EUROPHYSICS LETTERS 15 April 1999

Europhys. Lett., **46** (2), pp. 177-182 (1999)

Relative dispersion in fully developed turbulence: Lagrangian statistics in synthetic flows

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(received 7 September 1998; accepted in final form 27 January 1999)

PACS. 47.27Qb – Turbulent diffusion.
PACS. 47.27Gs – Isotropic turbulence; homogeneous turbulence.
PACS. 47.27Eq – Turbulence simulation and modeling.

Abstract. – The effect of Eulerian intermittency on the Lagrangian statistics of relative dispersion in fully developed turbulence is investigated. A scaling range spanning many decades is achieved by generating a multi-affine synthetic velocity field with prescribed intermittency features. The scaling laws for the Lagrangian statistics are found to depend on intermittency in agreement with a multifractal description. As a consequence of the Kolmogorov law, the Richardson law for the variance of pair separation is not affected by intermittency corrections.

Understanding the statistics of particle pairs dispersion in turbulent velocity fields is of great interest for both theoretical and practical implications. Since fully developed turbulence displays well-known, non-trivial universal features in the Eulerian statistics of velocity differences [1, 2], it represents a starting point for the investigation of the general problem of the relationship between Eulerian and Lagrangian characteristics.

Since the pioneering work by Richardson [3], many efforts have been done to confirm experimentally [1] or numerically [4-6] his law. Most of the previous works concerning the validation of the Richardson law have been focused mainly on the numerical prefactor (Richardson constant [1]). Also theoretically there are very few attempts to investigate possible corrections stemming from Eulerian intermittency [7-9].

This is quite surprising compared with the enormous amount of literature concerning the intermittency correction for the Eulerian statistics [2, 10]. The main obstacle to a deeper investigation of relative dispersion is essentially the lack of sufficient statistics due to technical difficulties in laboratory experiments and to moderate inertial range achieved in direct numerical simulations.

In this letter we present a detailed investigation of the statistics of relative dispersion, obtained by numerical simulations of the advection of particle pairs on a synthetic turbulent velocity field with prescribed intermittency features. Our main result is that there is evidence of intermittency corrections to Lagrangian scaling laws for relative dispersion, and these corrections are tightly related to the intermittency of the statistics of velocity differences.

The Richardson law says that, in fully developed turbulence,

$$\langle R^2(t)\rangle \sim t^3\,, \tag{1}$$

where R is the separation of a particle pair and the average is performed over many pair dispersion experiments. The scaling (1) can be obtained by a simple dimensional argument [1] starting from Kolmogorov similarity law for longitudinal velocity increments in fully developed turbulence

$$\left\langle \left|\delta v_{\parallel}^{(\mathrm{E})}(\mathbf{R})\right|\right\rangle = \left\langle \left|(\mathbf{v}(\mathbf{x}+\mathbf{R})-\mathbf{v}(\mathbf{x}))\cdot\frac{\mathbf{R}}{R}\right|\right\rangle \sim R^{1/3}\,, \tag{2}$$

with $R=|\mathbf{R}|$. The pair of particles separates according to

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \delta\mathbf{v}^{(\mathrm{L})}(\mathbf{R})\,, \tag{3}$$

where $\delta\mathbf{v}^{(\mathrm{L})}$ represents the velocity difference evaluated along the Lagrangian trajectories. Assuming $\delta v_{\parallel}^{(\mathrm{L})}(\mathbf{R}) \simeq |\delta v_{\parallel}^{(\mathrm{E})}(\mathbf{R})|$ from (2) one obtains $\mathrm{d}R^2/\mathrm{d}t \sim R\,\delta v_{\parallel}^{(\mathrm{L})}(R) \sim R^{4/3}$ and hence the Richardson law (1).

To investigate the role of Eulerian intermittency, we have performed extensive numerical investigations of relative dispersion at very large Reynolds numbers. To accomplish this purpose, we have developed a Lagrangian numerical code for particle pairs whose separation evolves according to (3) with a realistic turbulent velocity difference. We consider the Quasi-Lagrangian reference frame [11] moving with a reference particle. The second particle is advected by the relative velocity $\delta\mathbf{v}(\mathbf{r},t)$ which possesses the same single-time statistics of the Eulerian velocity, whenever one considers statistically stationary flows. A realistic velocity field in this reference frame is generated by extending a recently introduced stochastic algorithm for the generation of multiaffine processes [12]. For the sake of simplicity, we consider, as in [5,6], a two-dimensional velocity field. The reason is that the relevant aspect for the statistics of particle pairs separation are the scaling laws for the relative velocity $\delta\mathbf{v}(\mathbf{r},t)$, which we take equal to those of three-dimensional turbulence.

