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Turbulence in Two-dimensional Forced Electron-magnetohydrodynamics

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Abstract

2D-EMHD turbulence is studied in the freely decaying case and in the random forced one. The analogue of the Kolmogorov's four-fifths law [3] is derived under the assumptions of statical homogeneity and isotropy. Direct numerical simulations show that it holds true for different values of the adimensional electron inertial length scale d_e. In the $d_e \sim 1$ regime, the energy spectrum is found to be close to the Kolmogorov spectrum.

1. Basic equations

A plasma of cold electrons can be described in the framework of Electron-MagnetoHydrodynamics, hereafter EMHD, [1], which is a fluid-dynamical model representing the behaviour of electrons in a neutralizing background of motionless ions. In recent years, this model has received considerable interest for its application to the study of laser-plasma interactions and inertially confined plasmas. The (dimensionless) equation of motion is given by:

$$d_e^2 \frac{\partial \boldsymbol{v}}{\partial t} + d_e^2 \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p - \boldsymbol{E} - \boldsymbol{v} \times \boldsymbol{B} \quad (1)$$

where the fields are three-dimensional vectors, depending only on the plane coordinates, and $d_e^2 = c^2/\omega_{pe}^2 = mc^2/4\pi ne^2 L^2$ is the (square) dimensionless electron skin depth. To normalize the equation, the macroscopic size L was used as a scale length, B_0 is a characteristic magnetic field, the inverse gyrofrequency $\omega_{ce}^{-1} = mc/eB_0$ was taken as a scale time and the density n was supposed to be constant.

Neglecting dissipation effects, the electron equation reduces to a frozen-in law for the curl of the canonical momentum in the velocity flow [2]. Using Ampère equation ($\boldsymbol{v} = -(c/4\pi ne)\nabla \wedge \boldsymbol{B}$), the above statement can be written in the following dimensionless form:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \nabla \wedge (\nabla \wedge \boldsymbol{B} \wedge \boldsymbol{\Omega}) = 0, \quad (2)$$

where $\boldsymbol{\Omega} = (1 - d_e^2 \nabla^2)\boldsymbol{B}$ is the curl of the canonical momentum, sometimes called generalized vorticity. Using incompressibility for the velocity field and solenoidality for the magnetic field, it is easy to obtain two scalar equations:

$$\frac{\partial(\psi - d_e^2 \nabla^2 \psi)}{\partial t} + [\varphi, \psi - d_e^2 \nabla^2 \psi] = 0, \quad (3)$$

$$\frac{\partial(\varphi - d_e^2 \nabla^2 \varphi)}{\partial t} + [\varphi, \varphi - d_e^2 \nabla^2 \varphi] = [\psi, \psi - d_e^2 \nabla^2 \varphi], \quad (4)$$

where the Jacobian operator is defined as usual $[a, b] = \partial_x a \partial_y b - \partial_x b \partial_y a$ and $\boldsymbol{B}(x, y) = \varphi(x, y)\boldsymbol{e}_z + \nabla \wedge (\psi(x, y)\boldsymbol{e}_z)$, so that $\nabla^2 \psi$ and $\nabla^2 \varphi$ are respectively the current and the vorticity along z direction.

The following three quadratic quantities are conserved by eqs (1):

$$E = \int d^2x(\boldsymbol{v}^2 + \boldsymbol{B}^2) = \int d^2x \times (\varphi^2 + (\nabla\psi)^2 + d_e^2[(\nabla\varphi)^2 + (\nabla^2\psi)^2))], \quad (5)$$

$$H = \int d^2x(\psi - d_e^2 \nabla^2 \psi)^2, \quad (6)$$

$$K = \int d^2x(\psi - d_e^2 \nabla^2)(\varphi - d_e^2 \nabla^2 \varphi). \quad (7)$$

2. EMHD turbulence

Dissipation can be taken into account by means of a diffusive term, so that (1) becomes

$$d_e^2 \frac{\partial \boldsymbol{v}}{\partial t} + d_e^2 \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p - \boldsymbol{E} - \boldsymbol{v} \times \boldsymbol{B} + \nu_q(-\nabla)^{2q} \boldsymbol{v}. \quad (8)$$

The dissipation coefficient ν_q is the analogue of the inverse Reynolds number, normalized on the inverse electron gyrofrequency instead that on the eddy turnover time, as usually done. In our numerical simulations we actually used hyperviscosity ($q > 1$), while $q = 1$ corresponds to electron viscosity and $q = 0$ to resistivity.

