reflected into itself $\beta_g \mapsto \beta_0, \beta_{g-1} \mapsto \beta_1, \ldots$. So not only we proved that if β and α are roots then the reflection of β with respect to α is a root but also that the entire string of α through β is invariant under such reflection.

1.2.2.1 Angles between the roots

It is convenient to introduce the following notation of a *hook product*

$$\langle\,\beta\,,\,\alpha\,\rangle \;\equiv\; 2\,\alpha\,\frac{(\beta\,,\,\alpha)}{(\alpha\,,\,\alpha)} \tag{1.2.53}$$

¿From theorem 1.2.2 we have learned that $\langle\,\beta\,,\,\alpha\,\rangle \in \mathbb{Z}$, but at the same time also $\langle\,\alpha\,,\,\beta\,\rangle \in \mathbb{Z}$. Hence we conclude that

$$\langle\,\beta\,,\,\alpha\,\rangle\,\langle\,\alpha\,,\,\beta\,\rangle = 4\cos^2\theta_{\alpha\beta} \in \mathbb{Z} \tag{1.2.54}$$

where $\theta_{\alpha\beta}$ is the angle between the two roots.

Explicitely, the table of possible ...

$\theta_{\alpha\beta}$	$\cos\theta_{\alpha\beta}$	$\cos^2\theta_{\alpha\beta}$	$\lvert\beta\rvert^2/\lvert\alpha\rvert^2$	$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$
$\pi/6$	$\sqrt{3}/2$	$3/4$	3	1	3
$\pi/4$	$\sqrt{2}/2$	$1/2$	2	1	2
$\pi/3$	$1/2$	$1/4$	1	1	1
$\pi/2$	0	0	undet.	0	0
$2\pi/3$	$-1/2$	$1/4$	1	-1	-1
$3\pi/4$	$-\sqrt{2}/2$	$1/2$	2	-1	-2
$5\pi/6$	$-\sqrt{3}/2$	$3/4$	3	-1	-3

Table 1.1. Possible angles and ratio of norms between pairs of roots.

1.2.2.2 Root systems in rank 1 and 2

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1.2.3 Simple root systems

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1.2.3.1 Decomposable root systems

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1.2.3.2 Simple root systems

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1.2.4 The Weyl group

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1.2.5 The Cartan matrix

The Cartan matrix associated to a simple root system $\Delta = \{alpha_i\}$, $i = 1, \ldots, r$, of a simple Lie algebra of ramk r is defined by

$$C_{ij} = \langle \alpha_i, \alpha_j \rangle = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \,. \tag{1.2.55}$$

According to eq. , there are only the following possibilities

$$\begin{aligned} &C_{ii} = 2\,,\ \forall i\,, \\ &C_{ij} = 0, -1, -2, -3\,,\ \forall i \neq j\,. \end{aligned} \tag{1.2.56}$$

Notice that the Cartan matrix is in general not symmetric: $\langle \alpha_i, \alpha_j \rangle \neq \langle \alpha_j, \alpha_i \rangle$ unless the two roots have the same length. So the Cartan matrix is symmetric only if all the simple roots have the same length (in which case the algebra is said to be a *simply-laced* Lie algebra.

Example For instance, consider the root system B_2 of Fig. **??**. We have $\langle \alpha_1, \alpha_2 \rangle = -1$ and $\langle \alpha_2, \alpha_1 \rangle = -2$, so that the corresponding Cartan matrix is

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \,. \tag{1.2.57}$$

Example: the Lie algebra $A_2 \sim \mathrm{sl}(3, \mathbb{C})$ ...

Example: the G_2 algebra ...

1.3 Classification of the irreducible root systems

Having established that all possible irreducible root systems Φ are uniquely determined (up to isomorphisms) by the Cartan matrix we can classify *all the complex simple Lie algebras* by classifying all possible Cartan matrices.

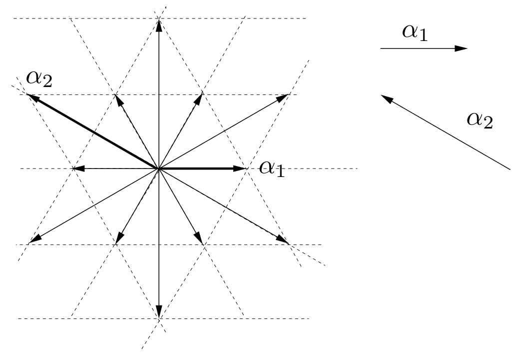


Figure 1.1. The root system G_2.

1.3.1 Dynkin diagrams

Each Cartan matrix can be given a graphical representation in the following way. To each simple root α_i we associate a circle ◯ as in fig.1.2 and then we link the i-th circle with the j-th circle

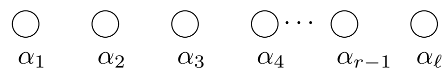


Figure 1.2. The simple roots α_i are represented by circles

by means of a line which is *simple*, *double* or *triple* depending on whether

$$< \alpha_i \, , \, \alpha_j >< \alpha_j \, , \, \alpha_i >= 4 \cos^2 \theta_{ij} = \begin{cases} 1 \\ 2 \\ 3 \end{cases} \tag{1.3.1}$$

having denoted θ_{ij} the angle between the two simple roots α_i and α_j. The corresponding graph is named a **Coxeter graph**.

If we consider the simplest case of two–dimensional Cartan matrices we have the four possible Coxeter graphs depicted in fig. 1.3 Given a Coxeter graph if it is *simply laced*, namely if there

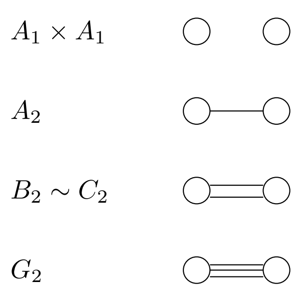
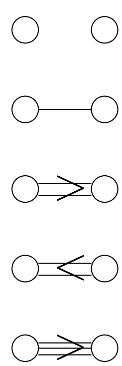


Figure 1.3. The four possible Coxeter graphs with two vertices

are only simple lines, then all the simple roots appearing in such a graph have the same *length* and the corresponding Cartan matrix is completely identified. On the other hand if the Coxeter graph involves double or triple lines, then, in order to identify the corresponding Cartan matrix, we need to specify which of the two roots sitting at the end points of each multiple line is the *long* root and which is the *short* one. This can be done by associating an arrow to each multiple line. By convention we decide that this *arrow points* in the direction of the *short root.* A Coxeter graph equipped with the necessary arrows is named a **Dynkin diagram**. Applying this convention to the case of the Coxeter graphs of fig. 1.3 we obtain the result displayed in fig. 1.4. The one-to-one



$$
\begin{array}{lcc}
A_1 \times A_1 & = & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\
A_2 & = & \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\
B_2 & = & \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \\
C_2 & = & \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \\
G_2 & = & \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}
\end{array}
$$

