The shallow-water nonlinear Schrödinger equation in Lagrangian coordinates

A. R. Osborne and G. Boffetta
Istituto di Fisica Generale dell'Università and Istituto di Cosmo-geofisica del C.N.R., Corso Fiume 4, Torino 10133, Italy

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The nonlinear Schrödinger equation in Lagrangian coordinates (LNLS) is derived from the Lagrangian form of the Korteweg–de Vries equation (LKdV) using a multiscale averaging approach. The resultant LNLS equation describes fluid particle motions for a small-amplitude, narrow-banded nonlinear wave field propagating in shallow water. Particle motion is discussed for LNLS in the context of the inverse scattering transform (IST) for both infinite-line and periodic boundary conditions and explicit expressions for the trajectories of fluid parcels are derived. It is demonstrated that, even in the presence of nonlinear effects, a particle trajectory can be orthogonally decomposed into nonlinear horizontal and vertical motions which, given the initial conditions, evolve independently of each other. It is shown how the presence of a long, slowly varying wave component, which is interpreted as radiation stress, influences the motion of fluid particles. Finally it is also shown how these results may be used in an experimental context for the study of data described approximately by the LNLS equation.

I. INTRODUCTION

During the last two decades a new area of mathematical physics has evolved as a consequence of the discovery of the soliton by Zabusky and Kruskal.[1] The approach, known as the inverse scattering (spectral) transform (IST), was first used to solve the KdV equation by Gardner *et al.*[2] and has subsequently led to the exact solutions of a number of other physically interesting nonlinear wave equations including the modified KdV, the nonlinear Schrödinger, and sine-Gordon equations.[3–8] Nonlinear partial differential equations of this type have been called "universal" because of their fundamental and ubiquitous occurrence in a wide variety of physical systems.[9] Solutions to these and many other nonlinear wave equations are now known for both infinite-line and periodic boundary conditions; the total number of integrable wave equations presently available is about 100.[10] The IST for these nonlinear systems provides the *general* solution and may be viewed as a *nonlinear generalization* of the linear Fourier transform solution to the classical Cauchy problem: Given the amplitude of the wave motion at some initial time, $\eta(x, t=0)$, determine the evolution of the system for all time thereafter, $\eta(x, t)$. A major consequence of the IST approach is the appearance of a wavenumber or frequency spectrum whose temporal evolution is simple (constant nonlinear Fourier amplitudes and $e^{-i\omega t}$ phase variations), but which nevertheless contains all the essential physics of the nonlinear wave motion, which is formulated as a spectral "inverse" problem.[2–8]

Probably the most well-known example of a nonlinear, integrable system is that given by the KdV equation which describes wave motion in shallow water,

$$\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0, \tag{1}$$

where the constant coefficients are given by

$$c_0 = (gh)^{1/2}, \tag{2}$$

$$\alpha = 3c_0/2h, \tag{3}$$

$$\beta = c_0h^2/6. \tag{4}$$

Here h is the depth and g is the acceleration of gravity. Subscripts refer to partial derivatives with respect to time t and space x. There are many other fluid mechanical applications of the KdV equation including internal wave motions[11] and geophysical fluid dynamical (GFD) motions.[12–17] When infinite-line boundary conditions are appropriate [$\eta(x,t) \to 0$ as $|x| \to \infty$ for $-\infty < x < \infty$], the solution to (1) is given by Gardner *et al.*[2] When periodic boundary conditions are applied [$\eta(x,t) = \eta(x+L,t)$ on $0 < x < L$, for L the period], the solution is given by Dubrovin and Novikov,[18] Flaschka and McLaughlin,[19] and McKean and Trubowitz[20] (see also Ablowitz and Segur[5]).

The periodic IST solution for the KdV equation, which can be viewed as a nonlinear Fourier series,[21] solves for the motion in terms of the interacting, nonlinear Fourier components of the KdV equation (hyperelliptic functions), which include sine waves, Stokes waves, cnoidal waves and solitons.[21,22] Thus while linear Fourier analysis consists of the linear superposition of ordinary sine waves, nonlinear Fourier analysis consists of the linear superposition of the nonlinearly interacting, nonlinear normal modes for the KdV equation.

The shallow-water Eulerian nonlinear Schrödinger equation is also known, e.g., take the shallow-water limit of the NLS equation derived by Hasimoto and Ono,[23] and find

$$i\psi_t + iC_g\psi_x + \mu\psi_{xx} + \nu|\psi|^2\psi = 0. \tag{5}$$

The constant coefficients are given by

$$C_g = \frac{\partial\omega_0}{\partial k_0} = c_0 - 3\beta k_0^2, \tag{6}$$

$$\mu = \frac{1}{2}\frac{\partial^2\omega_0}{\partial k_0^2} = -3\beta k_0, \tag{7}$$

$$\nu = 9c_0/16h^4k_0, \tag{8}$$

where C_g is the group velocity. The NLS equation (5) de-

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scribes small-amplitude, narrow-banded wave motion in shallow water. The carrier frequency and wavenumber are related by the shallow-water dispersion relation $\omega_0 = c_0 k_0 - \beta k_0^3$. A modulation envelope $A(x,t)$ and phase $\phi(x,t)$ are assumed to describe the dynamical motion. The complex field $\psi(x,t)$ is related to the envelope and phase by $\psi(x,t) = A(x,t)e^{i\omega' t + i\phi(x,t)}$, and the free surface elevation is given by

$$\eta(x,t) = -\frac{3A^2(x,t)}{4k_0^2h^3} + A(x,t)\cos\theta(x,t) + \frac{3A^2(x,t)}{4k_0^2h^3}\cos 2\theta(x,t) + \cdots, \quad (9)$$

where the phase $\theta = \theta(x,t)$ is

$$\theta = k_0 x - (\omega_0 + \omega')t + \phi(x,t). \quad (10)$$

The nonlinear frequency correction in (10) is

$$\omega' = 9c_0 \overline{A^2}/16k_0h^4, \quad (11)$$

where the overbar denotes spatial averaging. Note that (9) in its unmodulated form is just the shallow-water Stokes wave.The wave motions describable by KdV and NLS in shallow water can be quite different because the KdV equation generally covers a much larger class of solutions than that for small-amplitude, narrow-banded wave motion. However, generally speaking, the relationship simplifies when one also investigates small-amplitude, narrow-banded solutions to the KdV equation. The study of the relationship between these two nonlinear wave equations in Eulerian coordinates is interesting in its own right, both physically and spectrally, and is treated in detail elsewhere.[24–27]