One of the more interesting question is: how the ideal invariants (5), (6), (7) are nonlinearly transferred among the lengthscales? To answer this question we followed a statistical approach mainly due to A. N. Kolmogorov, who developed the three-dimensional hydrodynamic (3D-NS) turbulence theory [3, 4], showing that, in the limit of vanishing viscosity and due to the hypotheses of statistical homogeneity and isotropy, the turbulent regime is characterized by a forward energy cascade, by a finite value of energy dissipation and by the existence of a range (inertial range) of scales in which the energy flux is constant.

It was recently proved [5] that 2D-EMHD turbulence resembles 3D-NS turbulence, at least in the $d_e \sim 1$ regime.

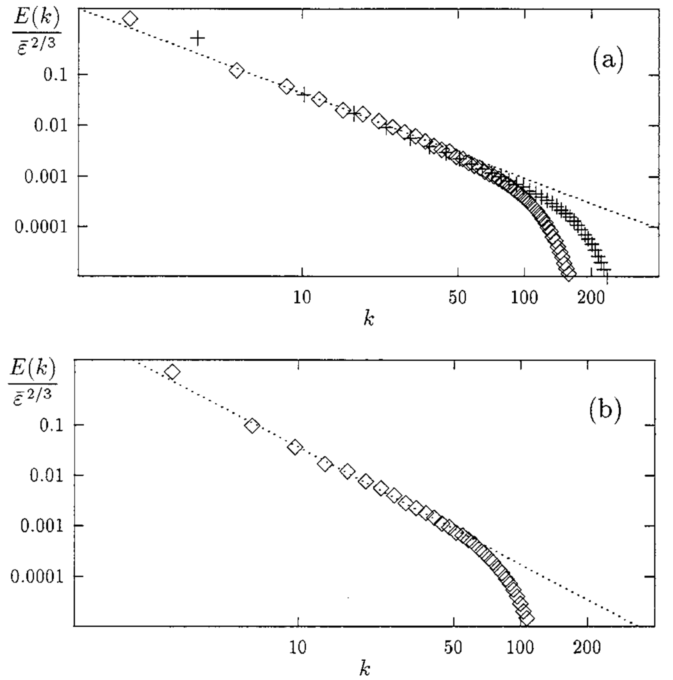

Fig. 1. Energy spectrum. (a) $d_e = 0.3$; Diamonds: $N = 512$; $\nu_4 = 10^{-12}$. Crosses: $N = 1024$; $\nu_4 = 10^{-13}$. The continuous line is the Kolmogorov spectrum (11) with $C_K = 2.0$. (b) $d_e = 0.02$; Diamonds: $N = 512$; $\nu_4 = 10^{-13}$. The continuous line is the spectrum (12) with $C'_K = 8.0$.

This is quite surprising, because many two-dimensional systems exhibit a reversed energy flux, from the small scales to the larger ones, as it happens for 2D-NS turbulence [6–9], Hasegawa-Mima turbulence [10] or equivalent-barotropic turbulence [11]. The case of 2D-MHD turbulence is different, since this is a direct energy cascade, but it is probably driven by the Alfvèn effect [12].

In order to investigate how the energy flows among the lengthscales, starting from the energy budget obtained by (1), we derived [13] the 2D-EMHD counterpart of Kolmogorov's four-fifths law [3, 4]. We made the following hypotheses: statistical homogeneity and isotropy, incompressibility and the existence of an inertial range in which the energy flux is constant ($\bar{\varepsilon}$). We found a relation that can be read as a scaling law for the third order structure function:

$$Q_3(l) = -\tfrac{2}{3}d_e^2\langle \delta v_\| \,\delta v_\| \,\delta v_\| \rangle - \tfrac{1}{2}d_e^2\langle \delta v_3 \,\delta v_3 \,\delta v_\| \rangle + \tfrac{1}{2}\langle \delta v_\| \,\delta B_\| \,\delta B_\| \rangle - \tfrac{1}{2}\langle \delta v_\| \,\delta B_\perp \,\delta B_\perp \rangle + \tfrac{1}{2}\langle \delta v_3 \,\delta B_3 \,\delta B_\| \rangle - \tfrac{1}{4}\langle \delta B_3 \,\delta B_3 \,\delta v_\| \rangle \simeq \bar{\varepsilon} l, \tag{9}$$

where differences of dynamical fields ($\delta \boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x} + \boldsymbol{l}) - \boldsymbol{v}(\boldsymbol{x})$) appear, the brackets mean ensemble averages and the subscripts express increments taken longitudinally ($\delta v_\| = \delta v_i l_i / l$) or transversally ($\delta v_\perp = \epsilon_{ij}\, \delta v_i l_j / l$). The linear scaling and the value of the numerical coefficient in (9) were confirmed by numerical simulations [13].