Figure 1.4. The distinct Cartan matrices in two dimensions (and therefore the simple Algebras in rank two) correspond to the Dynkin diagrams displayed above. We have distinguished a B_2 and a C_2 matrix since they are the limiting case for $\ell = 2$ of two series of Cartan matrices the B_ℓ and the C_ℓ series that for $\ell > 2$ are truly different. However B_2 is the transposed of C_2 so that they correspond to isomorphic algebras obtained one from the other by renaming the two simple roots $\alpha_1 \leftrightarrow \alpha_2$

correspondence between the Dynkin diagram and the associated Cartan matrix is illustrated by considering in some detail the case B_2 of fig. 1.4. By definition of the Cartan matrix we have:

$$
\begin{aligned}
2\,\frac{(\alpha_1\,,\,\alpha_2)}{(\alpha_2\,,\,\alpha_2)} &= 2\,\frac{|\alpha_1|}{|\alpha_2|}\,\cos\theta = \, -\,2 \\
2\,\frac{(\alpha_2\,,\,\alpha_1)}{(\alpha_1\,,\,\alpha_1)} &= 2\,\frac{|\alpha_2|}{|\alpha_1|}\,\cos\theta = \, -\,1
\end{aligned}
\tag{1.3.2}
$$

so that we conclude:

$$
|\alpha_1|^2 = 2\,|\alpha_2|^2 \tag{1.3.3}
$$

which shows that α_1 is a long root, while α_2 is a short one. Hence the arrow in the Dynkin diagram pointing towards the short root α_2 tells us that the matrix elements C_{12} is -2 while the matrix element C_{21} is -1. It happens the opposite in the example C_2.

1.3.2 The classification theorem

Having clarified the notation of Dynkin diagrams the basic classification theorem of *complex simple Lie algebras* is the following:

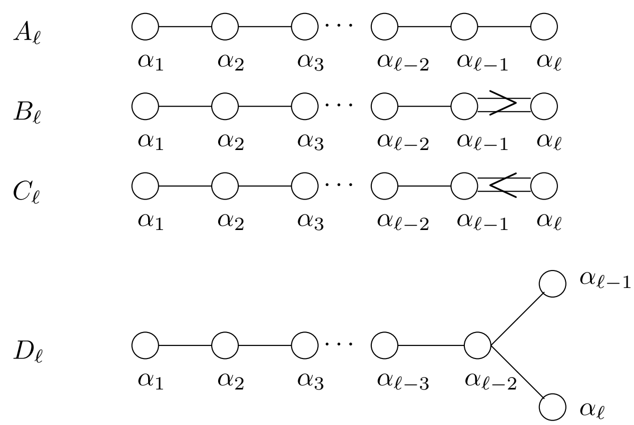


Figure 1.5. The Dynkin diagrams of the four infinite families of classical simple algebras

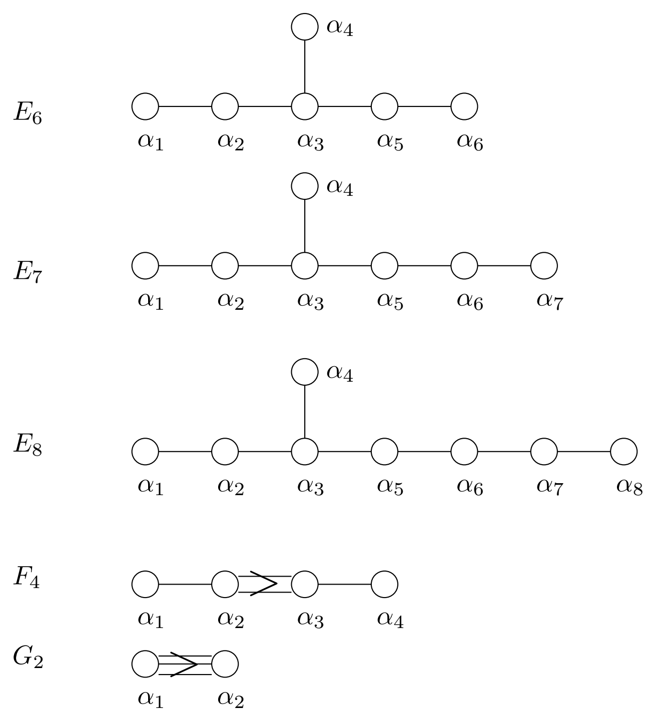


Figure 1.6. The Dynkin diagrams of the five exceptional algebras

Theorem 1.3.1. If Φ is an irreducible system of roots of rank ℓ then its Dynkin diagram is either one of those shown in fig.1.5 or for special values of ℓ is one of those shown in fig.1.6. There are no other irreducible root systems besides these ones.

Proof 1.3.1.1. Let us consider a Euclidean space E and in E let us consider set of vectors:

$$\mathcal{U} = \{\epsilon_1\,,\,\epsilon_2\,,\,\dots\,,\epsilon_\ell\} \tag{1.3.4}$$

that satisfy the following three conditions:

$$\begin{aligned}(\epsilon_i\,,\,\epsilon_i)\ &= 1\\ (\epsilon_i\,,\,\epsilon_j)\ &\leq 0 \quad i \neq j\\ 4\,(\epsilon_i\,\epsilon_j)^2\ &= 0,1,2,3 \quad i \neq j\end{aligned} \tag{1.3.5}$$

Such a system of vectors is named *admissible*. It is clear that each admissible system of vectors singles out a coxeter graph Γ. Indeed the vectors ϵ_i correspond to the simple roots α_i divided by their norm:

$$\epsilon_i = \frac{\alpha_i}{\sqrt{|\alpha_i|^2}} \tag{1.3.6}$$

Our task is that of classifying all connected Coxeter graphs.

We proceed through a series of steps.