Until recently the integrable wave equations solvable by IST have been exclusively studied in Eulerian coordinates, where the evolution of the system is described as a function of time at a particular fixed spatial location. An alternative formulation is in terms of Lagrangian coordinates, where the motions of individual particles of the fluid are followed. At the present time the Boussinesq,[28] the KdV, and the potential KdV[29] equations are known to have Lagrangian forms. The IST spectral solution to Lagrangian KdV (LKdV) has also been discussed.[29] Here we show that the Lagrangian form of NLS can be determined from LKdV using a multiscale averaging approach.[30,31] Not only do we gain insight into the behavior of certain nonlinear Schrödinger solutions for Lagrangian motions, but we also gain valuable understanding about the deep connection between the LKdV and the LNLS equations, both from the physical and spectral points of view. The nonlinear spectral formulation for the Eulerian case has been studied in some detail elsewhere.[25–27]

II. LAGRANGIAN MOTION DESCRIBED BY THE KdV EQUATION

Beginning with the work of Ursell,[28] Osborne *et al.*[29] developed a Lagrangian form for the KdV equation (LKdV) and discussed several examples of particle motions governed by this equation. We now give a terse discussion of LKdV and some of its implications; these serve to establish the notation and to set the stage for the derivation of NLS. The governing LKdV equation of motion is

$$\eta_t + c_0\eta_a + \alpha\eta\eta_a + \beta\eta_{aaa} = 0, \quad (12)$$

where subscripts refer to partial derivatives with respect to the Lagrangian position coordinate a and time t. The constant coefficients are the same as those appearing in (1). Here the wave field amplitude $\eta(a,t)$ is given by

$$\eta(a,t) = -h\frac{\partial[x(a,t) - a]}{\partial a} \equiv -h\frac{\partial X(a,t)}{\partial a}, \quad (13)$$

where $x(a,t)$ is the horizontal particle position whose derivative, with respect to the reference position a, is proportional to the field $\eta(a,t)$; the coordinate a may be considered as a label for the individual particles. For infinite-line boundary conditions, a is the particle position at $t = -\infty$; for periodic boundary conditions, a is the average horizontal position of the particle.

The vertical particle motion $y(a,b,t)$ is governed by

$$\eta(a,t) = (h/b)[y(a,b,t) - b] \equiv (h/b)Y(a,b,t). \quad (14)$$

Hence for infinite-line boundary conditions b is the vertical position of the particle at $t = -\infty$, while for periodic boundary conditions b is the average vertical position of the particle. When the average vertical position lies in the free surface $b = h$, $Y(a,h,t)$ may be identified as the free surface elevation of the fluid motion. From (13) and (14) the following constraint relation always holds: $Y = -b\,\partial X/\partial a$.

The fact that (13) and (14) give linear relations between the particle motion coordinates $[X(a,t), Y(a,b,t)]$ and the field amplitude $\eta(a,t)$ means that X, Y may be orthogonally decomposed, even if the motion is nonlinear. First use (13) in (12) to get the governing equation of motion for the horizontal coordinate $X(a,t) = x(a,t) - a$,

$$X_t + c_0X_a - (\alpha h/2)X_a^2 + \beta X_{aaa} + C = 0. \quad (15)$$

This is the Lagrangian form for the potential KdV (pKdV) equation, where

$$C = \left(\frac{\alpha}{2LH}\right)\int_a^{a+L} \eta^2(a',t)d\,da'. \quad (16)$$

This last equation is valid for both infinite-line and periodic boundary conditions. Specifically, in the infinite-line limit $L\to\infty$ and $C\to 0$.[29]

Finally use of (14) in (12) gives the KdV equation for the vertical $Y(a,b,t) = y(a,b,t) - b$ motion,

$$Y_t + c_0Y_a + (\alpha h/b)YY_a + \beta Y_{aaa} = 0. \quad (17)$$

When $b = h$ this latter equation becomes the KdV equation for the free surface; on the bottom $b = 0$ and no vertical motion occurs.

As pointed out by Osborne *et al.*[29] the motion of water particles in shallow water described by LKdV is integrable by IST. This is because the transformation from Eulerian KdV (1) to Lagrangian KdV (12) is given by the following simple form:

$$x\to a, \quad t\to t,$$

$$\eta(x,t)\to\eta(a,t) = -\frac{\partial[x(a,t) - a]}{\partial a} = -\frac{\partial X(a,t)}{\partial a}. \quad (18)$$

Thus by reinterpreting the spatial variable x in terms of the particle reference position a and using the last of (18) to connect the field $\eta(a,t)$ with the horizontal particle motion $X(a,t)$, one obtains the Lagrangian form of KdV. Therefore IST, for either infinite-line or periodic boundary conditions, may be used to solve LKdV by making a simple change of variables.

III. DERIVATION OF LAGRANGIAN NLS FROM LAGRANGIAN KdV

It is well known that the Eulerian nonlinear Schrödinger equation may be derived from the Eulerian KdV equation by application of the multiscale expansion technique.[25–27,30,31] The approach is to restrict attention to a limited subclass of possible motions describable by the KdV equation, i.e., to those in which a rapidly oscillating (nonlinear) carrier wave is of small amplitude and modulated on slow space and time scales. We now apply the multiscale approach to the Lagrangian KdV equation and its associated particle motion coordinates (12)–(14). We focus on analytic expressions for the particle coordinates $X(a,t)$ and $Y(a,b,t)$. First consider the following Fourier series expansions for the free surface amplitude $\eta(a,t)$ and the particle position coordinates $X(a,t)$ and $Y(a,b,t)$ in which the Fourier amplitudes $u_n(x,t)$, $X_n(a,t)$, and $Y_n(a,t)$ are allowed to vary in space and time,

