Experimental evidence of chaotic advection in a convective flow

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Abstract. – Lagrangian chaos is experimentally investigated in a convective flow by means of Particle Tracking Velocimetry. The Finite Size Lyapunov Exponent analysis is applied to quantify dispersion properties at different scales. In the range of parameters of the experiment, Lagrangian motion is found to be chaotic. Moreover, the Lyapunov exponent depends on the Rayleigh number as $\mathcal{R}a^{1/2}$. A simple dimensional argument for explaining the observed power law scaling is proposed.

The investigation of transport and mixing of passive tracers is of fundamental importance for many geophysical and engineering applications [1, 2]. It is now well established, and confirmed by several numerical [3] and experimental [4] evidence, that even in very simple Eulerian flow (*i.e.* laminar flow) the motion of Lagrangian tracers can be very complex due to Lagrangian Chaos [2, 5]. In such a situation, diffusion may be of little relevance for transport which is, on the contrary, strongly enhanced because of chaotic advection [3, 5].

In this letter, we address the problem of quantifying the dispersion of passive tracers in a relatively simple convective flow at various Rayleigh numbers $\mathcal{R}a$. By applying the Finite Size Lyapunov Exponent (FSLE) [6] analysis to the Lagrangian trajectories obtained from Particle Tracking Velocimetry (PTV) technique, we are able to estimate the dispersion properties at different scales. We find a clear power law dependence of the Lagrangian Lyapunov exponent on the Rayleigh number. This dependence is explained by a dimensional argument which excludes a role of diffusion in the dispersion process.

The experiment is performed in a rectangular tank $L = 15.0$ cm wide, 10.4 cm deep and $H = 6.0$ cm height, filled with water. Upper and lower surfaces are kept at constant temperature while the side walls are adiabatic. The convection is generated by an electrical circular

heater of radius 0.4 cm placed in the mid-line of the tank at 0.4 cm above the lower surface. The heater works at constant heat flux Q which is controlled by a feedback on the power supply. By changing the heat input we control the Rayleigh number $\mathcal{R}a = (g\beta Q H^3)/(\alpha\nu\kappa)$, where g is the gravitational acceleration, β the thermal expansion coefficient, α the thermal conductivity, ν the kinematic viscosity and κ the thermal diffusivity. In the parameter range explored in our experiments the flow consists of two main counter-rotating rolls divided by an ascending thermal plume above the heat source [7]. The upper end of the plume oscillates horizontally almost periodically with a frequency which depends on the Rayleigh number.

Lagrangian trajectories are obtained by PTV technique [8]. The fluid is seeded with a large number of small (50 μm in diameter), non-buoyant particles. The vertical plane in the middle of the tank and orthogonal to the heat source is illuminated by a thin laser light sheet. Single exposure images are taken by a CCD camera and subsequently digitized at 8.33 Hz rate with a 752×576 pixel resolution. Trajectories are then identified as time-ordered series of particle locations.

Each run lasts for 2700 s, corresponding to 22500 frames. Typically 900 particles are simultaneously tracked for each frame. In the following we analyze the trajectories obtained in 6 different runs with Rayleigh number in the range $6.87 \cdot 10^7 < \mathcal{R}a < 2.17 \cdot 10^9$.

The analysis of Lagrangian data have been done with the Finite Size Lyapunov Exponent tool which, introduced in the context of the predictability problem in turbulence [6], has already been demonstrated to be very efficient for the characterization of dispersion in bounded domains [9] and in the treatment of experimental data [10]. Let us recall the basic ideas on the FSLE; more details can be found in [6, 9]. The idea of the Finite Size Lyapunov Exponent is to generalize the Lyapunov exponent, which measures the average rate of divergence of two infinitesimally close trajectories, to finite separations. To this aim, we fix a set of thresholds $R_n = R_0 \rho^n$ ($n = 0, \ldots, N$) and we consider, at each time t, new couples of trajectories $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ found at separation $R(t) = |\mathbf{x}_1(t) - \mathbf{x}_2(t)| < R_0$. We follow the evolution of the trajectories and compute the "doubling time", $T_\rho(R_n)$, it takes for the separation to grow from scale R_n up to $R_{n+1} = \rho R_n$. Of course, it must be $\rho > 1$, but we take ρ not too large in order to avoid contributions from different scales.

