ELSEVIER

Physica A 280 (2000) 1–9

PHYSICA A

www.elsevier.com/locate/physa

Pair dispersion in turbulence

G. Boffetta[a,*], A. Celani[b]

[a] *Dipartimento di Fisica Generale and INFM Unità di Torino Università, via Pietro Giuria 1, 10125 Torino, Italy*

[b] *CNRS, Observatoire de la Côte d'Azur, B.P. 4229, 06304 Nice Cedex 4, France*

Abstract

We study the statistics of pair dispersion in two-dimensional turbulence. Direct numerical simulations show that the probability distribution function (pdf) of pair separations is in agreement with the Richardson prediction. The pdf of doubling times follows dimensional scaling and shows a long tail which is the signature of close approaches between particles initially seeded with a large separation. This phenomenon is related to the formation of fronts in passive scalar advection.

PACS: 47.27.Qb; 47.27.Eq

Keywords: Turbulent diffusion; Turbulence simulation

1. Introduction

The concentration of a dilute solution of a passive tracer in an incompressible flow obeys the scalar equation

$$\partial_t\theta + \mathbf{v}\cdot\nabla\theta = \kappa\Delta\theta + f\,, \tag{1}$$

where $\mathbf{v}$ is the divergenceless velocity field, κ is the molecular diffusivity, and f is the external source of tracer fluctuations. Eq. (1) can be solved by the method of characteristics to obtain the solution

$$\theta(\mathbf{x},t) = \int_{-\infty}^{t} \mathrm{d}s\, f(\boldsymbol{\rho}(s),s)$$

$$\dot{\rho}(s) = \mathbf{v}(\rho(s),s) + \sqrt{2\kappa}\,\eta(s), \quad \rho(t) = \mathbf{x}\,. \tag{2}$$

* Corresponding author. Fax: +39-011 660-4056.
E-mail address: boffetta@to.infn.it (G. Boffetta)

0378-4371/00/$ - see front matter
PII: S0378-4371(99)00613-5

The characteristics $\rho(t)$ are the Lagrangian trajectories of the fluid particles, and $\sqrt{2\kappa}\eta$ is the white-noise contribution due to molecular diffusion. There is an immense literature devoted to both Eulerian (1), and Lagrangian (2), descriptions of passive transport in turbulence [1]. What is relevant to our purposes is to keep in mind the tight relationship between these two complementary descriptions. As an instance, simultaneous two-point correlations of the scalar field can be written in terms of two-particle Lagrangian statistics as

$$\langle\theta(\mathbf{x}_1,t)\theta(\mathbf{x}_2,t)\rangle = \int_{-\infty}^{t} \mathrm{d}s_1 \int_{-\infty}^{t} \mathrm{d}s_2 \,\langle f(\rho(s_1),s_1) f(\rho(s_2),s_2)\rangle \,. \tag{3}$$

Properly choosing the form of the correlation function of the scalar forcing, e.g. $\langle f(\mathbf{x}_1,t_1) f(\mathbf{x}_2,t_2)\rangle = \chi(|\mathbf{x}_1-\mathbf{x}_2|)\delta(t_1-t_2)$, and exploiting space homogeneity, expression (3) can be further simplified to the form

$$\langle\theta(\mathbf{x},t)\theta(\mathbf{x}+\mathbf{R},t)\rangle = \int_{-\infty}^{t} \mathrm{d}s \int \mathrm{d}\mathbf{r}\chi(\mathbf{r})\, p(\mathbf{r},s|\mathbf{R},t)\,, \tag{4}$$

where $p(\mathbf{r},s|\mathbf{R},t)$ is the probability density function for a pair to be at a separation $\mathbf{r}$ at time s, under the condition that it has to have a separation $\mathbf{R}$ at time t.

Summarizing, the knowledge of the statistics of pair dispersion is sufficient to determine the values of the correlations of passive scalar for a given forcing. On the contrary, to extract the Lagrangian statistics from the Eulerian one it is necessary to know the scalar correlation functions for different forcings. In this sense, the Lagrangian information is more fundamental, and hereafter we shall restrict to this one.