Step 1 We note that by deleting a subset of vectors ϵ_i in an admissible system those that are left still form an admissible system whose Coxeter graph is obtained from the original one by deleting the corresponding vertices and all the lines that end in these vertices

Step 2 The number of pairs of vertices that are connected by at least one line is strictly less than the number of vectors ϵ_i namely strictly less than ℓ. Indeed let us set $\epsilon = \sum_{i=1}^{\ell} \epsilon_i$ and observe what follows. Since all the ϵ_i are independent we have $\epsilon \neq 0$ Hence

$$0 < (\epsilon\,,\,\epsilon) = \ell + 2 \sum_{i<j} (\epsilon_i\,,\,\epsilon_j) \tag{1.3.7}$$

If the i-th vertex is joined to the j-th vertex we have $4\,(e_i\,,\,\epsilon_j)^2 = 1,2,3$. Hence we can conclude that, in this case:

$$2\,(\epsilon_i\,,\,\epsilon_j) \leq -1 \tag{1.3.8}$$

On the other hand if the i-th vertex is not joined to the j-th vertex we have $2\,(\epsilon_i\,,\,\epsilon_j) = 0$. Naming N_J the number of pairs of vertices joined by at least one line we conclude that:

$$0 < (\epsilon\,,\,\epsilon) < \ell - N_J \quad \Rightarrow \quad N_J \leq \ell - 1 \tag{1.3.9}$$

which is what we have asserted

Step 3 The Coxeter graph Γ cannot contain any loop. Indeed if a loop existed this would constitute a subgraph Γ' for which the number of pairs joined by a line N_J would be larger than the number of vertices and this we have shown to be impossible.

Step 4 The number of lines that end up in any vertex can be at most three. Indeed let $\epsilon \in \mathcal{U}$ and let us denote $\eta_1, \eta_2, \dots, \eta_k$ the vectors connected to ϵ by some link. In other words we have $(\epsilon, \eta_i) < 0\,(\forall \eta_i)$. Since there are no loops in the graph it follows that no η_i can be connected to any other η_j, namely $(\eta_i\,,\,\eta - j) = 0 \forall i \neq j$. Since U is a set of linearly independent vectors there must exist a unit vector η_0 in the vector span of $\epsilon, \eta_1, \ldots, \eta_k$ which is orthogonal to $\eta_1, \ldots, \eta_k$. Obviously the projection of such a vector η_0 is non vanishing on ϵ, namely $(\epsilon\,,\,\eta_0) \neq 0$. The set $\eta_0, \eta_1, \ldots, \eta_k$ makes an orthogonal basis for the linear span of the vectors $\epsilon, \eta_1, \ldots, \eta_k$ and we can write:

$$\begin{aligned} \epsilon &= \sum_{i=0}^{k} (\epsilon\,,\,\eta_i)\,\eta_i \\ 1 = (\epsilon\,,\,\epsilon) &= \sum_{i=0}^{k} (\epsilon\,,\,\eta_i)^2 \end{aligned} \tag{1.3.10}$$

This reasoning implies that $\sum_{i=1}^{k} (\epsilon\,,\,\eta_i)^2 < 1$ and hence

$$4 \sum_{i=1}^{k} (\epsilon\,,\,\eta_i)^2 < 4 \tag{1.3.11}$$

On the other hand $4\,(\epsilon\,,\,\eta_i)^2$ is precisely the number of lines that link η_i to ϵ so that eq.(1.3.11) is precisely the statement we wanted to prove in **Step 4**

Step 5 The only connected Coxeter graph that contains a triple line is the G_2 graph of fig.1.3. This immediately follows from **Step 4**.

Step 6 Let $\{\epsilon_1, \ldots, \epsilon_k\} \subset \mathcal{U}$ be a subset of vectors corresponding to a simple line as in fig.1.7. Then the subset $\mathcal{U}' \equiv \{\mathcal{U} - \{\epsilon_1, \ldots, \epsilon_k\}\} \bigcup \{\epsilon\}$ where $\epsilon \equiv \sum_{i=1}^{k} \epsilon_i$ is still an admissible

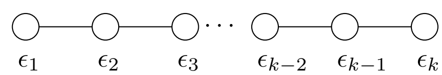


Figure 1.7. A simple line Coxeter graph

system. Graphically the operation of making the transition from the admissible system $\mathcal{U}$ to the admissible system $\mathcal{U}'$ corresponds to collapsing the entire simple line a single vertex. That this statement is true can be proved in the following way. That the vectors composing $\mathcal{U}'$ are linearly independent is obvious. By hypothesis of a simple chain we have:

$$2\,(\epsilon_i\,,\,\epsilon_{i+1}) = -1 \quad 1 \leq i \leq k-1 \tag{1.3.12}$$

so that

$$(\epsilon\,,\,\epsilon) = k + 2 \sum_{i<j} (\epsilon_i\,,\,\epsilon_j) = k - (k-1) = 1 \tag{1.3.13}$$

and hence ϵ is a unit vector. Furthermore each $\eta \in \mathcal{U} - \{\epsilon_1, \ldots, \epsilon_k\}$ can be joined at most to one of the vectors $\epsilon_1, \ldots, \epsilon_k$. Otherwise we would generate a loop. Hence we either have $(\eta\,,\,\epsilon) = 0$ or we have $(\eta\,,\,\epsilon_i)$ for some value of i. In any case we conclude $4\,(\eta, \epsilon_i)^2 = 0, 1, 2, 3$ which is what makes $\mathcal{U}'$ and admissible system.

Step 7 A Coxeter graph cannot contain subgraphs of the form displayed in fig. 1.8 Indeed in all these three cases, by using the property shown in **Step 6** and collapsing a simple chain

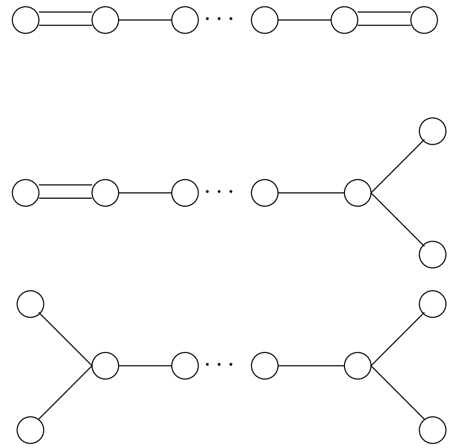


Figure 1.8. Prohibited subgraphs

we obtain a graph that contains a vertex where 4 lines converge. This was shown to be forbidden in **Step 4**

Step 8 Relying on the properties we have so far proven we are left with four types of possible Coxeter graphs, namely i) the simple chains of length ℓ corresponding to the A_ℓ Dynkin diagrams of fig.1.5, ii) the G_2 graph of fig.1.3 iii) the graphs of fig.1.9 with a double line and finally iv) the graphs of fig.1.10 with a node.