$$\eta(a,t) = \sum_{n=-\infty}^{\infty} u_n(a,t) e^{in\theta(a,t)}, \tag{19}$$

$$X(a,t) = \sum_{n=-\infty}^{\infty} X_n(a,t) e^{in\theta(a,t)}, \tag{20}$$

$$Y(a,t) = \sum_{n=-\infty}^{\infty} Y_n(a,t) e^{in\theta(a,t)}, \tag{21}$$

where the phase is given by

$$\theta(a,t) = k_0 a - \omega_0 t \tag{22}$$

and the shallow-water dispersion relation is

$$\omega_0 = c_0 k_0 - \beta k_0^3. \tag{23}$$

We further impose the conditions $u^*_{-n} = u_n$, $X^*_{-n} = X_n$, and $Y^*_{-n} = Y_n$ on the coefficients of (19)–(21) to ensure real solutions for the wave amplitude $\eta(a,t)$ and particle coordinates $X(a,t), Y(a,b,t)$. Now use (19) in (12), equate equal coefficients of the $e^{in\theta}$, and find for each n,

$$\left(\frac{\partial}{\partial t} - in\omega_0\right)u_n + c_0\left(\frac{\partial}{\partial a} + ink_0\right)u_n + \frac{\alpha}{2}\left(\frac{\partial}{\partial a} + ink_0\right) \times \sum_q u_q u_{u-q} + \beta\left(\frac{\partial}{\partial a} + ink_0\right)^3 u_n = 0. \tag{24}$$

Introduce slow space and time scales (a', t') into the coefficients u_n by

$$u_n(a,t) = \epsilon^{\alpha(n)} u_n'(a',t'), \tag{25}$$

where

$$\alpha(0) = 2, \quad \alpha(n) = |n|, \quad |n| \geqslant 1, \tag{26}$$

and

$$a' = \epsilon\left(a - \frac{\partial \omega_0}{\partial k_0} t\right) = \epsilon[a + (3\beta k_0^2 - c_0 t)], \tag{27}$$

$$t' = \epsilon^2 \frac{\partial^2 \omega_0}{\partial k_0^2} t = -6\beta k_0 \epsilon^2 t. \tag{28}$$

Note that the first of (26) forces the low-frequency field u_0 to occur at $O(\epsilon^2)$ [see (32) below]. Using (25) in (24) we find

$$\left[-6\beta k_0 \epsilon^2 \frac{\partial}{\partial t'} + (3\beta k_0^2 - c_0)\frac{\partial}{\partial a'} - in\omega_0\right]\epsilon^{\alpha(n)} u_n' + c_0\left[\epsilon\frac{\partial}{\partial a'} + ink_0\right]\epsilon^{\alpha(n)}u_n' + \frac{\alpha}{2}\left[\epsilon\frac{\partial}{\partial a'} + ink_0\right] \times \sum_{m=-\infty}^{\infty} \epsilon^{\alpha(m)+\alpha(n-m)} u_m' u_{n-m}' + \beta\left[\epsilon\frac{\partial}{\partial a'} + ink_0\right]^3 \epsilon^{\alpha(n)} u_n' = 0. \tag{29}$$

It is clear that as $\epsilon \to 0$ the lowest-order terms in ϵ must balance in (29). Excluding the terms $n = 0$ and $n = \pm 1$, this means that the following terms in (29) must balance: (i) the term $(-in\omega_0)$ in the first bracket, (ii) the term $(ic_0 nk_0)$ in the second bracket, (iii) the term $(i\alpha nk_0/2)$ term in the third bracket, and (iv) the term $\beta(ink_0)^3$ in the fourth bracket. Thus we have the following expression valid for $n \geqslant 2$:

$$\beta k_0^2(1-n^2)\epsilon^{\alpha(n)}u_n' + \frac{\alpha}{2}\sum_{m=-\infty}^{\infty} \epsilon^{\alpha(m)+\alpha(n-m)} u_m' u_{n-m}' = 0, \tag{30}$$

which we rewrite in the form

$$u_n' = \frac{3\lambda}{k_0^2(n^2-1)} \sum_m{}' u_m' u_{n-m}', \quad n \geqslant 2, \tag{31}$$

where $\lambda = \alpha/6\beta$ and the prime on the summation implies that only the terms for which $\alpha(m) + \alpha(n-m) = \alpha(n)$ are summed over m.

We now consider terms at dominant order in (29) and obain, for $n = 0$ (ϵ^3),

$$u_0' = -(\alpha/3\beta k_0^2)|u_1'|^2 + f_0, \tag{32}$$

where f_0 is an arbitrary constant of integration that physically corresponds to a vertical displacement of the mean surface level. We shall make a physical choice for f_0 below. For $n = 1$ (ϵ^4) in (29) we have

$$3\beta k_0\left(i\frac{\partial^2 u_1'}{\partial a'^2} - 2\frac{\partial u_1'}{\partial t'}\right) = -i\alpha k_0(u_0' u_1' + u_2' u_{-1}'), \tag{33}$$

and finally with $n = 2$ (ϵ^2) in (29),

$$u_2' = (\alpha/6\beta k_0^2)u_1'^2. \tag{34}$$

Notice that (33) describes the wave motion of the field u_1' with nonlinear coupling on the right-hand side to the $O(\epsilon^2)$ fields u_0' and u_2'. But (32) and (34) constrain the latter fields to follow exactly the motion of u_1'; thus we use (32) and (34) in (33), together with the reality condition $u^*_{-n} = u_n$ and find

$$i\frac{\partial u_1'}{\partial t'} + \frac{1}{2}\frac{\partial^2 u_1'}{\partial a'^2} - \left(\frac{\alpha}{6\beta k_0}\right)^2 |u_1'|^2 u_1' = -\frac{\alpha}{6\beta} f_0 u_1'. \tag{35}$$