2. Statistics of pair dispersion

The dispersion of a particles' pair in turbulence can be phenomenologically described in terms of a diffusion equation for the probability distribution of pair separations

$$\frac{\partial p(\mathbf{r},t)}{\partial t} = \frac{\partial}{\partial r_j}\left(K(r,t)\frac{\partial p(\mathbf{r},t)}{\partial r_j}\right) \tag{5}$$

with a space- and time-dependent diffusion coefficient $K(r,t)$. In general, the description by means of a diffusion equation is a drastic simplification. The only case in which it can be proven that particle separations obey (5) is when the advecting velocity field is rapidly changing in time [2,3].

The original Richardson proposal, obtained from experimental data in the atmosphere, is $K(r,t) = K(r) \sim r^{4/3}$ [4]. This leads to the well-known non-Gaussian distribution

$$p(\mathbf{r},t) \simeq t^{-9/2} \exp(-Cr^{2/3}/t)\,. \tag{6}$$

From the Richardson distribution (6) one has immediately that the mean square particle separation grows as

$$R^2(t) \equiv \langle r^2(t)\rangle \sim t^3\,. \tag{7}$$

The "t^3" law, which is known as the Richardson law, has been observed, although with large uncertainty, in direct numerical simulations [5] and, more recently, in laboratory experiments [6].

The diffusion equation (6) is not the unique possibility which leads to the "t^3" law. Batchelor [7] assuming that the diffusion coefficient should depend on average quantities, proposed that $K(r,t) = K(t) \sim \langle r^2(t)\rangle^{2/3} \sim t^2$. In this case the distribution of pair distances is Gaussian with a superballistic growing variance

$$p(\mathbf{r},t) \simeq t^{-9/2} \exp(-Cr^2/t^3)\,. \tag{8}$$

Of course, this is not the end of the story. Formally, any diffusion coefficient of the form $K(r,t) \sim r^a t^b$ with $3a+2b=4$ is compatible with the "t^3" law but gives different distribution function [1,8]. Early experimental data were in favor of the Batchelor–Gaussian distribution (8) [1], but recent laboratory experiments are more in the direction of the Richardson original proposal [6].

The Richardson law can be derived by a simple dimensional argument which makes use of the Kolmogorov similarity law for the Eulerian velocity increments in fully developed turbulence. By definition one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}R^2(t) = \langle \mathbf{r}\cdot\delta\mathbf{v}^{(L)}(\mathbf{r})\rangle = \langle r\,\delta v_{\|}^{(L)}(r)\rangle\,, \tag{9}$$

where $\delta\mathbf{v}^{(L)}(\mathbf{r}) = \mathbf{v}(\mathbf{x}(t)+\mathbf{r}) - \mathbf{v}(\mathbf{x}(t))$ is the Lagrangian velocity increment and $\delta v_{\|}$ is its projection on the $\mathbf{r}$ direction. Assuming Kolmogorov scaling for Lagrangian velocity differences, $\delta v_{\|}(r) \sim r^{1/3}$, one obtains from (9) the Richardson law (7). The assumption that the Lagrangian velocity difference has the same Kolmogorov scaling of the Eulerian one relies on the intuitive idea that the main contribution to the separation rate follows from eddies with size comparable with the separation itself.

Let us conclude this section with a remark. The description of relative dispersion in terms of the diffusion equation (5) assumes a self-similarity in the process. Of course this could not be the case, e.g. in intermittent three-dimensional fully developed turbulence. As a matter of fact, in presence of intermittency of the velocity field, one can expect a kind of "Lagrangian intermittency", in the sense that different moments $\langle r^p(t)\rangle$ have different scaling exponents:

$$\langle r^p(t)\rangle \sim t^{\alpha_p} \tag{10}$$

with $\alpha_p \neq 3/2\,p$. This problem has been discussed in several papers [9–12] with different conclusions. Recent detailed investigations with a synthetic turbulent model gave the evidence of Lagrangian intermittency with scaling exponents α_p linked to the Eulerian intermittent scaling exponents [13].

