myjournal manuscript No.
(will be inserted by the editor)

Acceleration and velocity statistics of Lagrangian particles in turbulence

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Received: date / Revised version: date

Abstract The statistics of Lagrangian tracers is a fundamental problem in fully developed turbulence. On the basis of high resolution direct numerical simulations, velocity and acceleration statistics will be discussed. The first part will be devoted to ideal fluid tracers, while the second part will consider the more realistic case of finite size particles with inertia.

1 Introduction

The knowledge of the statistical properties of Lagrangian tracers advected by a turbulent flow is not only a fundamental problem in the theory of fully developed turbulence but also a fundamental ingredients for the development of stochastic models for different application. Despite the importance of this problem, there are still relatively few experimental studies of Lagrangian turbulence [1,2]. This is mainly due to the intrinsic difficulty to follow tracers for long times at high resolution in a turbulent flow. An alternative approach is given by direct numerical simulations, which have clear advantages in terms of accuracy and possibility to make simultaneous measurement of different statistical quantities albeit at a smaller Reynolds number.

This contribution discusses the statistics of Lagrangian velocity fluctuations and accelerations in turbulent flows on the basis of high resolution direct numerical simulations. Most of the results presented here were published in previous papers [3–6] where the interested reader can find more details.

N	R_λ	T_E/τ_η	T/T_E	$L/\delta x$	$\eta/\delta x$
512	183	43.8	2.4	523	0.83
1024	284	54.6	2.4	1047	0.83

Table 1 Parameters of the numerical simulations. Resolution N, microscale Reynolds number R_λ, large-eddy turnover time $T_E = L/u_{rms}$, Kolmogorov timescale $\tau_\eta = (\nu/\varepsilon)^{1/2}$, total integration time T, box size L, grid spacing δx, Kolmogorov lengthscale $\eta = (\nu^3/\varepsilon)^{1/4}$

2 Numerical method

Direct numerical simulations of turbulent flow were done by using a parallel, fully dealiased, pseudo-spectral code on an IBM-SP4 parallel computer at Cineca at resolution up to 1024^3. Energy is injected at the average rate ϵ by keeping constant the total energy in each of the first two wavenumber shells [7] and is dissipated by a normal viscosity operator. In stationary conditions, particles are injected into the flow and their trajectories integrated according to [8]:

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{v}(t) \\ \frac{d\mathbf{v}}{dt} &= -\frac{\mathbf{v}(t) - \mathbf{u}(\mathbf{X}(t),t)}{\tau_s} \end{aligned} \tag{1}$$

where $\tau_s = r^2/(3\beta\nu)$ is the response time of a particle of radius r and density ρ_p in a fluid of viscosity ν and density ρ_f, and $\beta = 3\rho_f/(\rho_f + 2\rho_p)$. These equations are valid for a dilute suspensions of heavy ($\beta \ll 1$), small, spherical particles. In the limit of $\tau_s \to 0$, equations (1) simplifies to $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}(t),t)$, i.e. the motion of fluid particles. Lagrangian velocity was calculated using linear interpolation on the Eulerian grid. Particles' positions, velocities and accelerations have been recorded along the particle paths about every $0.1\tau_\eta$.

The range of Stokes number investigated is $0 \le St \le 3.31$ with $1.9 \cdot 10^6$ particles at $St = 0$ and $0.5 \cdot 10^6$ for each $St > 0$. Table 1 contains the most important numerical parameters. Details can be found in [5,6].

3 Fluid particle statistics

The simplest statistical object of interest in Lagrangian turbulence is single particle velocity increment $\delta_t\mathbf{v} \equiv \mathbf{v}(t) - \mathbf{v}(0)$ following a Lagrangian trajectory. In homogeneous, isotropic fully developed turbulence, dimensional analysis predicts [9]

$$\langle \delta_t v_i \delta_t v_j \rangle = C_0 \varepsilon t \delta_{ij} \tag{2}$$

where ε is the mean energy dissipation and C_0 is a dimensionless constant. The remarkable coincidence that the variance of $\delta_t v$ grows linearly with time is the physical basis for the development of stochastic models of particle dispersion. It is important to recall that the diffusive nature of (2) is purely incidental, consequence of Kolmogorov scaling in the inertial range of turbulence.