In terms of the fast space-time variables (x,t), together with

$\psi = 2\epsilon u_1', g_0 = \epsilon^2 f_0$, we have

$$i\frac{\partial\psi}{\partial t} + i\left(\frac{\partial\omega_0}{\partial k_0}\right)\frac{\partial\psi}{\partial a} + \frac{1}{2}\left(\frac{\partial^2\omega_0}{\partial k_0^2}\right)\frac{\partial^2\psi}{\partial a^2} + \left(\frac{\alpha^2}{24\beta k_0}\right)|\psi|^2\psi - \alpha k_0 g_0 \psi = 0, \quad (36)$$

where the constant coefficients are found by

$$C_g \equiv \frac{\partial\omega_0}{\partial k_0} = c_0 - 3\beta k_0^2 = c_0\left(1 - \frac{h^2k_0^2}{2}\right), \quad (37)$$

$$\mu \equiv \frac{1}{2}\frac{\partial^2\omega_0}{\partial k_0^2} = -3\beta k_0 = -c_0\frac{h^2k_0}{2}, \quad (38)$$

$$\nu \equiv \alpha^2/24\beta k_0 = 9c_0/16h^4k_0; \quad (39)$$

C_g is the group velocity. The transformation $\psi \rightarrow \psi e^{-i\omega_g' t}$ reduces (36) to the shallow-water Lagrangian nonlinear Schrödinger equation in the "laboratory" coordinate frame,

$$i\frac{\partial\psi}{\partial t} + iC_g\frac{\partial\psi}{\partial a} + \mu\frac{\partial^2\psi}{\partial a^2} + \nu|\psi|^2\psi = 0, \quad (40)$$

where we take

$$\omega_g' = \alpha k_0 g_0. \quad (41)$$

In what follows we shall assume the following form for $\psi(a,t)$:

$$\psi(a,t) = A(a,t)e^{-\omega' t + i\phi(a,t)}, \quad (42)$$

where

$$\omega' = \omega_c' + \omega_g'. \quad (43)$$

Here ω_c' is given by

$$\omega_c' = -9c_0\overline{A^2}/16k_0h^4, \quad (44)$$

where the overbar denotes the average

$$\overline{A^2} = \frac{1}{L}\int_0^L A^2(a,t)da. \quad (45)$$

Note that ω_g' is still undefined since we have yet to select the arbitrary constant g_0. This will be done in the following section. Note that $A(a,t)$ is the (real) envelope of the wave field and $\phi(a,t)$ is the (real) phase. The selection (42) marries the mathematics of NLS with its associated periodic spectral transform solution; we refer to the NLS spectrum as being "empty" when $A(a,t)$ and $\phi(a,t)$ are constants, i.e., when there is no modulation of the wave train. Another motivation for the choice (42) is that it allows an interpretation of the free surface elevation $\eta(a,t)$ in terms of a modulated Stokes field (see the discussion in the following section). Note that (42) also provides an interpretation of the phase in terms of a part that varies linearly in time, $-\omega' t$ and another part, $\phi(a,t)$, which represents fluctuations about $-\omega' t$. The frequency ω' is the amplitude-dependent frequency correction to the dispersion relation that normally appears in the Stokes wave; it will be computed explicitly for the present case in the following section.

The free surface elevation $\eta(a,t)$ (19), an approximate solution to the KdV equation under the assumptions of small amplitudes and narrow bandedness, may be written in terms of the solution to NLS, $\psi(a,t)$, by

$$\eta(a,t) = u_0(a,t) + [u_1(a,t)e^{i\theta(x,t)} + u_2(a,t)e^{i2\theta(x,t)} + \text{c.c.}] + \cdots$$
$$= \frac{1}{2}(\psi z + \psi^* z^*) + (\alpha/24\beta k_0^2)(\psi^2 z^2 + \psi^{*2} z^{*2} - 2|\psi|^2) + g_0, \quad (46)$$

where

$$z = e^{i(k_0 a - \omega_0 t)}. \quad (47)$$

Notice that the constant g_0 serves only to determine the mean level of the free surface elevation in the second half of (46). We refer to the term with u_1 $[O(\epsilon)]$ as the "first harmonic," $u_2[O(\epsilon^2)]$ the "second harmonic," and $u_0[O(\epsilon^2)]$ as the low-frequency contribution, which, as shown below, is the slow mean sea level variation resulting from radiation stress. By virtue of (32) and (34) the fields u_0 and u_2 are related directly to u_1 (and hence ψ); u_1 is governed by the NLS equation. Thus the second-order fields u_0 and u_2 are "locked" to the first-order NLS field u_1, which of course dominates the wave motion energetically in the approximation considered here.

IV. PHYSICAL CONSEQUENCES OF NARROW-BANDED WAVE MOTION DESCRIBED BY THE KdV EQUATION

Here we discuss some of the physical consequences of the above derivation of the NLS equation from the KdV equation. Among these are the predictions by the theory: (i) that the free surface elevation is described by a modulated Stokes field (Sec. IV A), (ii) that the Stokes field may be asymptotically summed to all orders n for leading order in ϵ (Sec. IV B), (iii) that there is a slow mean sea level variation resulting from radiation stress, which is given as proportional to $-A^2(a,t)$ (Sec. IV C), and (iv) that phase locking occurs between the carrier wave and the second- and higher-order harmonics in the Fourier spectrum (Sec. IV D).

A. The Stokes field as an approximate solution of the KdV equation

Now we elaborate on the fact that the free surface elevation, as given by (46), can be interpreted as a second-order modulated Stokes wave (Stokes field). This is obtained by using (42) in (46) to find the relationship between the wave elevation and the [complex solution $\psi(a,t)$] to NLS:

$$\eta(a,t) = -\frac{3[A^2(a,t) - \overline{A}^2]}{4k_0^2h^3} + A(a,t)\cos\theta(a,t) + \frac{3A^2(a,t)}{4k_0^2h^3}\cos 2\theta(a,t) + \cdots, \quad (48)$$

where the phase $\theta = \theta(a,t)$ is given by

$$\theta = k_0 a - (\omega_0 + \omega')t + \phi(a,t). \quad (49)$$

Here we have taken g_0 to give a zero mean for (48),

$$g_0 = 3\overline{A^2}/4k_0^2h^3. \quad (50)$$

With this choice for g_0 in (41) we find for the nonlinear frequency correction in (49), as given by (43),

$$\omega' = 9c_0\overline{A^2}/16k_0h^4. \quad (51)$$

Note that (48) reduces to an ordinary KdV Stokes wave in the absence of modulation, i.e., when the envelope $A(a,t) = A_0 = \text{const}$ and the phase $\phi(a,t) = \text{const}$:

$$\eta(a,t) = A_0 \cos \theta(a,t) + \frac{3A_0^2}{4k_0^2 h^3} \cos 2\theta(a,t) + \cdots,$$
$$\theta = k_0 a - (\omega_0 + \omega')t + \phi_0, \qquad (52)$$
$$\omega' = 9c_0 A_0^2 / 16 k_0 h^4.$$