3. Pair dispersion in two-dimensional turbulence

Pair dispersion statistics has been investigated by direct numerical simulation of the inverse energy cascade in two-dimensional turbulence. There are several reasons

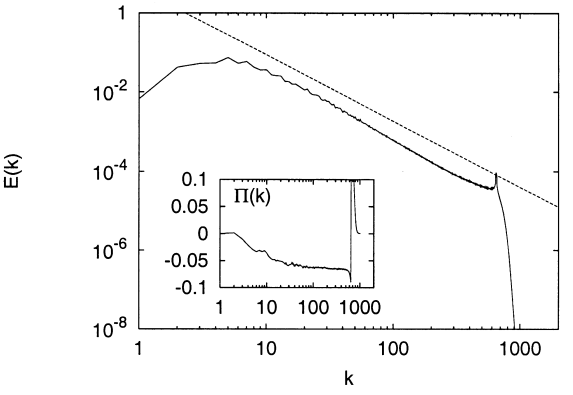

Fig. 1. Energy spectrum $E(k)$. The dashed line is the Kolmogorov scaling $E(k) \simeq k^{-5/3}$. In the inset is shown the energy flux $\Pi(k)$.

for considering 2D turbulence. From an applicative point of view, two-dimensional Navier–Stokes equations are among the simplest systems of geophysical interest. The observed absence of intermittency [14,15] makes the 2D inverse energy cascade an ideal framework for the study of Richardson scaling. Moreover, the dimensionality of the problem makes feasible direct numerical simulations at high Reynolds numbers.

The 2D Navier–Stokes equations for the vorticity $\omega = \nabla \times \mathbf{v} = -\Delta\psi$ are

$$\partial_t \omega + J(\omega, \psi) = \nu\Delta\omega - \alpha\omega + \phi \,, \tag{11}$$

where ψ is the stream function and J denotes the Jacobian. The friction linear term $-\alpha\omega$ extracts energy from the system to avoid Bose–Einstein condensation at the gravest modes [16]. The forcing is active only on a typical scale l_f and is δ-correlated in time to ensure the control of the energy injection rate. The viscous term has the role of removing enstrophy at scales smaller than l_f and, as customary, it is numerically more convenient to substitute it by a hyperviscous term (of order eight in our simulations). Numerical integration of (11) is performed by a standard pseudospectral method on a doubly periodic square domain of $N^2 = 2048^2$ grid points. All the results presented are obtained in conditions of stationary turbulence.

In Fig. 1, we present the energy spectrum, which displays Kolmogorov scaling $E(k) = C\varepsilon^{2/3}k^{-5/3}$ over about two decades with Kolmogorov constant $C \simeq 6.0$. The inertial range corresponds to the region of constant flux, also plotted in Fig. 1. Previous numerical investigation has shown that velocity differences statistics in the inverse cascade is almost Gaussian with Kolmogorov scaling not affected by intermittency corrections [15]. In this case we expect the Lagrangian statistics to be self-similar with Richardson scaling [13].

In Fig. 2, we plot relative dispersion $R^2(t)$ obtained after averaging over 64 000 particle pairs for two different initial conditions $R^2(0)$. The Richardson t^3 law is observed in a limited time interval, especially for the largest $R^2(0)$ run. It is remarkable that

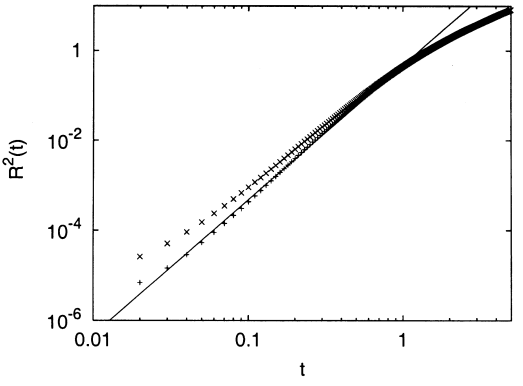

Fig. 2. Relative dispersion $R^2(t)$ for two initial separation $R(0) = 1.5 \times 10^{-3}(+)$ and $R(0) = 3 \times 10^{-3}(\times)$. The continuous line is the Richardson law $R^2(t) \simeq t^3$.