We introduce the stream function $\psi(\mathbf{r},t)$ which, in isotropic conditions, can be decomposed using polar coordinates as

$$\psi(r,\theta,t) = \sum_{i=1}^{N}\sum_{j=1}^{n}\frac{\phi_{i,j}(t)}{k_i}F(k_i r)G_{i,j}(\theta). \tag{4}$$

Being interested in velocity fields possessing scaling laws on a large number of decades, we use $k_i = 2^i k_0$. The width of the "inertial range" is thus of order 2^N. The $\phi_{i,j}(t)$ are stochastic processes with characteristic times $\tau_i = 2^{-2i/3}\,\tau_0$ and zero mean. In order to obtain a multiaffine field, the moments of the stochastic processes should scale as $\langle|\phi_{i,j}|^p\rangle \sim k_i^{-\zeta_p}$, where ζ_p is an increasing convex function of p. An efficient way of to generate $\phi_{i,j}$ is [12]

$$\phi_{i,j}(t) = g_{i,j}(t)\,z_{1,j}(t)\,z_{2,j}(t)\cdots z_{i,j}(t)\,, \tag{5}$$

where the $z_{k,j}$ are independent, positive definite, identically distributed random processes with characteristic time τ_k, while the $g_{i,j}$ are independent stochastic processes with zero mean,

$\langle g_{i,j}^2 \rangle \sim k_i^{-2/3}$ and characteristic time τ_i. With this choice the exponents ζ_p are determined by the probability distribution of $z_{i,j}$ via

$$\zeta_p = \frac{p}{3} - \log_2 \langle z^p \rangle. \tag{6}$$

For a fully developed turbulent velocity field we expect the scaling $\langle |\psi(r,\theta)|^p \rangle \sim r^{\zeta_p + p}$ which can be simply achieved by demanding that the radial function $F(x)$ has support only for $x \simeq 1$ and choosing the scaling of the random processes $\phi_{i,j}$ as above described. A simple choice is

$$F(x) = x^2(1-x) \qquad \text{for} \qquad 0 \le x \le 1 \tag{7}$$

and zero otherwise,

$$G_{i,1}(\theta) = 1, \qquad G_{i,2}(\theta) = \cos(2\theta + \phi_i)\,, \tag{8}$$

and $G_{i,j} = 0$ for $j > 2$ (ϕ_i is a quenched random phase).

The intermittency in the velocity field can be tuned by the set of parameters entering into the construction of $\phi_{i,j}(t)$. In this letter we shall consider synthetic turbulent fields whose intermittency corrections to the Kolmogorov scaling, *i.e.*, non-linear ζ_p, are close to the experimental exponents [13], *i.e.* $\zeta_1 = 0.39$, $\zeta_2 = 0.72$ and so on.

It is worth remarking that a velocity difference field built according to (4) is not statistically homogeneous: velocity differences computed between the origin and an arbitrary point behave differently than those computed between two points away from the origin. This lack of homogeneity does not anyway affect the study of pair dispersion, since in the dynamics of pair separation (3) there appears only the velocity difference between the origin, where the reference particle lies, and the point where the second particle is located. Unless one is concerned with the absolute dispersion or the relative dispersion of three or more particles, this want of homogeneity is not a shortcoming.

We can extend the dimensional argument for the Richardson law to the intermittent case by using the multifractal representation [2, 14]. Following, with a few changes, Novikov [7] we assume, in the spirit of the refined similarity hypothesis (RSH) of Kolmogorov, that $\delta v^{(\mathrm{L})}(R(t)) \sim (\epsilon_{R(t)}\, t)^{1/2}$ and $R(t) \sim (\epsilon_{R(t)}\, t^3)^{1/2}$, where ϵ_R is the energy density dissipation at scale R. Assuming that $\epsilon_R \sim (\delta v^{(\mathrm{E})}(R))^3/R$, a simple calculation leads to

$$\langle R^p(t) \rangle \sim \int \mathrm{d}h\, t^{[3+p-D(h)]/[1-h]}. \tag{9}$$

In the limit of time t much smaller than the eddy turnover time at large scale the integral can be performed by steepest-descent method, and we obtain the scaling laws $\langle R(t)^p \rangle \simeq t^{\alpha_p}$, where the exponents are given by