It could be useful to recall the argument leading to the scaling in (2). Consider the velocity $v(t)$ advecting the Lagrangian tracer as the superposition of the different velocity contributions coming from turbulent eddies. After a time lag t the components associated with the smaller (and faster) eddies, below a certain scale r are decorrelated and thus at the leading order one has $\delta_t v = \delta_r v$. Within Kolmogorov scaling, velocity fluctuations at scale r is given by $\delta_r v \sim V(r/L)^{1/3}$ where V represents the typical velocity at the largest scale L. The correlation time of $\delta_r v$ scales as $\tau_r \sim \tau_0(r/L)^{2/3}$ and thus one obtains the scaling in (2) with $\varepsilon = V^2/\tau_0$.

Equation (2) can be generalized to higher-order moments with the introduction of a set of temporal scaling exponents $\xi(p)$:

$$\langle \delta_t v^p \rangle \sim (\varepsilon t)^{xi(p)} \tag{3}$$

The dimensional estimation sketched above gives the prediction $\xi(p) = p/2$ but one might expect deviations in the presence of intermittency. In this case, a generalization can be easily developed on the basis of the multifractal model of turbulence [10,11,3]. The above dimensional argument is repeated for the local scaling exponent h, giving $\delta_t v \sim V(t/\tau_0)^{h/(1-h)}$. Integrating over the distribution of h one obtains the prediction [3]:

$$\xi(p) = \min_h \left[\frac{ph - D(h) + 3}{1 - h} \right] \tag{4}$$

The set of fractal dimensions $D(h)$ is related to the Eulerian structure function scaling exponents $\zeta(q)$ by the Legendre transform [10] $\zeta(q) = \min_h[qh - D(h) + 3]$. The standard inequality in the multifractal model $D(h) \leq 3h + 2$ implies for (4) that even in presence of intermittency, $\xi(2) = 1$. This is a direct consequence of the fact that energy dissipation enters into (2) at the first power.

Recent experimental results [2] have shown that Lagrangian velocity fluctuations are intermittent and display anomalous scaling exponents, as predicted by the above arguments. We remark that, despite the high Reynolds number of the experiments, the scaling range in temporal domain is very small. This is due to the presence of *trapping events* in which particles are trapped for relatively long times within small-scale vortices thus contaminating the inertial range scaling [5]. Therefore, an estimate of the scaling exponent $\xi(p)$ can be done only relatively to a reference moment (the ESS procedure [12].

Figure 1 shows the Lagrangian structure functions as obtained from our DNS for one component of the velocity. The inset shows that the relative exponents, as obtained from the ESS procedure, are in very well agreement with the multifractal prediction (4).