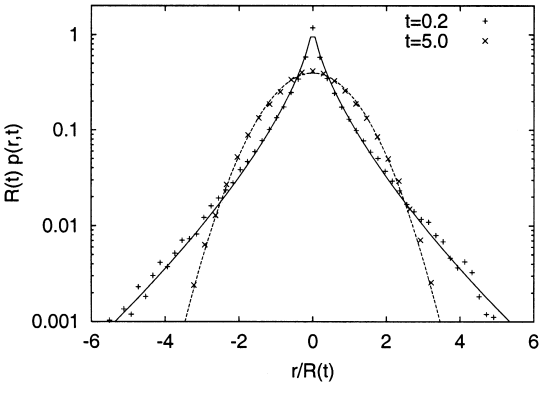

Fig. 3. Probability density function of relative separation at times t=0.2 and 5.0 rescaled with $R(t)=\langle r^2(t)\rangle^{1/2}$. The continuous line is the Richardson prediction (6), the dashed line is a Gaussian distribution.

the relative separation law displays such a strong dependence on the initial conditions even in this high-resolution runs.

The probability density functions of pair separations is plotted in Fig. 3 at two different times. For short times ($t = 0.2$) in which the relative dispersion is in the Richardson regime (see Fig. 2) we see that the Richardson distribution (6) fits well the numerical data. This is, we think, a clear evidence of the substantial validity of the original Richardson description. Let us observe that until now this point was not clear: recent laboratory data [6] pointed to the Richardson distribution but there were strong deviations from (6). At long time $t = 5.0$ relative separation distribution is described by a Gaussian distribution, but this has no relation with the Batchelor proposal (8) because it is not in the scaling range. At time $t = 5$ the average separation is of the size of the computational box and we have normal dispersion *à la* Taylor.

The Richardson pdf has a strong cusp at $r=0$ which signals the high probability for a pair to reach a very small separation compared to the typical value $R(t)$. As we shall see later on, this effect is highlighted by considering the statistics of first exit times.

4. Doubling time statistics

We have seen in the previous section that the Richardson picture seems to be confirmed in the inverse energy cascade in two dimensions. Nevertheless, we have seen that it is difficult to observe the Richardson scaling law even in our high-resolution direct numerical simulations. To understand this effect consider a series of particle pair dispersion experiments, in which a couple of particles is released at time $t=0$ with initial separation $R(0)$. At a *fixed time* t one performs an average over all different experiments and computes $R^2(t)$. It is clear that, unless t is large enough that all particle pairs have "forgotten" their initial conditions, the average will be biased. This is at the origin of the flattening of $R^2(t)$ for small times, which we can call a crossover from initial condition to self-similar regime. Of course, at larger $R(0)$ correspond longer crossover regimes (see Fig. 2). A similar effect is observed for times of the order of the integral time scale since some particle pairs might have reached a separation larger than the integral scale and thus diffuse normally, biasing the average, so that the curve $R^2(t)$ flattens again.

To overcome this difficulty we use an alternative approach based on statistics at *fixed scale* [17]. The method has been successfully applied to the analysis of relative dispersion in synthetic turbulent flow [13] and in experimental convective laminar flow [18]. The method works as follows. Given a set of thresholds $R_n = r^n R(0)$ within the inertial range, one computes the "doubling time" $T_r(R_n)$ defined as the time it takes for the particle pair separation to grow from threshold R_n to the next one R_{n+1}. Averages are then performed over many dispersion experiments, i.e., particle pairs. The outstanding advantage of this kind of averaging at fixed scale separation, as opposite to a fixed time, is that it removes crossover effects since all sampled particle pairs belong to the inertial range.

The scaling property of the doubling time statistics in fully developed turbulence is obtained by a simple dimensional argument. The time it takes for the particle pair separation to grow from R to rR can be dimensionally estimated as $T_r(R) \sim R/\delta v_{\|}^{(L)}(R)$; we thus expect for the inverse doubling times the scaling $\langle T_r(R)\rangle \simeq R^{2/3}$. From the definition, doubling times depend on the threshold ratio r. It is then useful to consider the normalized quantity

$$\lambda(R) = \frac{1}{\langle T(R)\rangle}\log r\,, \tag{12}$$

which is called the finite size Lyapunov exponent (FSLE) because is reduces to the (Lagrangian) Lyapunov exponent in the limit of small separation $R \to 0$ [19].