After performing a large number of doubling time experiments (over the possible different couples in the run) we average the doubling time at each scale R from which we define the Finite Size Lyapunov Exponent

$$\lambda(R) = \frac{1}{\langle T(R)\rangle_e} \ln \rho \, , \tag{1}$$

where $\langle [\ldots] \rangle_e$ indicates the average performed on the doubling time experiments (see refs. [6, 9] for further details). The FSLE is a generalization of the Lyapunov exponent λ [6], in the sense that

$$\lim_{R\to 0} \lambda(R) = \lambda \, ; \tag{2}$$

physically speaking, $\lambda(R) \approx \lambda$ for $R \leq l_{\mathrm{E}}$, where l_{E} is the smallest Eulerian characteristic length. For larger values of R, $\lambda(R)$ gives information on the mechanism governing the dispersion at scale R. For example, in the case of standard diffusion, on the basis of dimensional considerations one expects that

$$\lambda(R) \simeq D/R^2 \, , \tag{3}$$

where D is the diffusion coefficient.

The use of the FSLE is particularly useful for studying the dispersion properties of passive tracers in closed basins [9], and, therefore, fits very well with the considered flow. In such
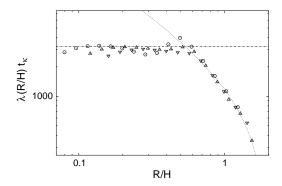
Fig. 1. – $\lambda(R)$ *vs.* R for different initial thresholds $R_0 = 0.067\,H$ ($\circ$), $0.1\,H$ ($\triangle$), $0.13\,H$ ($\triangledown$) at $\mathcal{R}a = 2.39 \cdot 10^8$. The straight line is the Lyapunov exponent $\lambda = 3100 \pm 200\,t_\kappa^{-1}$ and the curve is the saturation regime (4) with $\tau_{\mathrm{R}} = 8.0 \cdot 10^{-4}\,t_\kappa$ and $R_{\max} = 1.9\,H$.

a situation, asymptotic regimes like diffusion (3) might have never been reached due to the presence of boundaries. Denoting by R_{max} the average couple separation in the asymptotic uniform distribution, it has been found that for a large class of systems, for R close to $R_{\max}$, $\lambda(R)$ fits the following universal behavior:

$$\lambda(R) \simeq \frac{1}{\tau_{\mathrm{R}}} \frac{R_{\max} - R}{R}, \tag{4}$$

where τ_{R} has the physical meaning of the characteristic time of relaxation to the uniform distribution. Equation (4) can be obtained assuming an exponential relaxation of tracers' concentration to the uniform distribution [9].

We applied the FSLE analysis to the Lagrangian trajectories experimentally obtained. In order to increase the statistics at large separations R, we have computed $\lambda(R)$ for different values of the smallest scale R_0 ($R_0 = 0.4$ cm, 0.6 cm, 0.8 cm). The threshold ratio is $\rho = 1.2$ for all the analysis. For the results presented below we use $H = 6$ cm as unit length and the diffusive time $t_\kappa = H^2/\kappa \simeq 25000$ s as unit time.

In fig. 1 we show the $\lambda(R)$ *vs.* R computed for the run at $\mathcal{R}a = 2.39 \cdot 10^8$. The first important result is the convergence of $\lambda(R)$ to the constant value $\lambda \simeq 3100\,t_\kappa^{-1}$ at small R. This corresponds to an exponential divergence of close trajectories, *i.e.* a direct evidence of Lagrangian chaos. The large value of the Lyapunov exponent (in units of inverse diffusive time) indicates that chaotic advection is the dominant mechanism for particle dispersion at small scales.

For larger separation $\lambda(R)$ drops to smaller values, indicating a slowing down in the separation growth. This is quantitatively well described by the saturation regime (4). The collapse of the curves at different R_0 confirms that sufficient high statistics is reached even at these large scales. Fluctuation among different R_0 curves can be taken as an estimation of the error for $\lambda(R)$. By fitting the large-scale behavior of $\lambda(R)$ with (4), we obtain $R_{\max} \simeq 1.9\,H$ and $\tau_{\mathrm{R}} \simeq 8.0 \cdot 10^{-4} t_\kappa$. Thus also for the late stage of relaxation to the uniform distribution, diffusion seems to play a marginal role. The characteristic Eulerian scale in the flow l_{E} can be estimated by the end of the exponential regime (plateau $\lambda(R) \simeq \lambda$) at which the non-linear effects start to dominate. We find $l_{\mathrm{E}} \simeq 0.5\,H$, indeed not too far from the saturation value. In this condition there is no room for the development of a diffusive regime [9], as fig. 1 clearly shows. In addition, let us mention that by comparing $\lambda(R)$ computed at different $\mathcal{R}a$, we find that both the characteristic scales l_{E} and the saturation scale $R_{\max}$ are independent of the Rayleigh number.
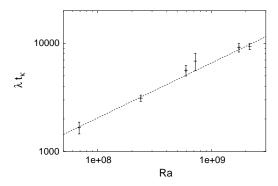
Fig. 2. – Lagrangian Lyapunov exponent dependence on the Rayleigh number $\mathcal{R}a$. The errors are estimated by the fluctuations corresponding to different initial R_0. The line is the best fit $\lambda \sim \mathcal{R}a^{0.51}$.