This form for the Stokes wave of the KdV equation is discussed by Whitham (1974).[30]

Hence the KdV equation and its dynamical description of shallow-water wave motion for a small-amplitude, narrow-banded Fourier spectrum, may be reduced to that of the dynamics of the NLS equation (40) whose solution $\psi(a,t) = A(a,t)e^{-i\omega' t + i\phi(a,t)}$ furnishes the nonlinear evolution of the complex envelope of a modulated second-order Stokes field (48), (49), and (51). Thus either the KdV equation (12) or NLS [(40) plus its modulated second-order Stokes field (48)] form equivalent descriptions of Lagrangian shallow-water wave dynamics in two dimensions for small-amplitude waves with narrow-banded Fourier spectra. Hence LKdV and LNLS give two ways of describing the same nonlinear physical system, but in terms of different physical variables, e.g., the free surface elevation $\eta(a,t)$ for the LKdV equation (12) and the complex envelope $\psi(a,t) = e^{-\omega' t + \phi(a,t)}$ for the LNLS equations [(46) and (48)].

B. Summing the modulated Stokes series

We now demonstrate that the Stokes fields, (19)–(21), can be summed by selecting terms of leading order in ϵ for each n and then summing over all n. This surprising result occurs because the recursion relations (31) can be evaluated to give the following simple, but explicit, form:

$$u_n = n\left[\frac{\lambda}{2k_0^2}\right]^{n-1} u_1^n, \quad n \geqslant 2. \qquad (53)$$

Thus all harmonics for $n \geqslant 2$ are locked to the first harmonics $n = 1$. Recall that $u_1 = \psi/2 = (A/2)e^{i\Phi}$, where $\Phi = -\omega' t + \phi$, hence

$$u_n = \frac{n}{2^{2n-1}} (\lambda / k_0^2)^{n-1} A^n e^{in\phi}$$
$$= \tfrac{1}{2} n A(a,t) U^{n-1}(a,t) e^{in\Phi}, \qquad (54)$$

where the slowly varying Ursell number is

$$U(a,t) = 3A(a,t)/8k_0^2 h^3. \qquad (55)$$

These results, when substituted into (19) [after removing the mean $\overline{A^2}$, as in (48)], gives the following free surface displacement:

$$\eta(a,t) = -\frac{3[A^2(a,t) - \overline{A^2}]}{4k_0^2 h^3} + A(a,t) \sum_{n=1}^{\infty} n U^{n-1}(a,t) \cos n\theta. \qquad (56)$$

Thus the series solution to the KdV equation (19), to leading order in ϵ for each n can be computed to all orders for $1 \leqslant n \leqslant \infty$. This series can be "locally" summed by using the fact that $A(x,t)$ and $U(x,t)$ are slowly varying in space and time, and may therefore by treated as constants in (56). The summation in the last equation is evaluated as follows:

$$S = \frac{4k_0^2}{\lambda} U(x,t) \sum_{n=1}^{\infty} n U^{n-1}(x,t) \cos n\theta$$
$$= \frac{2k_0^2}{\lambda} U(x,t) \sum_{n=1}^{\infty} n U^{n-1}(x,t) e^{in\theta} + \text{c.c.},$$
$$S = \frac{2k_0^2}{\lambda} X \sum_{n=1}^{\infty} n X^{n-1} + \text{c.c.}$$
$$= \frac{2k_0^2}{\lambda} \frac{X}{(1-X)^2} + \text{c.c.}, \qquad (57)$$

where $X = U(x,t)\, e^{i\theta}$. Then (56) is found to have the closed form expression,

$$\eta(a,t) = -\frac{3[A^2(a,t) - \bar{A}^2]}{4k_0^2 h^3} + A(a,t) \times \left(\frac{[1 + U^2(a,t)]\cos\theta(a,t) - 2U(a,t)}{[1 + U^2(a,t) - 2U(a,t)\cos\theta(a,t)]^2}\right). \qquad (58)$$

This simple compact form for the free surface amplitude is an approximation to the cnoidal wave for a large range of the modulus m (see Fig. 1 and the text below).

The forms for the particle coordinates follow from the latter equation (58) and from (13) and (14),

$$X(a,t) = \frac{3}{4k_0^2 h^4} \int_a [A^2(a,t) - \bar{A}^2] da - \frac{A(a,t)}{k_0 h} \times \left(\frac{\sin\theta(a,t)}{1 + U^2(a,t) - 2U(a,t)\cos\theta(a,t)}\right), \qquad (59)$$

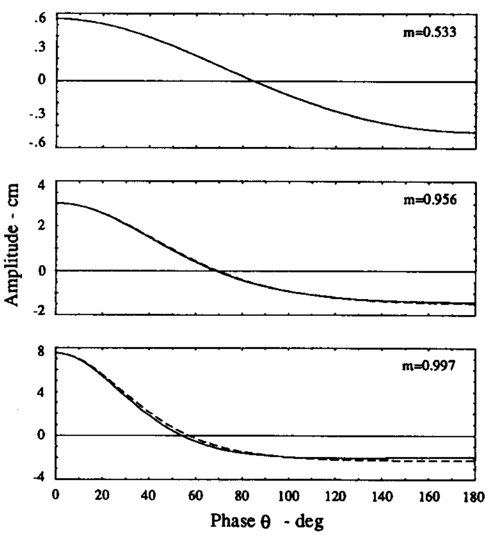

FIG. 1. Comparison of the approximate vertical particle motion $Y(a,b,t)$ (60) (solid line) with the cnoidal wave elliptic function solution (62) (dotted line) of the KdV equation. The depth is $h = 10$ cm, wavelength $L = 100$ cm, $a = 0$ cm, and $b = 10$ cm.

$$Y(a,t) = \frac{3b\,[A^2(a,t) - \bar{A}^2]}{4k_0^2h^4} + \frac{bA(a,t)}{h} \times\left(\frac{[1+U^2(a,t)]\cos\theta(a,t) - 2U(a,t)}{[1+U^2(a,t) - 2U(a,t)\cos\theta(a,t)]^2}\right). \quad (60)$$