In Fig. 4, the FSLE is shown for the same simulation of Fig. 2. At small scales $R < 0.01$ there appears a constant plateau corresponding to the Lagrangian Lyapunov

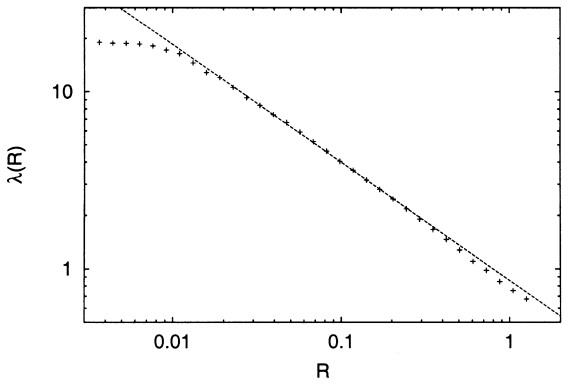

Fig. 4. Finite size Lyapunov exponent (12) for the same trajectories of Fig. 2. The initial threshold is $R(0) = 0.0031$ and the ratio is $r = 1.2$. The line is the theoretical Richardson scaling $R^{-2/3}$.

exponent $\lambda(R) \simeq 19$. At larger R, we observe the power law $\lambda(R) \simeq R^{-2/3}$ on a scaling range which is well enhanced with respect to the relative dispersion of Fig. 2.

The scaling of doubling times gives also information about the two-point correlations of passive scalar. Indeed, assuming that the correlation of the scalar forcing decays rapidly to zero with a typical scale L, by means of Eq. (4) we obtain

$$\langle \theta(\mathbf{x},t)\theta(\mathbf{x}+\mathbf{R},t)\rangle \simeq \int_{-\infty}^{t} \mathrm{d}s \int_{|\mathbf{r}|<L} \mathrm{d}\mathbf{r}\; p(\mathbf{r},s|\mathbf{R},t) \tag{13}$$

that is the average time that a particle pair released at a separation R spends below the scale L. Thus, our results for doubling times enable us to express the scalar correlation as

$$\langle \theta(\mathbf{x},t)\theta(\mathbf{x}+\mathbf{R},t)\rangle \sim L^{2/3} - R^{2/3} \tag{14}$$

for any L and R in the scaling range of the velocity field. Direct numerical simulations of passive scalar advection, Eq. (1), in the Navier–Stokes flow generated by Eq. (11) confirm that the exponent of scalar correlations is indeed indistinguishable from $\frac{2}{3}$ [20].

Beyond scaling properties of averaged quantities, the inspection of the pdf of doubling times is very insightful to capture the main features of pair dispersion.

In Fig. 5 the pdf is shown of doubling times rescaled with their average values. The normalized pdf's at different separations in the inertial range collapse, indicating the self-similarity of the Lagrangian dispersion. Most important, there is a large number of events for which the pair wanders for 20–30 times the average value before exiting. This effect is a reflection of the strong cusp observed in the pdf of pair separations. In these events the particles which are initially at a separation lying well inside the inertial range can approach each other as close as the diffusive scale. In the language of the passive scalar field, since trajectories originating from widely separated regions of space can carry very different values of the concentration field θ, these approaches generate steep gradients of scalar across small scales. These structures, known as

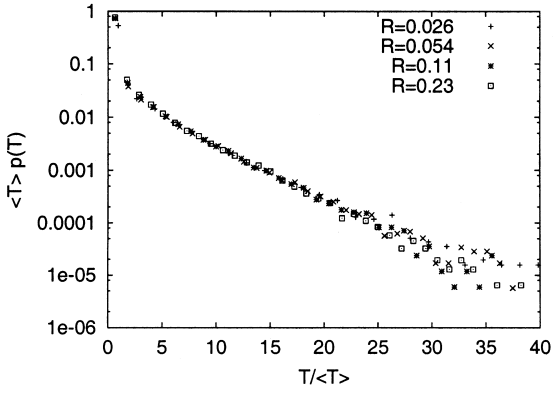

Fig. 5. Probability distribution functions of doubling times rescaled with the average doubling time $\langle T(R)\rangle$ for different R in the inertial range.