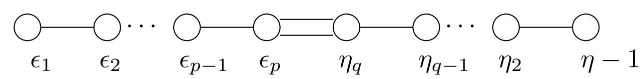


Figure 1.9. Coxeter graph with a double link that is preceded by a simple chain of length p and followed by a simple chain of length q

Step 9 If we consider the graphs of the type shown in fig.1.9 there are only two solutions namely:

$$\begin{array}{rclcrclccc} p & = & 2 & ; & q & = & 2 & \Rightarrow & F_4 & \text{Dynkin diagram} \\ p & = & \ell \in \mathbb{N} & ; & q & = & 1 & \Rightarrow & B_\ell \;\; or \;\; C_\ell & \text{Dynkin diagrams} \end{array} \tag{1.3.14}$$

Indeed let us set $\epsilon = \sum_{i=1}^{p} i\, \epsilon_i$ and $\eta = \sum_{i=1}^{q} i\, \eta_i$ By hypothesis of simple chains we have $2\,(\epsilon_i\,,\,\epsilon_{i+1}) = -1$, $2\,(\eta_i\,,\,\eta_{i+1}) = -1$ and all the other pairs of vectors are mutually orthogonal. In this way we obtain:

$$\begin{aligned} (\epsilon\,,\,\epsilon) &= \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i-1) = p\,\frac{p-1}{2} \\ (\eta\,,\,\eta) &= \sum_{i=1}^{q} i^2 - \sum_{i=1}^{q-1} i(i-1) = q\,\frac{q-1}{2} \end{aligned} \tag{1.3.15}$$

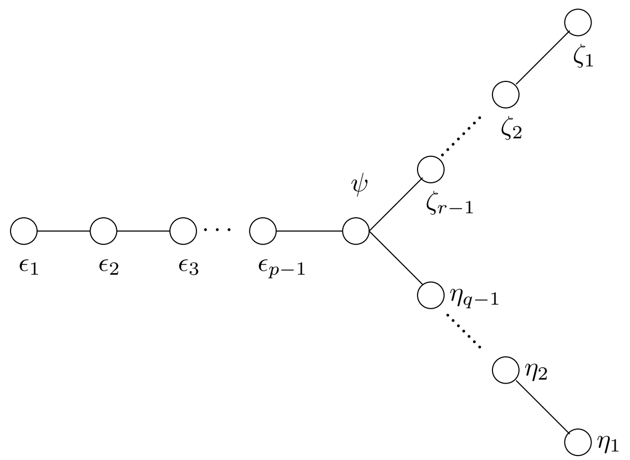


Figure 1.10. Coxeter graph with a node. The unit vector in the node is named ψ while the unit vectors along the three simple lines departing from the node are respectively named $\epsilon_1, \dots, \epsilon_{p-1}$, $\eta_1, \dots, \eta_{q-1}$, $\zeta_1, \dots, \zeta_{r-1}$. The graph is characterized by the three integer numbers p, q, r that denote the lengths of the three simple lines departing from the node

and since by hypothesis of double line we have: $4\,(\epsilon_p\,,\,\eta_q)^2 = 2$ it follows that

$$(\epsilon\,,\,\eta)^2 = p^2\,q^2\,\left(\epsilon_p\,,\,\eta_q\right)^2 = \frac{1}{2}\,p^2\,q^2 \tag{1.3.16}$$

On the other from the triangular Schwarz inequality of Euclidean geometry we have:

$$\begin{array}{c} (\epsilon\,,\,\eta) \,<\, (\epsilon\,,\,\epsilon)\,\,(\eta\,,\,\eta) \\ \Downarrow \\ (p-1)\,\,(q-1)\,<2 \end{array} \tag{1.3.17}$$

which for positive integers p, q admits only the two solutions advocated in eq.(1.3.14). The first solution leads to the Dynkin diagram of the exceptional Lie algebra F_4, while the second solution leads to the two infinite series of classical Lie algebras B_ℓ and C_ℓ.

Step 10 Let us finally consider the Coxeter graphs of the type shown in fig.1.10. We claim that the only possible solutions are:

$$(p,q,r) = \left\{ \begin{array}{lclllcl} (\ell,1,1) & \Rightarrow & A_\ell & \text{Dynkin diagrams} & \ell & \in & \mathbb{N} \\ (\ell-2,2,2) & \Rightarrow & D_\ell & \text{Dynkin diagrams} & 4 \le \ell & \in & \mathbb{N} \\ (3,3,2) & \Rightarrow & E_6 & \text{Dynkin diagram} & & & \\ (4,3,2) & \Rightarrow & E_7 & \text{Dynkin diagram} & & & \\ (5,3,2) & \Rightarrow & E_8 & \text{Dynkin diagram} & & & \end{array} \right. \tag{1.3.18}$$

To prove this statement we follow a strategy similar to that used in the proof of **Step 9** and we define the following three vectors:

$$\epsilon = \sum_{i=1}^{p-1} i\,\epsilon_i \quad ; \quad \eta = \sum_{i=1}^{q-1} ii\,\eta_i \quad ; \quad \sum_{i=1}^{r-1} i\,\zeta_i \tag{1.3.19}$$

Clearly ϵ, η, ζ are mutually orthogonal and ψ, the vector in the node is not in the subspace generated by ϵ, η, ζ. Hence if in the linear span of $\{\psi, \epsilon, \eta, \zeta\}$ we construct a vector γ that is orthogonal to $\{\epsilon, \eta, \zeta\}$ we obtain that $(\gamma\,,\,\psi) \neq 0$. Normalizing this vector to 1 we can write:

$$\psi = (\psi\,,\,\gamma)\;\gamma + \frac{(\psi\,,\,\epsilon)}{\sqrt{(\epsilon\,,\,\epsilon)}}\,\epsilon + \frac{(\psi\,,\,\eta)}{\sqrt{(\eta\,,\,\eta)}}\,\eta + \frac{(\psi\,,\,\zeta)}{\sqrt{(\zeta\,,\,\zeta)}}\,\zeta \tag{1.3.20}$$

and we obtain:

$$(\psi\,,\,\psi) = 1 = (\psi\,,\,\gamma)^2 + \frac{(\psi\,,\,\epsilon)^2}{(\epsilon\,,\,\epsilon)} + \frac{(\psi\,,\,\eta)^2}{(\eta\,,\,\eta)} + \frac{(\psi\,,\,\zeta)^2}{(\zeta\,,\,\zeta)} \tag{1.3.21}$$

that implies the inequality:

$$1 > \frac{(\psi\,,\,\epsilon)^2}{(\epsilon\,,\,\epsilon)} + \frac{(\psi\,,\,\eta)^2}{(\eta\,,\,\eta)} + \frac{(\psi\,,\,\zeta)^2}{(\zeta\,,\,\zeta)} \tag{1.3.22}$$