The integral in (59) has the limits $(-\infty,a)$ on the infinite line and $(0,a)$ (for $0\leqslant a\leqslant L$) on the periodic interval. Expressions (59) and (60) are modulated nonlinear wave trains with two sources of nonlinearity: (i) the carrier wave is a nonlinear Stokes field and (ii) the envelope function is a solution to the nonlinear Schrödinger equation. It is instructive to consider how closely $X(a,t)$ in (59) and $Y(a,b,t)$ in (60) (in their unmodulated forms) approximate the elliptic function solutions to the pKdV (15) and KdV (17) equations. These exact particle motions are given by[29]

$$X(a,t) = -\frac{2kh^2}{3}\frac{\theta_4'[(ka-\omega t)/2]}{\theta_4[(ka-\omega t)/2]} + \frac{\bar{\eta}L}{h}, \quad (61)$$

$$Y(a,b,t) = (2\eta_c b/h)cn^2\{K(m)(ka-\omega\tau)/\pi;m\} - b\bar{\eta}/h, \quad (62)$$

where θ_4 is the Jacobi theta function.[32] Here the elliptic function modulus m is related to the Ursell number of the wave train by

$$mK^2(m) = (3\pi^2/2K^2h^3)\eta_c.$$

The dispersion relation has the form

$$\omega = c_0k - \beta k^3 + c_0k(-\eta_c/h + (k^2h^2/6\pi^2)\{\pi^2 - 4K(m)[3E(m) - 2K(m)]\}),$$

where $k = 2\pi/L$; the constant $\bar{\eta}$ is given by

$$\bar{\eta} = 2\eta_c + (4k^2h^3/3\pi^2)K(m)[E(m) - K(m)].$$

Advantages of the approximate theoretical formulation given by (59) and (60), over the exact solutions (61) and (62), are (i) the simplicity of these equations compared to the elliptic function solutions to KdV (61) and (62) and (ii) the fact that the complex envelope function is an exact solution of the NLS equation (which is exactly integrable in terms of the inverse scattering transform). Figure 1 shows comparisons of the $Y(a,t)$ motions of an unmodulated wave train for the exact (62) and approximate (60) equations, while in Fig. 2 the $X(a,b,t)$ motion for both exact (61) and approximate (59) forms is shown. The simple expressions for the orthogonal decompositions of the $X(a,t)$ and $Y(a,b,t)$ particle motions given by (59) and (60), together with an *integrable* equation [NLS (40)] for the envelope function provide a simple, nonlinear theory for particle motions beneath a modulated wave train. The agreement between the motions associated with a small-amplitude unmodulated (asymptotically summed) carrier wave, (59) and (60), and the exact unmodulated carrier, (61) and (62), is rather remarkable in our opinion.

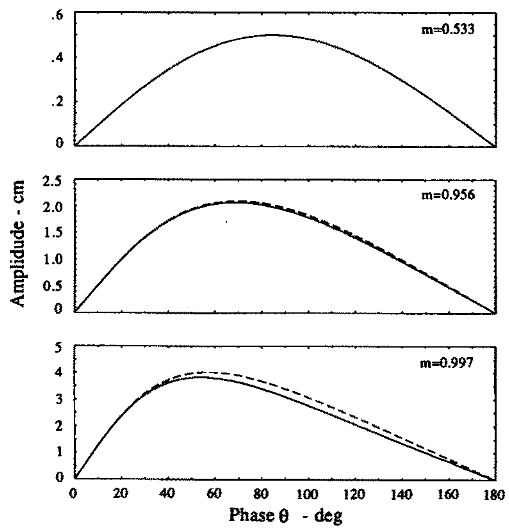

FIG. 2. Comparison of the approximate horizontal particle motion $X(a,b,t)$ (59) (solid line) with the cnoidal wave elliptic function solution (61) (dotted line) of the potential KdV equation. The depth is $h = 10$ cm, wavelength $L = 100$ cm, $a = 0$ cm, and $b = 10$ cm.

C. Radiation stress

We now note the connection between the modulated Stokes field (58) and the associated particle motions (59) and (60) and radiation stress.[33] We consider the "running average" as given by

$$\langle\eta(a,t)\rangle = \frac{1}{L}\int_a^{a+L}\eta(a',t)da'. \quad (63)$$

Applying the latter to (58) integrates out the rapidly oscillating part [i.e., the term ( ) in (58)] to give

$$\langle\eta(a,t)\rangle = -3[A^2(a,t) - \overline{A^2}]/4k_0^2h^3. \quad (64)$$

This is the slow, mean free surface elevation associated with radiation stress, as first determined by Longuet-Higgins and Stewart[33] in Eulerian coordinates for shallow water. Subsequent application of (63) to (59) and (60) gives the contribution of radiation stress to the particle trajectories

$$\langle X(a,t)\rangle = \frac{3}{4k_0^2h^4}\int_a [A^2(a,t) - \overline{A^2}]da, \quad (65)$$

$$\langle Y(a,t)\rangle = -3b\,[A^2(a,t) - \overline{A^2}]/4k_0^2h^4. \quad (66)$$

The limits on the integral in (65) are $(-\infty,a)$ on the infinite line and $(0,a)$ on the periodic interval. The expression (64) for radiation stress may be interpreted as slow variations in the mean sea level that are proportional to the squared envelope $A^2(a,t)$ of a slowly modulated wave field. Equations (65) and (66) give the influence of these slow variations on the particle trajectories.

D. Phase speeds and phase locking

As a result of the fact that $\eta(a,t)$ is a modulated Stokes wave, one finds that the wave phase speed at the carrier frequency and at the second and higher harmonic are "phase locked." This arises because of the definition of phase speed,

$$c = \frac{\omega}{k} = -\frac{\partial\theta/\partial t}{\partial\theta/\partial a} = -\frac{\partial 2\theta/\partial t}{\partial 2\theta/\partial a} = \cdots. \quad (67)$$

Equations (58)–(60) [see also the Fourier series in (56)] imply that the Fourier spectra of the Stokes field and associated particle motions have peaks at the first harmonics, the second and higher harmonics, and at low frequency. These results demonstrate that the Lagrangian KdV equation contains phase locking to the approximation considered here and that the associated description in terms of LNLS also predicts this behavior.