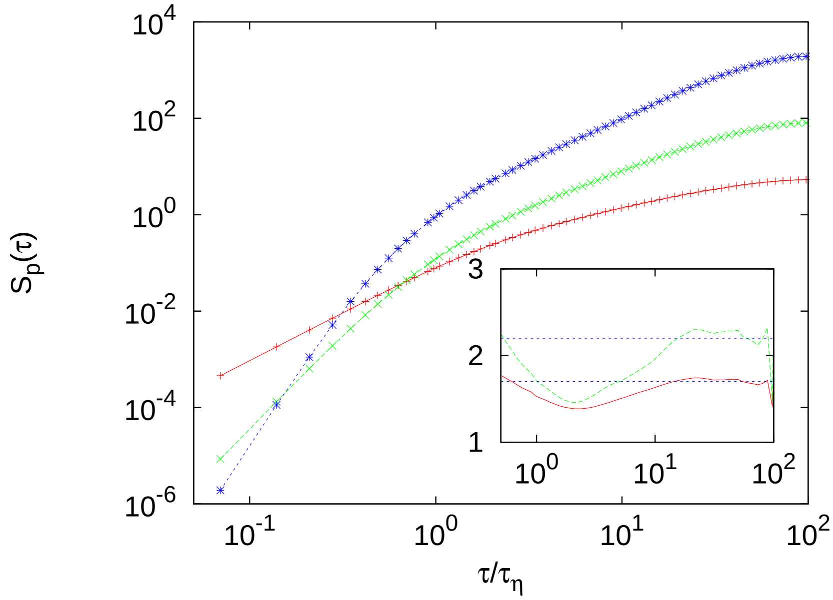

Fig. 1 Lagrangian structure functions of orders $p = 2, 4, 6$ (bottom to top) as a function of τ in log-log coordinates. Inset: ESS local slopes with respect to the second order structure function for $p = 4, 6$ (bottom to top). Straight lines correspond to the Lagrangian multifractal prediction (4) with the set of fractal dimensions $D(h)$ obtained from Eulerian velocity structure functions. Data refer to $R_\lambda = 284$.

For very small time increments, $\delta_t v$ reproduces the acceleration of transported particles. It is now well known that turbulent acceleration is an extremely intermittent quantity, with a probability density function (pdf) characterized by large tails corresponding to fluctuations up to 80 times the root mean square value a_{rms} [1]. The multifractal description of turbulence can be used also for predicting the shape of acceleration pdf. The basic idea [4] is to define the acceleration as the velocity increments at the smallest Kolmogorov scale, $a = \delta_{\tau_\eta} v/\tau_\eta$. Taking into account the fluctuations of the Kolmogorov scale and integrating over the distribution of large-scale velocity fluctuations, one ends with the prediction for the pdf of dimensionless Lagrangian acceleration $\tilde{a} = a/\sigma_a$:

$$\mathcal{P}(\tilde{a}) \sim \int \tilde{a}^{\frac{(h-5+D(h))}{3}} R_\lambda^{y(h)} \exp\left(-\frac{1}{2}\tilde{a}^{\frac{2(1+h)}{3}} R_\lambda^{z(h)}\right) dh \qquad (5)$$

where $y(h) = \chi(h-5+D(h))/6+2(2D(h)+2h-7)/3$ and $z(h) = \chi(1+h)/3+4(2h-1)/3$. We remark that the above expression contains an unphysical divergence for $a \to 0$ for several models $D(h)$. This is due to the fact that in general multifractal model cannot be used to describe small velocity (and acceleration) increments [4]. Therefore, we have to limit $\tilde{a}$ in a range of value

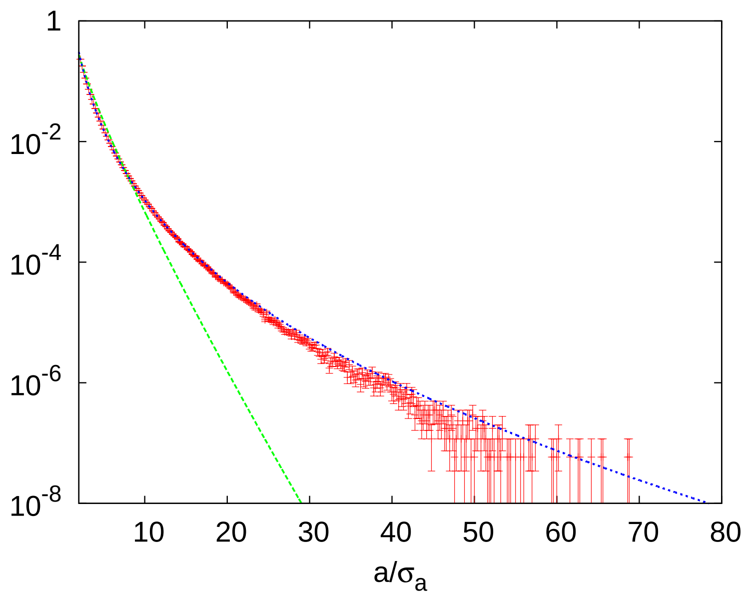

Fig. 2 Lin-log plot of the acceleration pdf. Points are the DNS data at $R_\lambda = 284$, solid line is the multifractal prediction and the dashed line if the K41 prediction.

above $\tilde{a}_{min} = O(1)$. This is the only free parameter in (5), as the set $D(h)$ is given from Eulerian measurements.