By definition of the Coxeter graph in fig.1.10 we have:

$$\begin{aligned} (\psi\,,\,\epsilon) = (p-1)\;(\epsilon_{p-1}\,,\,\psi) \quad \Rightarrow \quad (\psi\,,\,\epsilon)^2 &= \frac{(p-1)^2}{4} \\ (\epsilon\,,\,\epsilon) &= \frac{p(p-1)}{2} \end{aligned} \tag{1.3.23}$$

and similarly for the scalar products associated with the other chains. Inserting these results into the inequality of eq.(1.3.22) we obtain the Diophantine inequality:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 \tag{1.3.24}$$

whose independent solutions are those displayed in eq.(1.3.18). To this effect it is sufficient to note that eq.(1.3.24) has an obvious permutational symmetry in the three numbers p, q, r. To avoid double counting of solutions we break this symmetry by setting $p \geq q \geq r$ and then we see that the only possibilities are those listed in eq.(1.3.18).

■

1.4 An isomorphic problem: the ADE classification of Kleinian groups

We begin by considering one parameter subgroups of the rotation group in three dimensions, namely SO(3). These are singled out by a rotation axis, namely by a point on the two–sphere S^2. Explicitly let us consider a solution (ℓ, m, n) to the sphere equation :

$$\ell^2 + m^2 + n^2 = 1 \tag{1.4.25}$$

The triplet of real numbers (ℓ, m, n) parametrize the direction of a possible rotation angle. The generator of infinitesimal rotations around such an axis is given by the following matrix

$$A = A_{\ell,m,n} = \begin{pmatrix} 0 & -n & m \\ n & 0 & -\ell \\ -m & \ell & 0 \end{pmatrix} = -A^T \tag{1.4.26}$$

which being antisymmetric belongs to the SO(3) Lie algebra. The matrix A has the property that $A^3 = -A$ and explicitly we have:

$$A^2 = \begin{pmatrix} -1+\ell^2 & \ell\, m & \ell\, n \\ \ell\, m & -1+m^2 & m\, n \\ \ell\, n & m\, n & -1+n^2 \end{pmatrix} \tag{1.4.27}$$

Hence a finite element of the group SO(3) corresponding to a rotation of an angle θ around this axis is given by:

$$\mathcal{O} = \exp[\theta A] = \mathbf{1} + \sin\theta\, A \,+\, (1-\cos\theta)\, A^2 \tag{1.4.28}$$

Setting

$$\begin{aligned} \lambda &= \ell\, \sin\frac{\theta}{2} \\ \mu &= m\, \sin\frac{\theta}{2} \\ \nu &= n\, \sin\frac{\theta}{2} \\ \rho &= \cos\frac{\theta}{2} \end{aligned} \tag{1.4.29}$$

the corresponding SU(2) finite group elements, realizing the double covering are:

$$\mathcal{U} = \pm \begin{pmatrix} \rho + \mathrm{i}\nu & \mu - \mathrm{i}\lambda \\ -\mu - \mathrm{i}\lambda & \rho - \mathrm{i}\nu \end{pmatrix} \tag{1.4.30}$$

We can now consider the argument that leads to the ADE classification of the finite subgroups of SU(2). Let us consider the action of the SU(2) matrices on $\mathbb{C}^2$. A generic 2×2 matrix that belongs to $SU(2)$ can be written as follows

$$\mathcal{U} = \begin{pmatrix} \alpha & \mathrm{i}\beta \\ \mathrm{i}\overline{\beta} & \overline{\alpha} \end{pmatrix} \tag{1.4.31}$$

in terms of two complex numbers α, β satisfying the constraint:

$$|\alpha|^2 + |\beta|^2 = 1 \tag{1.4.32}$$

the corresponding action of $\mathcal{U}$ on a $\mathbb{C}^2$-vector $\overrightarrow{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is given by the usual matrix multiplication $\mathcal{U}\overrightarrow{z}$. Each element $\mathcal{U} \in$ SU(2) has two eigenvectors $\overrightarrow{z}_1$ and $\overrightarrow{z}_2$, such that

$$\begin{aligned} \mathcal{U}\, \overrightarrow{z}_1 &= \exp[\mathrm{i}\theta]\, \overrightarrow{z}_1 \\ \mathcal{U}\, \overrightarrow{z}_2 &= \exp[-\mathrm{i}\theta]\, \overrightarrow{z}_2 \end{aligned} \tag{1.4.33}$$

where θ is some (half)-rotation angle. Namely for each $\mathcal{U} \in$ SU(2) we can find an orthogonal basis where $\mathcal{U}$ is diagonal and given by:

$$\mathcal{U} = \begin{pmatrix} \exp[\mathrm{i}\theta] & 0 \\ 0 & \exp[-\mathrm{i}\theta] \end{pmatrix} \tag{1.4.34}$$

for some angle θ. Then let us consider the rays $\{\lambda\, \overrightarrow{z}_1\}$ and $\{\mu\, \overrightarrow{z}_2\}$ where $\lambda, \mu \in \mathbb{C}$ are arbitrary complex numbers. Since $\overrightarrow{z}_1 \cdot \overrightarrow{z}_2 = \overrightarrow{z}_1^{\dagger} \overrightarrow{z}_2 = 0$ it follows that each element of SU(2) singles out two rays, hereafter named *poles* that are determined one from the other by the orthogonality relation. This concept of pole is the basic item in the argument leading to the Kleinian groups classification.