The most remarkable feature of (9) is that it is valid for any values of d_e. Using a Kolmogorov-type analyis (i.e. supposing the cascade to be self-similar), it is easy to estimate the slopes of the energy spectra in two limiting cases, shown in Fig. 1.

If the energy is dominated by its kinematic contribution ($d_e \gg 1$), from (9) we have:

$$\delta v(l) \propto \bar{\varepsilon}^{1/3} l^{1/3} \tag{10}$$

which leads to the energy spectrum:

$$E(k) = C_K \bar{\varepsilon}^{2/3} k^{-5/3}. \tag{11}$$

In the opposite regime, the magnetic energy dominates ($d_e \ll 1$) in (9) so that:

$$\delta B(l) \propto \bar{\varepsilon}^{1/3} l^{2/3}$$

and the spectrum is given by:

$$E(k) = C'_K \bar{\varepsilon}^{2/3} k^{-7/3}. \tag{12}$$

The presence of a third lengthscale (d_e), in addition to the inertial scale and the Kolmogorov scale which are the upper and lower limit of the inertial range, and the existence of the quantity (6) which is not only passively advected, makes 2D-EMHD turbulence different from the other 2D turbulent systems.

In the $d_e \gg 1$ regime, we observed the decoupling of the planar motion and the axial one: eq. (4) reduces to the vorticity equation in 2D-NS turbulence, while the current along z-axis is advected (3) as a passive scalar by the planar velocity field. Axial kinetic energy (6) and planar kinetic energy (5) are both (ideally) conserved, but the decoupling introduces new planar invariants, such as enstrophy.

In the $d_e \ll 1$ regime, mean square potential (6) and magnetic energy (5) are conserved, so that one can expect an inverse cascade of H and a direct cascade of E, as it happens in 2D-MHD, which is consistent with the energy spectrum slope found [14].

It is worth noticing one more case: for $d_e \sim 1$, energy is dominated by its kinematic part, but the forcing term in (4) is not negligible, since it behaves like an enstrophy source active at any lengthscale in the inertial range [13].

3. Is there a Whistler effect?

It is known [1] that linearizing eq. (1) against a uniform background magnetic field $\boldsymbol{B}_0$ one finds the whistler waves, sometimes called helicons, whose dispersion relation is given by

$$\omega(k) = \frac{k k_\| \, d_e^2 \, \omega_{ce}}{1 + k^2 d_e^2} \tag{13}$$

where $\omega_{ce} = eB_0/m_e c$ is the electron girofrequency and $k_\|$ means that whistlers propagate along the equilibrium magnetic field B_0. The role of whistlers in EMHD could be analogous to the role of Alfvén waves in MHD: they are responsible of a mechanism of energy transfer in competition to the non linear one. Following a Kraichnan-type [12] analysis, it is possible to evaluate the corrections to the energy spectrum due to the "whistler effect". Contrary to Alfvén waves, whistlers are dispersive waves, thus we can expect different results in the asymptotic regimes that we studied above.

In the magnetic regime ($kd_e \ll 1$), relation dispersion (13) reduces to

$$\omega(k) \approx k^2 d_e^2 \, \omega_{ce}$$

and whistlers propagate with group velocity

$$v_g \approx 2 d_e^2 \, \omega_{ce} k. \tag{14}$$

Then supposing the energy to be transferred in the inertial range during the non linear time of interaction of whistler wavepackets, one finds the following energy spectrum

$$E(k) \sim \bar{\varepsilon}^{1/2} d_e \omega_{ce}^{1/2} k^{-2}. \qquad (15)$$

In the kinematic regime ($kd_e \gg 1$) (13) becomes

$$\omega(k) \approx \omega_{ce}$$

and whistler waves reduce to non propagating oscillations: there cannot exist a whistler effect.