It is simple to recover from (5) the prediction in the case of non-intermittent Kolmogorov scaling. Assuming $h = 1/3$ with $D(h) = 3$ one obtains the form

$$\mathcal{P}(\tilde{a}) \sim \tilde{a}^{-5/9} R_\lambda^{-1/2} \exp\left(-\tilde{a}^{8/9}/2\right) \tag{6}$$

Figure 2 shows the pdf obtained from numerical data compared with the theoretical predictions. The agreement with (5) is remarkable, expecially considered the range of fluctuations from 1 to $70\sigma_a$.

4 Heavy particle acceleration

We now consider the case of inertial particles with $St > 0$. It is well known that inertial particles spontaneously concentrate on inhomogeneous sets, a phenomenon called *preferential concentration* [13]. The clustering of inertial particles has important physical applications, from rain generation [14] to planet formation [15].

Preferential concentration has dramatic consequence on Lagrangian statistics, in particular on the acceleration as inertial particles sample the turbulent flow in non-homogeneous way. It is relatively simple to predict that in general turbulent acceleration for inertial particles will be reduced with

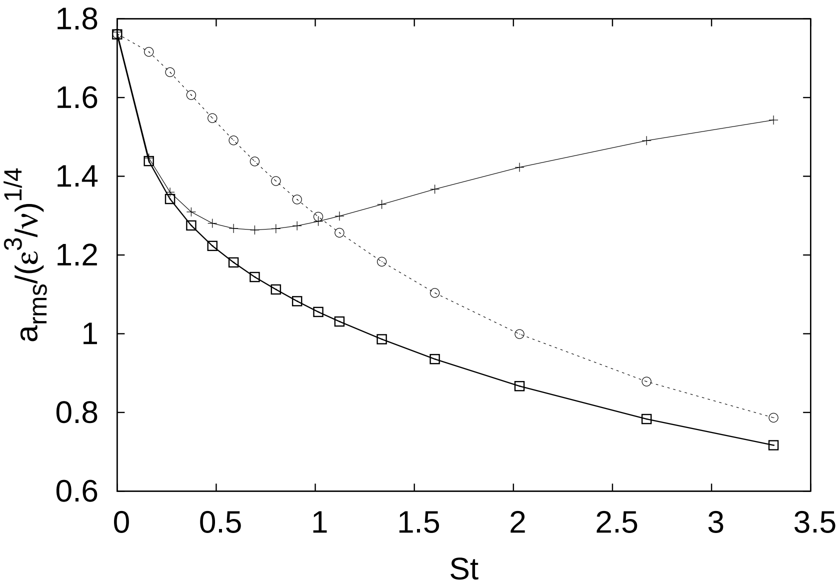

Fig. 3 Normalized acceleration variance $a_{rms}/(\varepsilon^3/\nu)^{1/4}$ as a function of Stokes number for $R_\lambda = 185$ (square). Acceleration of the fluid tracer conditioned to particle positions (crosses) and acceleration obtained from filtered velocity (circles).

respect to fluid. This is due to two different effects. From one hand, centrifugal forces will expel particles from most intense vortices. Therefore we expect a preferential concentration on the region of minor pressure gradient (i.e. minor acceleration). On the other hand, the formal solution to (1) gives for the acceleration [6]

$$\mathbf{a}(t) = \frac{1}{\tau_s^2} \int_{-\infty}^{t} e^{-(t-s)/\tau_s} [\mathbf{v}(\mathbf{X}(t), t) - \mathbf{v}(\mathbf{X}(s), s)]\, ds \ . \tag{7}$$

therefore inertial particles acceleration is the result of a low-pass filtering of velocity differences and thus we expect the suppression of fluctuations at higher frequencies. These two mechanisms, preferential concentration and filtering, act in the same direction in a reduction of acceleration fluctuations.