Let $H \subset \mathrm{SU}(2)$ be a finite, discrete subgroup of $SU(2)$. Then the order of H is some positive integer number:

$$|H| = n \in \mathbb{N} \tag{1.4.35}$$

The total number of poles associated with H is:

$$\# \,\text{of poles} = 2n - 2 \tag{1.4.36}$$

since $n-1$ is the number of elements in H that are different from the identity. Let us then adopt the notation

$$p_i \equiv \{\lambda \overrightarrow{z}_i\} \tag{1.4.37}$$

for the pole or ray singled out by the eigenvector $\overrightarrow{z}_i$. We say that two poles are equivalent if there exists an element of the group H that maps one into the other:

$$p_i \sim p_j \quad \text{iff} \quad \exists \gamma \in H \,/\, \gamma p_i = p_j \tag{1.4.38}$$

Let us distribute the poles p_i into equivalence classes:

$$\mathcal{C}_\alpha = \left\{p_1^\alpha, \dots, p_{m_\alpha}^\alpha\right\} \quad ; \quad \alpha = 1, \dots, r \tag{1.4.39}$$

and name m_α the cardinality of the equivalence class $\mathcal{C}_\alpha$, namely the number of poles it contains.

Each pole $p \in \mathcal{C}_\alpha$ has a stability subgroup $K_p \subset H$:

$$\forall\, h \,\in\, K_p \quad h\, p \,=\, p \tag{1.4.40}$$

that is a finite, abelian and cyclic of order k_α. Indeed it must be finite since it is a subgroup of a finite group, it must be abelian since in the basis $\overrightarrow{z}_1, \overrightarrow{z}_2$ the SU(2) matrices that preserve the poles $\lambda \overrightarrow{z}_1$ and $\mu \overrightarrow{z}_2$ are, of the form (1.4.34) and therefore it is cyclic of some order. The H group can be decomposed into cosets according to the subgroup K_p:

$$H = K_p + v_1\, K_p + \ldots + v_{m_\alpha}\, K_p \quad m_\alpha \in \mathbb{N} \tag{1.4.41}$$

Consider now an element $x_i \,\in v_i\, K_p$ belonging to one of the equivalence classes and define the group conjugate to K_p through x_i:

$$K_{(xp)_i} = x_i\, K_p\, x_i^{-1} \tag{1.4.42}$$

Each element $h \in K_{(xp)_i}$ admits a pole p_x:

$$h\, p_x = p_x \tag{1.4.43}$$

that is given by:

$$p_x = x_i\, p \tag{1.4.44}$$

since

$$h\, p_x = x\, h_p x\, x^{-1}\, p = x\, h_p\, p = x\, p = p_x \tag{1.4.45}$$

Hence the set of poles $\{p, v_1\, p, v_2\, p, \dots v_{m_\alpha}\, p\}$ are all equivalent and each of them has a stability group K_{p_i} conjugate to K_p which implies that all K_{p_i} are finite of the same order:

$$\forall v_i\, p \quad |K_{p_i}| = k_\alpha \tag{1.4.46}$$

Because of this we must have:

$$k_\alpha \, m_\alpha = n \tag{1.4.47}$$

and both k_α and m_α depend only on the equivalence class. The total number of poles we have in the equivalence class $\mathcal{C}_\alpha$ (counting coincidences) is:

$$\#\,\text{of poles in class } \mathcal{C}_\alpha \; = m_\alpha \; (k_\alpha - 1) \tag{1.4.48}$$

since the numebr of elements in K_p different from the identity is $k_\alpha - 1$. Hence we find

$$2\,n \,-\, 2 = \sum_{\alpha=1}^{r} m_\alpha \; (k_\alpha - 1) \tag{1.4.49}$$

Dividing by n we obtain:

$$2\,\left(1 - \frac{1}{n}\right) = \sum_{\alpha=1}^{r} m_\alpha \, \left(1 - \frac{1}{k_\alpha}\right) \tag{1.4.50}$$

We consider next the possible solutions to the diophantine equation (1.4.50) and to this effect we rewrite it as follows:

$$r + \frac{2}{n} - 2 = \sum_{\alpha=1}^{r} \frac{1}{k_\alpha} \tag{1.4.51}$$

We observe that $k_\alpha \geq 2$. Indeed each pole admits at least two group elements that keep it fixed, the identity and the non trivial group element that defines it by diagonalization. Hence we have the bound:

$$r + \frac{2}{n} - 2 \leq \frac{r}{2} \tag{1.4.52}$$

which implies:

$$r \leq 4 - \frac{4}{n} \quad \Rightarrow \quad r = 1, 2, 3 \tag{1.4.53}$$

On the other hand we also have $k_p \leq n$ so that:

$$r + \frac{2}{n} - 2 \geq \frac{r}{n} \quad \Rightarrow \quad r\,\left(1 - \frac{1}{n}\right) \geq 2\,\left(1 - \frac{1}{n}\right) \quad \Rightarrow \quad r \geq 2 \tag{1.4.54}$$

Therefore there are only two possible cases:

$$r = 2 \quad \text{or} \quad r = 3 \tag{1.4.55}$$

1.5 Identification of the classical and exceptional Lie algebras

In the previous sections we have classified the allowed Dynkin diagrams and hence the allowed simple root systems. We have not shown that all of them do indeed indeed exist. This is what we do in the present section by explicit construction. Furthermore we identify the classical or exceptional complex Lie algebra that corresponds to each of the constructed root systems.

Table 1.2. Comparison between the ADE classification of simply laced Lie Algebras and the ADE classification of Kleinian subgroups of SU(2)

	Simple Lie Algebras	Kleinian groups
r	rank	# of conjugacy classes of poles
k_α	lengths of simple chains in Dynkin diagram	order of stability subgroups of poles
m_α		# of poles in the α-th conjugacy class

The 1st case Choosing $r = 2$, the diophantine equation (1.4.51) reduces to:

$$\frac{2}{n} = \frac{1}{k_1} + \frac{1}{k_2} \tag{1.4.56}$$

Since we have $k_{1,2} \leq n$, the only solution of (1.4.56) is $k_1 = k_2 = n$, with n arbitrary. Since the order of the cyclic stability subgroup of the two poles coincides with the order of the full group H it follows that H itself is a cyclic subgroup of $SU(2)$ of order n. The two equivalence classes are given by the two eigenvectors of the cyclic group generator.

A_ℓ

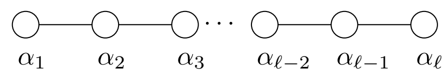


Figure 1.11. The Dynkin diagram of A_ℓ type

1.5.1 The A_ℓ root system and the corresponding Lie algebra

The Dynkin diagram is that recalled in fig.1.11. We want to perform the explicit construction of a root system that admits a basis corresponding to such a diagram.