In order to explore the dependence of the Lagrangian statistics on the Eulerian characteristics, we have performed the FSLE analysis for Rayleigh number varying over more than one order of magnitude. The dependence of the Lagrangian Lyapunov exponent λ (computed from the plateau of $\lambda(R)$ at small R) on $\mathcal{R}a$ is shown in fig. 2. A clear scaling is observed, indicating a power law dependence

$$\lambda \sim \mathcal{R}a^\gamma \tag{5}$$

with $\gamma = 0.51 \pm 0.02$.

It is worth noting that because of the geometry of our experiment, the flow shows qualitatively the same pattern for the whole range of $\mathcal{R}a$ explored. This is confirmed by the independence of l_{E} and $R_{\max}$ on $\mathcal{R}a$ as discussed above. In these conditions, the scaling of λ on $\mathcal{R}a$ can be supported by the following dimensional argument. The equations of motion in the Boussinesq approximation, made non-dimensional by H and t_κ, are [11]

$$\frac{1}{Pr}\left[\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial}{\partial x_\alpha}p\right] = \frac{\partial^2 u_\alpha}{\partial x^2} - \mathcal{R}aTz_\alpha \,, \tag{6}$$

$$\frac{\partial T}{\partial t} + u_\beta \frac{\partial T}{\partial x_\beta} = \frac{\partial^2 T}{\partial x^2} \,, \tag{7}$$
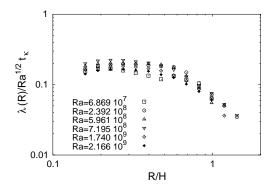

Fig. 3. – Data collapse of $\lambda(R)$ at different $\mathcal{R}a$ rescaled with $\mathcal{R}a^{1/2}$.

where $Pr = \nu/\kappa$ is the Prandtl number (which is kept constant in our experiments). It is now easy to verify that performing the following rescaling:

$$u_\alpha \to \Lambda u_\alpha, \qquad t \to \Lambda^{-1}t, \qquad \mathcal{R}a \to \Lambda^2 \mathcal{R}a, \qquad (8)$$

where Λ is an arbitrary factor, eqs. (6)-(7) remain unchanged, if the diffusive terms are neglected (as is suggested by the previous results). Let us stress that this rescaling does not involve neither the space (as is suggested by the above discussion) nor the Prandtl number (because in our experiments we change Q, keeping both ν and κ constant). The consequence of the Eulerian scaling invariance on the Lagrangian motion, governed by

$$\frac{\mathrm{d}\mathbf{x(t)}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x(t)}, t) \qquad (9)$$

is that Lagrangian trajectories are independent of the Rayleigh number. The Lyapunov exponent, which is dimensionally the inverse of a time, thus scales as $\mathcal{R}a^{1/2}$, according to the result shown in fig. 2.

As a consequence of the dimensional argument (8)-(9), the FSLE should scale with $\mathcal{R}a$ both in the linear and in the saturation ranges, *i.e.* $\lambda(R)/\mathcal{R}a^{1/2}$ is a $\mathcal{R}a$-independent function. In order to verify the scaling argument, in fig. 3 we plot the FSLE compensated with $\mathcal{R}a^{1/2}$ for the different runs. The collapse is fairly good, and the fluctuations are of the order of the error reported in fig. 2. As a further argument in favor of the scaling (8), let us observe that the transitions to the saturation regime occur at a scale independent of $\mathcal{R}a$.

As a conclusion, let us stress again that the approximate agreement with the dimensional scaling argument and the large values of the Lyapunov exponent in units of diffusion time give a coherent picture of Lagrangian dispersion ruled by chaotic advection.

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