V. EXAMPLE WAVE MOTIONS

We now give examples of particle motions described by the Lagrangian NLS equation. We select for the envelope a particular periodic solution to the NLS equation, the snoidal wave, which is a single nonlinear Fourier component of the NLS spectrum. To arrive at this form of the envelope we set

$$\psi(a,t) = A(a,t)e^{i\omega t}, \tag{68}$$

where $A(a,t)$ is a real field, so that (40) gives

$$A_t + C_g A_a = 0, \tag{69}$$

$$\omega A - \mu A_{aa} - \nu A^3 = 0, \tag{70}$$

the solution of the latter is in terms of the sn elliptic function

$$A(a,t) = A_0 \,\mathrm{sn}[k(a - C_g t);m], \tag{71}$$

where m is the modulus and

$$k = (-\nu/2\mu)^{1/2}(A_0/\sqrt{m}), \tag{72}$$

$$\omega = (\nu A_0^2/2)[(1+m)/m]. \tag{73}$$

Note that when the modulus of the snoidal wave is small, $m \ll 1$ and the envelope is approximated by a sine wave; when the modulus approaches one, $m \sim 1$ and the envelope approaches a tanh function. The first example we choose is an unmodulated small-amplitude sine wave; we use formulas (59) and (60) to graph a single period of the particle motions under the wave (Fig. 3). The maximum amplitudes of

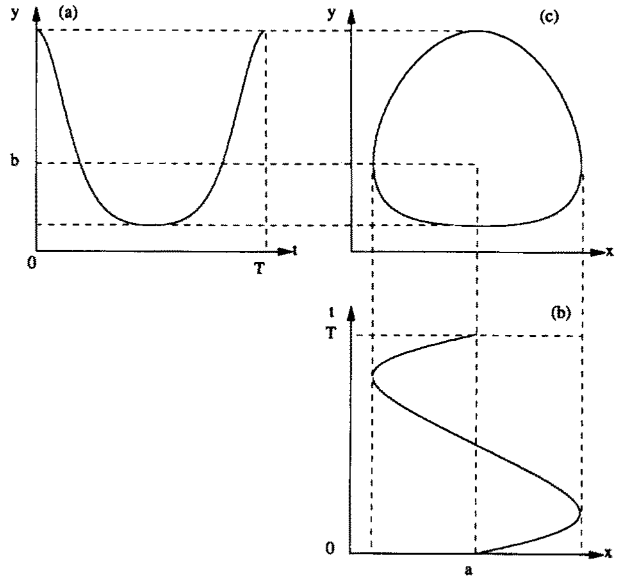

FIG. 4. Orthogonal decomposition of the nonlinear particle motion during the passage of a small-amplitude cnoidal wave with Ursell number 0.19, amplitude $a_0 = 20.0$ cm, wavelength $L = 10.0$ m in depth $h = 1.0$ m. The (a) vertical motion combines with the (b) horizontal motion to give an obviously (c) nonelliptic particle orbit. The relative horizontal and vertical particle maximum amplitudes are normalized to unity; consequently deviations from a circular orbit are due to nonlinear effects.

the particle motions have been normalized to one; the unnormalized particle motion is elliptical rather than circular, as in Fig. 3(c). With this normalization, nonlinearity is viewed as a deformation from the circular shape; on this basis the

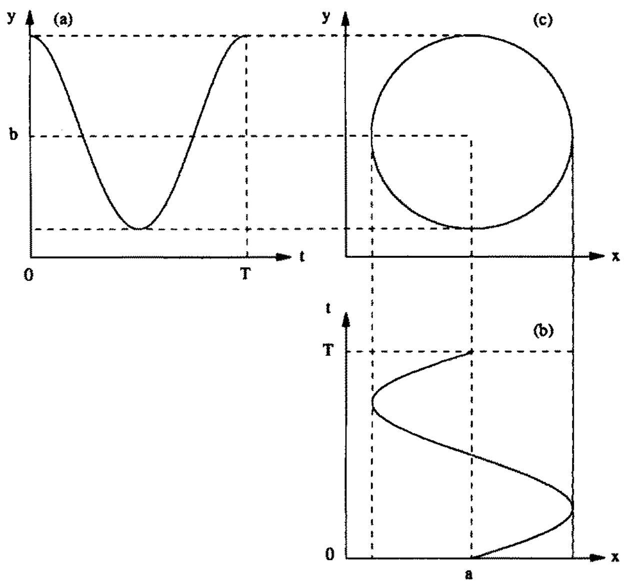

FIG. 3. Orthogonal decomposition of the nonlinear particle motion during the passage of a small-amplitude cnoidal wave with Ursell number 0.019, amplitude $a_0 = 2.0$ cm, wavelength $L = 10.0$ m in depth $h = 1.0$ m. The motion is nearly a cosine in the (a) vertical and a sine in the (b) horizontal. These motions combine to give a nearly elliptical particle orbit, displayed here as a circle resulting from normalization of the relative particle maximum amplitudes to 1(c).

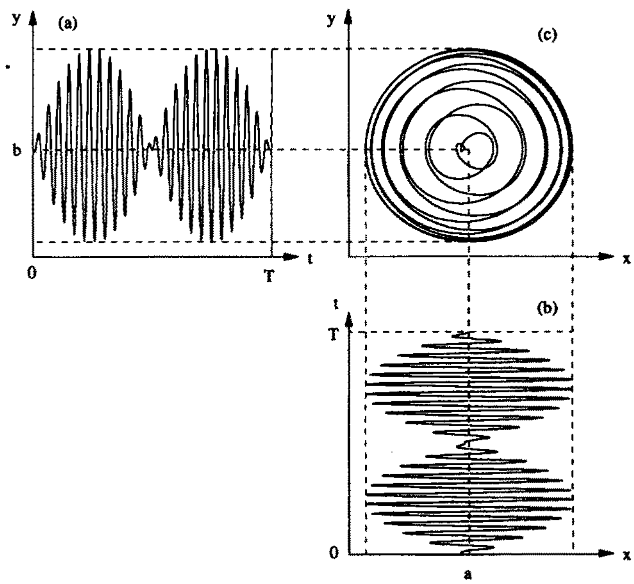

FIG. 5. Orthogonal decomposition of the nonlinear particle motion during the passage of a modulated, small-amplitude cnoidal wave *in the absence of radiation stress*: the amplitude $a_0 = 1.0$ cm, wavelength $L = 5.0$ m in depth $h = 0.5$ m. The modulation envelope is a snoidal wave of modulus $m = 0.49$ and wavelength 100 m. The (a) vertical motion combines with the (b) horizontal motion to give the (c) particle orbit. The particle orbit shown is that over the first half-cycle of a single modulation period.