To this effect consider the $\ell + 1$–dimensional Euclidean space $\mathbb{R}^{\ell+1}$ and let $\overrightarrow{\epsilon}_1, \ldots, \overrightarrow{\epsilon}_{\ell+1}$ denote the unit vectors along the $\ell + 1$ axes:

$$\overrightarrow{\epsilon}_1 = \begin{pmatrix} 1 \\ 0 \\ \ldots \\ \ldots \\ 0 \end{pmatrix} \quad , \quad \overrightarrow{\epsilon}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \quad \ldots \quad \overrightarrow{\epsilon}_{\ell+1} = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ \ldots \\ 1 \end{pmatrix} \tag{1.5.57}$$

Define the vector $\overrightarrow{v} = \epsilon_1 + \epsilon_2 + \ldots + \overrightarrow{\epsilon}_{\ell+1}$:

$$\overrightarrow{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \ldots \\ 1 \end{pmatrix} \tag{1.5.58}$$

and define $\mathbb{I} \subset \mathbb{R}^{\ell+1}$ to be the $\ell + 1$ –dimensional cubic lattice immersed in $\mathbb{R}^{\ell+1}$:

$$\mathbb{I} = \left\{ \overrightarrow{x} \in \mathbb{R}^{\ell+1} \,/\, \overrightarrow{x} = n^i \, \overrightarrow{\epsilon}_i \,, \quad n^i \in \mathbb{Z} \right\} \tag{1.5.59}$$

In the cubic lattice $\mathbb{I}$ consider the sublattice:

$$\mathbb{I}' = \mathbb{I} \bigcap E \tag{1.5.60}$$

where E is the hyperplane of vectors orthogonal to the vector $\overrightarrow{v}$:

$$E = \left\{ \overrightarrow{y} \in \mathbb{R}^{\ell+1} \,/\, (\overrightarrow{v}\,,\,\overrightarrow{y}) = 0 \right\} \tag{1.5.61}$$

Finally in the sublattice $\mathbb{I}'$ consider the finite set of vectors whose norm is $\sqrt{2}$:

$$\Phi = \{ \overrightarrow{\alpha} \in \mathbb{I}' \,/\, (\overrightarrow{\alpha}\,,\,\overrightarrow{\alpha}) = 2 \} \tag{1.5.62}$$

Theorem 1.5.1. The above defined set Φ is a root system and it corresponds to the A_ℓ Dynkin diagram

Proof 1.5.1.1. To prove this proposition let us first summarize the properties of Φ. We have:

$$\overrightarrow{\alpha} \in \Phi \Rightarrow \left\{ \begin{array}{lrcllrcl} 1) & \overrightarrow{\alpha} & = & n^i\,\overrightarrow{\epsilon}_i & & n^i & \in & \mathbb{Z} \\ 2) & (\overrightarrow{\alpha}\,,\,\overrightarrow{v}) & = & 0 & \Leftrightarrow & \sum_{i=1}^{\ell+1} n^i & = & 0 \\ 3) & (\overrightarrow{\alpha}\,,\,\overrightarrow{\alpha}) & = & 2 & \Leftrightarrow & \sum_{i=1}^{\ell+1} \left(n^i\right)^2 & = & 2 \end{array} \right. \tag{1.5.63}$$

These diophantine equations have the following solutions:

$$\overrightarrow{\alpha} = \overrightarrow{\epsilon}_i - \overrightarrow{\epsilon}_j \quad (i \neq j) \tag{1.5.64}$$

The number of such solutions is equal to twice the number of pairs (ij) in $\ell+1$–dimensional space[1]:

$$\#\,\alpha = 2\,\frac{1}{2}\,(\ell+1)\,(\ell+1-1) = \ell\,(\ell+1) = (\ell+1)^2 - 1 - \ell \tag{1.5.65}$$

We verify that this finite set of vectors is a root system. First we check that for all pairs $\alpha, \beta \in \Phi$ their hook product is an integer. Indeed we have:

$$\begin{aligned} <\alpha\,,\,\beta> &= 2\,\frac{(\alpha\,,\,\beta)}{(\beta\,,\,\beta)} = (\alpha\,,\,\beta) \\ &= (\epsilon_i - \epsilon_j\,,\,\epsilon_k - \epsilon_\ell) = \delta_{ik} - \delta_{jk} - \delta_{i\ell} + \delta_{j\ell} \in \mathbb{Z} \end{aligned} \tag{1.5.66}$$

Secondly we check that the reflection of any candidate root $\beta \in \Phi$ with respect to any other candidate root $\alpha \in \Phi$ belongs to the same set Φ:

$$\begin{aligned} \sigma_\alpha\,(\beta) &= \beta - (\alpha\,,\,\beta) \\ &= \epsilon_k - \epsilon_\ell - (\delta_{ik} - \delta_{jk} - \delta_{i\ell} + \delta_{j\ell})\,(\epsilon_i - \epsilon_j) \end{aligned} \tag{1.5.67}$$

If (k,ℓ) are both different from (i,j) then $\sigma_\alpha(\beta) = \beta \in \Phi$, so the statement is true. If $k = i$ then, necessarily $k \neq j$ and $i \neq \ell$, so that:

$$\sigma_\alpha\,(\beta) = \epsilon_k - \epsilon_\ell - (\delta_{ik} + \delta_{jl})\,(\epsilon_i - \epsilon_j) \tag{1.5.68}$$

If $j \neq \ell$ then:

$$\sigma_\alpha\,(\beta) = \epsilon_k - \epsilon_\ell - (1)\,(\epsilon_i - \epsilon_j) = \epsilon_j - \epsilon_\ell \in \phi \tag{1.5.69}$$

[1] In the sequel we omit the arrow on the vectors since the notation is by now clear

If $j = \ell$ then

$$\sigma_\alpha(\beta) = \epsilon_k - \epsilon_\ell - 2(\epsilon_k - \epsilon_\ell) = \epsilon_\ell - \epsilon_k \in \Phi \tag{1.5.70}$$

which exhausts all possible cases.
Hence, in $\mathbb{R}^{\ell+1}$ we have constructed a root system of $\ell(\ell+1)$ roots.
Consider the roots:

$$\alpha_i = (\epsilon_i - \epsilon_{i+1}) \quad , \quad (i = 1, \dots, \ell) \tag{1.5.71}$$

These roots are clearly linearly independent and given a root $\alpha \in \Phi$ it can be expressed as a linear combination of the α_i. Subdivide the set of roots into a positive and negative set according to the following rule:

$$\begin{array}{lcll} \Phi & = & \Phi_+ \bigcup \Phi_- & \\ \alpha & \in & \Phi_+ & : \ \{\alpha = \epsilon_i - \epsilon_j \quad , \quad i < j\} \\ \alpha & \in & \Phi_- & : \ \{\alpha = \epsilon_j - \epsilon_i \quad , \quad i < j\} \end{array} \tag{1.5.72}$$