Figure 3 show the behavior of the acceleration variance as a function of Stokes number. At the maximum $St = 3.3$ the acceleration rms has been reduced by a factor about 2.5 with respect the fluid case $St = 0$. In Fig. 3 we also show the two different contributions discussed above. The contribution of preferential concentration has been estimated by computing fluid acceleration conditioned to heavy particle positions. The agreement of this quantity with the inertial particle acceleration indicates that this is the main mechanism for $St < 0.5$. The second contribution has been computed by filtering the Lagrangian velocity with a low-pass filter which suppresses frequencies above τ_s^{-1} and the computing the acceleration as the time derivative of the

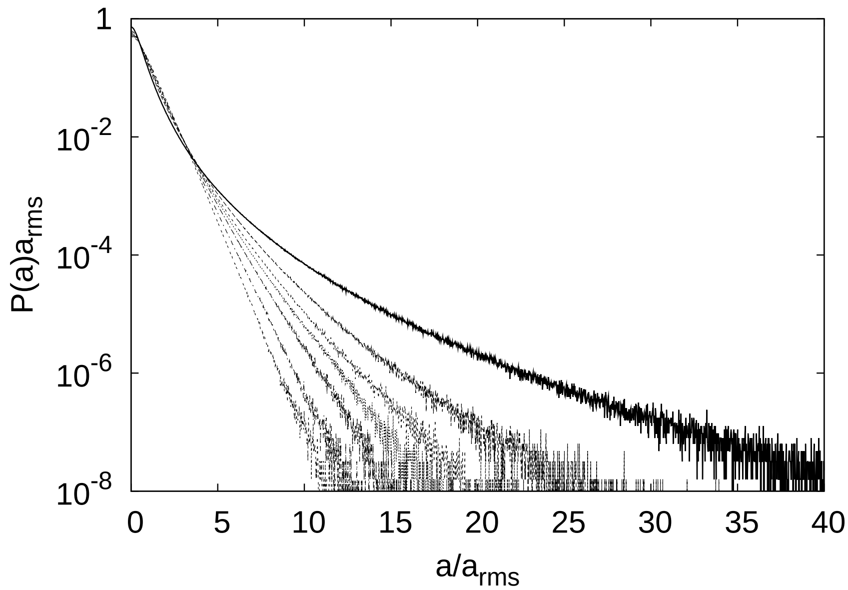

Fig. 4 Acceleration pdf in lin-log plot for inertial particles at $St = 0, 0.16, 0.37, 0.58, 1.01, 2.03, 3.31$ (from top to bottom) for the simulation at $R_\lambda = 185$.

filtered velocity. Figure 3 shows that filtered acceleration recovers inertial particle acceleration for large St. In conclusion, the two described mechanisms, preferential concentration and filtering, are complementary as they become important in two limit of Stokes number.

The effects of inertia on acceleration pdf is shown in Fig. 4. Increasing St inertial particle acceleration become less and less intermittent with a flatness which decreases from $F \simeq 30$ at $St = 0$ to $F \simeq 5$ at $St = 3.31$. The change in the shape of the pdf can be qualitatively captured by an argument similar to the one discussed for a_{rms} [6].

5 Conclusions

In conclusion, we have shown that single particle Lagrangian statistics in turbulence can be described by a simple extension of the multifractal formalism. Compared with other existing models, our proposal is very simple as it is based on the assumption that Lagrangian velocity increments are dimensionally related to Eulerian velocity increments.

In the case of inertial particles, we have shown that acceleration statistics is modified by two different mechanism, namely preferential concentration and filtering and we have discussed which mechanism is dominant in the small and large Stokes number regimes. Of course, our comprehension of the effects of inertia on acceleration is here only at a qualitative level. It would be extremely interesting to develop also in this case a quantitative prediction based on an extension of the multifractal formalism.

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