$$\alpha_p = \inf_h \left[\frac{p + 3 - D(h)}{1-h} \right]. \tag{10}$$

From the above argument we thus expect that, in general, relative dispersion displays anomalous scaling in time (non-linear α_p). However there is an interesting result, already obtained in [7], for the case $p = 2$. From the general multifractal formalism one has that $3 - D(h) \ge 1 - 3h$ and the equality is satisfied for the scaling exponent h_3 which realizes the third-order structure function $\zeta_3 = 1$. From (10) it follows that $\alpha_2 = 3$ and thus we have that the Richardson law $\langle R^2 \rangle \sim t^3$ is not affected by intermittency corrections. We note that the Lagrangian RSH argument leading to (9) is just a one-dimensional reasonable assumption which can be justified

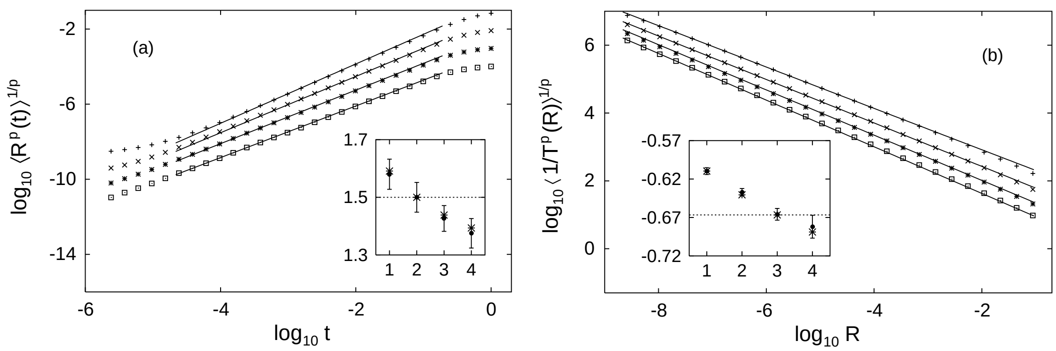

Fig. 1. – Relative dispersion $\langle R^p(t)\rangle^{1/p}$ (a) and inverse doubling times $\langle 1/T^p(R)\rangle^{1/p}$ (b) for $p = 1, 2, 3, 4$ (from top to bottom) for $N = 30$ shells averaged over 10^5 realizations. Continuous lines represent the theoretical scaling. In the inset we plot the theoretical and numerical exponents as a function of the moment p (the dashed line is the non-intermittent prediction).

only *a posteriori* by numerical simulations. Other different assumptions are possible [8, 9, 7] leading to different predictions.

In fig. 1a we plot the result of $\langle R^p(t)\rangle^{1/p}$ for different moments p. We indeed observe for $p = 2$ a t^3 law, while for higher moments we observe that the non-linear exponent α_p obtained from (10) gives a better fit than the linear scaling $\alpha_p = 3p/2$.

The scaling exponents satisfy the inequality $\alpha_p/p < 3/2$ for $p > 2$: this amounts to saying that, as time goes by, the right tail of the pdf of the separation $R(t)$ becomes less and less broad. In other words, due to the effect of intermittency, particle pairs are more likely to stay close to each other than to experience a large separation.

Figure 1a also shows that the power law scaling regime for $\langle R^p(t)\rangle \sim t^{\alpha_p}$ is observed only well inside the inertial range. This follows from the dependence of the integration of (3) on the smallest scale in the inertial range. Furthermore there is also a crossover to normal diffusion for separations comparable with the integral scale. This effect is particularly evident for lower Reynolds numbers, as shown in fig. 2 for a simulation with $Re \simeq 10^6$. This correction to a pure power law is far from being negligible. For instance in experimental data the inertial range is generally limited due to the Reynolds number and the experimental apparatus.