Clearly positive roots can be written as follows:

$$\alpha = \epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1} \tag{1.5.73}$$

and, consequentely, have integer positive components in the $\{\alpha, \dots, \alpha_\ell\}$ basis. Negative roots have negative integer components. It follows that $\{\alpha, \dots, \alpha_\ell\}$ form a basis of simple roots.
Let us compute the Cartan matrix:

$$\begin{aligned} (\alpha_i \, , \, \alpha_j) &= (\epsilon_i - \epsilon_j \, , \, \epsilon_j - \epsilon_{j+1}) \quad , \quad i < j \\ &= \delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1} + \delta_{i+1,j+1} \end{aligned} \tag{1.5.74}$$

$$\begin{array}{llrr} \text{If} & i = j & (\alpha_i \, , \, \alpha_i) = & 2 \\ \text{If} & j = i+1 & (\alpha_i \, , \, \alpha_{i+1}) = & -1 \\ \text{If} & j = i-1 & (\alpha_i \, , \, \alpha_{i-1}) = & -1 \end{array}$$

Hence we have precisely the Dynkin diagram of fig.1.11.
This concludes the proof of our theorem ■

Theorem 1.5.2. The root system A_ℓ corresponds to the complex Lie algebra $\mathrm{SL}(\ell+1, \mathbb{C})$ of traceless matrices in $\ell + 1$ dimensions.

Proof 1.5.2.1. Note that the dimension of the $\mathrm{SL}(\ell+1, \mathbb{C})$ Lie algebra is:

$$\dim \mathrm{SL}(\ell+1, \mathbb{C}) = (\ell+1)^2 - 1 \tag{1.5.75}$$

since on the $(\ell+1) \times (\ell+1)$ matrix A we just impose one scalar condition, namely $\mathrm{tr}A = 0$. This agrees with the number of roots in the system Φ

$$\mathrm{card}\,\Phi = \ell(\ell+1) = (\ell+1)^2 - \ell - 1 \tag{1.5.76}$$

if the rank of $\mathrm{SL}(\ell+1, \mathbb{C})$ is precisely ℓ:

$$\mathrm{card}\,\Phi = \dim \mathbb{G} - \dim \mathcal{H} \tag{1.5.77}$$

$\mathcal{H}$ being the Cartan subalgebra.

This is indeed the case. Let e_{ij} denote the $(\ell+1)\times(\ell+1)$ matrix whose entries are all zero except for the ij-th entry which is one

$$
e_{ij} = \begin{array}{cc}
\begin{pmatrix}
0 & 0 & \dots & \dots & \dots & 0 \\
0 & 0 & \dots & \dots & \dots & 0 \\
\dots & \dots & 0 & 0 & 0 & \dots \\
\dots & \dots & 0 & 1 & 0 & \dots \\
\dots & \dots & 0 & 0 & 0 & \dots \\
0 & 0 & \dots & \dots & \dots & 0
\end{pmatrix} & \begin{array}{c} \\ \\ \\ i\text{-th} \\ \\ \\ \end{array} \\
\begin{array}{cccccc} & & & \text{j-th} & & \end{array} &
\end{array}
\tag{1.5.78}
$$

and define:

$$
\begin{aligned}
H_i &= e_{ii} - e_{\ell+1,\ell+1} \quad (i = 1,\dots,\ell) \\
\mathcal{E}_{ij} &= e_{ij} \quad (i \neq j)
\end{aligned}
\tag{1.5.79}
$$

Since:

$$
e_{ij} \cdot e_{km} = \delta_{jk}\, e_{im} \tag{1.5.80}
$$

we have:

$$
\begin{aligned}
[H_i\,,\,H_j] &= 0 \\
[H_i\,,\,\mathcal{E}_{rs}] &= \delta_{ir}\, e_{is} - \delta_{si}\, e_{ri} - \delta_{\ell+1,r}\, e_{\ell+1,s} + \delta_{s,\ell+1} e_{r,\ell+1} \\
&= \left(\delta_{ir} - \delta_{si} - \delta_{\ell+1,r} + \delta_{s,\ell+1}\right) \mathcal{E}_{rs}
\end{aligned}
\tag{1.5.81}
$$

Now observe that a basis for the space E of vectors orthogonal to $v = (1,1,\dots,1)$ is provided by:

$$
\overrightarrow{u}_i = \overrightarrow{\epsilon}_i - \overrightarrow{\epsilon}_{\ell+1} \quad (i = 1,\dots,\ell) \tag{1.5.82}
$$

Indeed this is a system of ℓ linearly independent vectors in an ℓ–dimensional space. Hence we can identify:

$$
\left(\delta_{ir} - \delta_{si} - \delta_{\ell+1,r} + \delta_{s,\ell+1}\right) = \left(\overrightarrow{\epsilon}_r - \overrightarrow{\epsilon}_s\,,\,\overrightarrow{u}_i\right) \quad ; \quad (r,s = 1,\dots,\ell+1)\ ,\ (i = 1,\dots,\ell) \tag{1.5.83}
$$

This implies that to every Cartan subalgebra element $h = \omega^r\, H_r$ we can associate the vector $\overrightarrow{\omega} \equiv \omega^r\, \overrightarrow{u}_r \in E$ and to every root $\overrightarrow{\epsilon}_r - \overrightarrow{\epsilon}_s$ we can associate the linear functional:

$$
\left[\overrightarrow{\epsilon}_r - \overrightarrow{\epsilon}_s\right](\overrightarrow{\omega}) = \left(\overrightarrow{\epsilon}_r - \overrightarrow{\epsilon}_s\,,\,\overrightarrow{\omega}\right) \tag{1.5.84}
$$

With such identifications the $\mathrm{SL}(\ell+1,\mathbb{C})$ is cast into the canonical Weyl form of eq.s (1.2.38) and our theorem is proved. ■

1.5.2 The B_ℓ root system and the corresponding Lie algebra

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1.5.3 The C_ℓ root system and the corresponding Lie algebra

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1.5.4 The D_ℓ root system and the corresponding Lie algebra

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1.5.5 The exceptional root systems and the corresponding Lie algebras

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1.6 Real forms of the Lie algebras

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1.6.0.1 The Weyl theorem

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