"cliffs", have been actually observed both experimentally [21–23] and numerically [24,25] for the temperature field in a turbulent flow.

5. Conclusion

Passive scalar transport in turbulence can be described in two complementary ways. It is possible to adopt the field description, and think in terms of correlation functions of the scalar field, either to prefer the particle description, and thus ask questions about the statistics of relative separations. These two aspects complete each other. We have investigated the Lagrangian properties of transport in two-dimensional turbulence. Our results show that pair dispersion statistics is not intermittent, since the velocity field is self-similar and the geometric content of two-particle configurations is trivial. This result makes contact with the analytical result that can be derived in the rapid-change model, where two-point scalar correlations show no anomaly. This will not be the case in three-dimensional turbulence, a case which is under current analysis.

Although on average particle pairs separate, probability density functions of pair separations and of doubling times clearly display the fingerprint of frequent close approaches between particles. These events occur with a relatively high probability, and are for the formation of quasi-discontinuities in the scalar field (the cliffs).

Acknowledgements

We thank M. Vergassola for useful discussions. The support by INFM (Progetto di Ricerca Avanzata TURBO) is acknowledged. The numerical simulation were perfomed at CINECA under the INFM contract "Lagrangian and Eulerian statistics in fully developed turbulence".

References

[1] A. Monin, A. Yaglom, Statistical Fluid Mechanics, MIT Press, Cambridge, MA, 1975.
[2] R.H. Kraichnan, Phys. Fluids 11 (1968) 945.
[3] D. Bernard, K. Gawedzki, A. Kupiainen, J. Stat. Phys. 90 (1998) 519.
[4] L.F. Richardson, Proc. Roy. Soc. A 110 (1926) 709.
[5] N. Zovari, A. Babiano, Physica D 76 (1994) 318.
[6] M.C. Jullien, J. Paret, P. Tabeling, Phys. Rev. Lett. 82 (1999) 2872.
[7] G.K. Batchelor, Proc. Cambridge Philos. Soc. 48 (1952) 345.
[8] J. Klafter, A. Blumen, M.F. Shlesinger, Phys. Rev. A 35 (1987) 3081.
[9] S. Grossmann, I. Procaccia, Phys. Rev. A 29 (1984) 1358.
[10] A. Crisanti, G. Paladin, A. Vulpiani, Phys. Lett. A 126 (1987) 120.
[11] M.F. Shlesinger, B.J. West, J. Klafter, Phys. Rev. Lett. 58 (1987) 1100.
[12] E.A. Novikov, Phys. Fluids A 1 (1989) 326.
[13] G. Boffetta, A. Celani, A. Crisanti, A. Vulpiani, Europhys. Lett. 46 (1999) 177.
[14] J. Paret, P. Tabeling, Phys. Fluids 10 (1998) 3126.
[15] G. Boffetta, A. Celani, M. Vergassola, Phys. Rev. E 61 (2000) R29.
[16] L. Smith, V. Yakhot, Phys. Rev. Lett. 71 (1993) 352.
[17] V. Artale, G. Boffetta, A. Celani, M. Cencini, A. Vulpiani, Phys. Fluids A 9 (1997) 3162.
[18] G. Boffetta, M. Cencini, S. Espa, G. Querzoli, Europhys. Lett. 48 (1999) 629.
[19] E. Aurell, G. Boffetta, A. Crisanti, G. Paladin, A. Vulpiani, J. Phys. A 30 (1997) 1.
[20] A. Celani, A. Lanotte, A. Mazzino, M. Vergassola, chao-dyn/9909038.
[21] C. Gibson, C. Freihe, S. McConnell, Phys. Fluids 20 (1977) S156.
[22] P.G. Mestayer, J. Fluid Mech. 125 (1982) 475.
[23] K.R. Sreenivasan, Proc. Roy. Soc. London A 434 (1991) 165.
[24] M. Holzer, E. Siggia, Phys. Fluids 6 (1994) 1820.
[25] A. Pumir, Phys. Fluids 6 (1994) 2118.