4. Forced EHMD turbulence

2D-EMHD turbulence was proven [5, 13] to exhibit a Kolmogorov energy spectrum, for $d_e \sim 1$, in the freely decaying case. It is interesting to test how strong is this behaviour, namely to find a corresponding statistically stationary state. Taking into account forcing and dissipation, eq. (1) becomes

$$d_e^2 \frac{\partial \boldsymbol{v}}{\partial t} + d_e^2 \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p - \boldsymbol{E} - \boldsymbol{v} \times \boldsymbol{B} + \nu_q (-\nabla)^{2q} \boldsymbol{v} + \boldsymbol{F} \qquad (16)$$

To solve (3) and (4) we used a pseudo-spectral code in a square box, with periodic boundary conditions and 2/3-rule deliasing. We chose a large scale forcing in order to achieve a stationary state, in which the turbulent cascade is maintained by externally injected energy. To avoid accumulation of energy at large scale, which could be due to the inverse cascade of H, we forced the field $\psi - d_e^2 \nabla^2 \psi$ using a Langevin-type scheme

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + \eta(t) \qquad (17)$$

where the random noise $\eta(t)$ is delta-correlated in time and only wavevectors with ($k \leqslant 3$) were forced. The field $\phi - d_e^2 \nabla^2 \phi$ was left freely decaying, with the aim of investigating the role of the Lorentz forcing term in (4). Starting from an initial condition in which both fields were zero, we observed the raising up of H, due to the random forcing, followed by that of energy E. After a linear phase, when H grows linearly in time, a stationary state was reached in which total energy and dissipation oscillated around a finite value (Fig. 2(a)). During this stationary phase, the energy spectrum was found to be close to the Kolmogorov one, as shown in Fig. 2(b). Direct plots of the fields (see Fig. 3) are clearly recognizable as fully developed turbulent state.

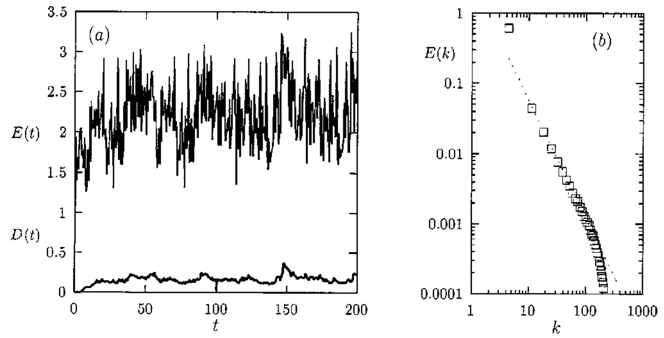

Fig. 2. (a) Total energy and dissipation versus time for a Langevin forced run. Output time corresponds to ω_{ce}^{-1}. $d_e = 0.3$, $\nu_4 = 10^{-10}$, $N = 256$. At $t = 100$ ν_4 was reduced to a half. (b) Energy spectrum. $d_e = 0.3$, $\nu_4 = 10^{-13}$, $N = 1024$. The continuous line is the Kolmogorov spectrum (11).

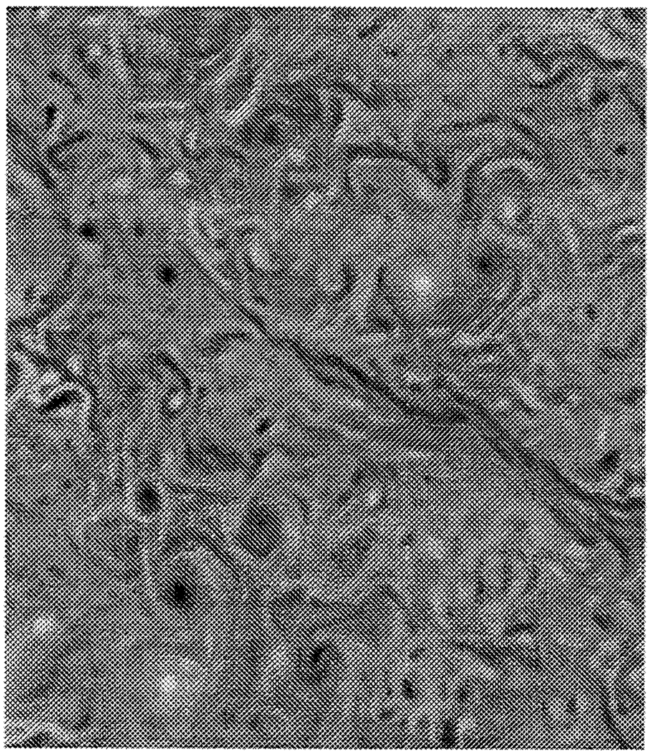

Fig. 3. Direct plot of the field $\psi - de^2 \nabla^2 \psi$ during the stationary phase shown in Fig. 2(a).

This behaviour does not hold out for an asymptotically long time: in fact, when the Lorentz force in (4) starts to act as a consistent source of planar kinematic energy, the energy spectrum slope becomes steeper than the Kolmogorov one. The Lorentz force cannot be treated as a large scale forcing, since it injects energy at all lengthscales and the standard picture à la Richardson needs to be modified.

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