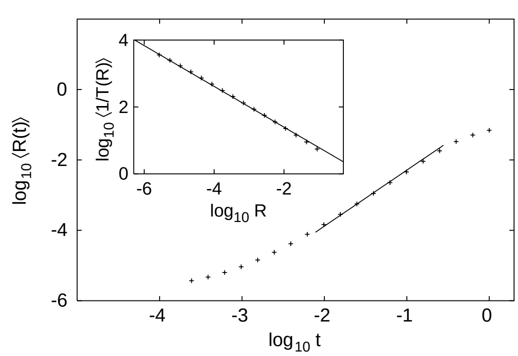

Fig. 2. – Relative dispersion $\langle R(t)\rangle$ for for $N = 20$ shells averaged over 10^4 realizations. In the inset we show the corresponding average inverse doubling time $\langle 1/T\rangle$. Observe the enhancement of the scaling range in the latter case.

References [6, 15] show quite clearly the difficulties that may arise in numerical simulations with the standard approach.

Within this framework we propose an alternative approach which is based on the statistics at fixed scale. It has the clear advantage over the fixed time statistics that when one samples all pairs at a separation within the inertial range, there are no spurious contributions due to the crossover between inertial scales and scales smaller or larger than the inertial ones. The method is in the spirit of a recently introduced generalization of the Lyapunov exponent to finite-size perturbation (Finite-Size Lyapunov Exponent) which has been successfully applied in the predictability problem [16] and in the diffusion problem [17]. Given a set of thresholds $R_n = R_0 2^n$ within the inertial range, we compute the "doubling time" $T(R_n)$ defined as the time it takes for the particle separation to grow from one threshold R_n to the next one R_{n+1}. We can give a dimensional estimate of this time as $T(R) \sim R/\delta v(R)$ and thus see that it fluctuates with the velocity fluctuations. After averaging over many realizations we can write

$$\left\langle \frac{1}{T^p(R)} \right\rangle \simeq \int \mathrm{d}h R^{p(h-1)} R^{3-D(h)} \simeq R^{\zeta_p - p}\,, \tag{11}$$

from which it follows that the doubling time statistics contains the same information on the velocity intermittency as the relative dispersion exponents (10).

As reported in fig. 1b prediction (11) is very well verified in our simulations. Note that the scaling region is wider than that of fig. 1a and the scaling exponent can be measured with higher accuracy, especially in the case of moderate inertial range simulation (fig. 2). For this reason we suggest that this kind of analysis should be preferred when dealing with experimental data. Also in this case there is an exponent, $\zeta_3 - 3 = -2$, unaffected by intermittency.

In the construction of the synthetic turbulent field we have implicitly assumed that the Lagrangian time τ_L and the Eulerian time τ_E are of the same order of magnitude, an assumption consistent with the experimental data and theoretical arguments (see, *e.g.*, [18]). Nevertheless it should be interesting to study the relevance of the ratio $\tau_\mathrm{L}/\tau_\mathrm{E}$ on the intermittent corrections. One could, indeed, expect that for $\tau_\mathrm{L}/\tau_\mathrm{E} \gg 1$ the Lagrangian intermittency disappears.

Finally we note that non-intermittent turbulence, *i.e.* $\zeta_p = p/3$, is recovered by keeping $z_{i,j} = 1$ fixed. In this case one may ask whether our results are realistic also for the Richardson constant G_Δ defined from the pair dispersion law $R^2(t) = G_\Delta\, \overline{\epsilon}\, t^3$, where $\overline{\epsilon}$ is the average energy dissipation rate. The value of $\overline{\epsilon}$ can be obtained from the second-order Eulerian structure function, which reads $S_2^{(\mathrm{E})}(R) = \langle |\delta v_\parallel^{(\mathrm{E})}(R)|^2 \rangle = C_\mathrm{L}\, \overline{\epsilon}^{2/3}\, R^{2/3}$, where C_L is a universal constant related to the Kolmogorov constant. According to the experimental measurements, we fix $C_\mathrm{L} = 2.0$, leading to $G_\Delta = 0.190 \pm 0.005$ for the Richardson constant which is in agreement with previous values [1, 5].

In this letter, using a synthetic turbulence model, we have the first evidence that the relative dispersion statistics for Lagrangian tracers in fully developed turbulence is affected by intermittency of the velocity field. We have suggested a new approach based on the Lagrangian doubling times which seems very promising for data analysis. The present work is a first step towards the clarification of Lagrangian-Eulerian relationship in fully developed turbulence. It would be extremely interesting to check our claims by means of direct numerical simulations or laboratory experiments.

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We thank L. Biferale for useful discussions in the early stage of the work. This work has been partially supported by INFM (Progetto di Ricerca Avanzata TURBO), by MURST (program number 9702265437) and by CEE (contract number FMRX-CT98-0175).

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