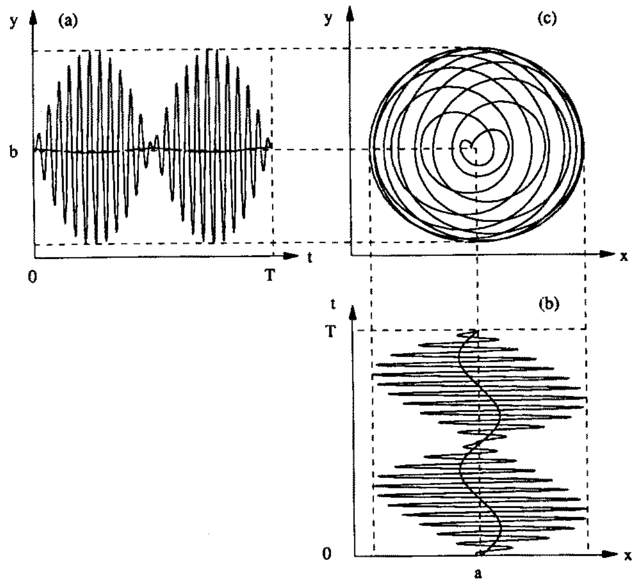

FIG. 6. The orthogonal decomposition of the nonlinear particle motion during the passage of the modulated, small-amplitude cnoidal wave of Fig. 5 *in the presence of radiation stress*: the amplitude $a_0 = 1.0$ cm, wavelength $L = 5.0$ m in depth $h = 0.5$ m. The modulation envelope is a snoidal wave of modulus $m = 0.49$ and wavelength 100 m. The (a) vertical motion combines with the (b) horizontal motion to give (c) particle orbit. The radiation stress contributions to the particle motions are the low-frequency, small-amplitude waves shown beneath the rapidly oscillating wave trains in (a) and (b). The particle orbit shown is that over the first half-cycle of a single modulation period.

motion in Fig. 3 is seen to be essentially linear. The second example (Fig. 4) is a small-amplitude cnoidal wave that is considerably more nonlinear than that of Fig. 3. Note the deviations of the particle motions from sinusoidal; furthermore, the particle orbit is quite distorted from circular, indicating that nonlinear effects are clearly present in ths example.

We now consider a case where the carrier is modulated by a snoidal wave of modulus $m = 0.49$ (Fig. 5). We first show the results for the particle motions when radiation stress is excluded from the calculations. The time series for horizontal and particle motions are quite similar. The particle orbit, circular near the maximum in the modulation envelope, spirals cleanly to the center as the modulation falls to zero between two consecutive packets. In Fig. 6 we repeat the results of Fig. 5 but we now include the effects of radiation stress. In Fig. 6(a) the total particle motion includes the contribution resulting from radiation stress, here shown as a long, small-amplitude, low-frequency variation beneath the rapidly oscillating wave train. For the horizontal motions [Fig. 6(b)] radiation stress is seen to be quite larger than that for the vertical motions and the result is a quite skewed horizontal motion compared to the previous case without radiation stress [Fig. 5(b)]. Comparing Figs. 5 and 6 it is clear that there are substantial differences in the motion, particularly in the horizontal component, when the effects of radiation stress are included. Another effect, which we found quite surprising, is that the orientation of the particle ellipses [Fig. 6(c)], as viewed locally under each wave, are *not* horizontal. This occurs because of the nonlinearity in the radiation stress contribution to the motion. This is seen by comparing Figs. 5(c) (no radiation stress) and 6(c) (radiation stress included). The average orientation of the particle ellipses over a modulation period is, however, zero.

We now consider examples of particle motions that are quite nonlinear; the snoidal wave modulation envelope has modulus $m = 0.98$ (Fig. 7), where we first exclude the effects of radiation stress. Note that the modulation envelopes of the particle motions are distinctly nonsinusoidal. Note also that a greater number of waves have amplitudes near the maximum modulation relative to the linear case of Fig. 5. This effect is seen in the particle orbit where the motion is observed to stay a substantial amount of the time near the maximum amplitude. In Fig. 8 we include the effects of radiation stress on the particle motions. The radiation stress contribution is shown in Figs. 8(a) and 8(b) as long, small-amplitude particle motions. Note that the presence of radiation stress skews the horizontal motions substantially [Fig. 8(b)]. The particle orbit that includes radiation stress [Fig. 8(c)] differs substantially from that in the absence of radiation stress [Fig. 7(c)]. Inclinations of the particle ellipses to the horizontal are seen to be greater in this case than in the example of Fig. 6. Increasing the nonlinearity increases the local orbital inclination.

VI. EXPERIMENTAL CONSIDERATIONS

It is now appropriate to discuss LNLS as a tool for the time series analysis of data. The approach consists of application of the inverse scattering transform, a method that furnishes robust mathematical machinery for the nonlinear

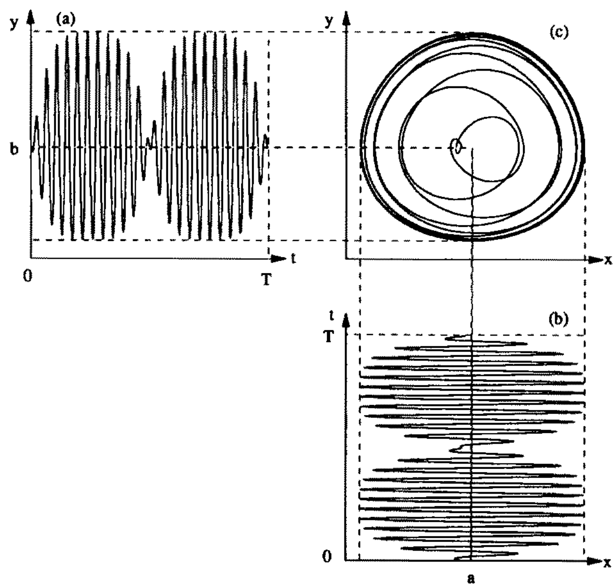

FIG. 7. The orthogonal decomposition of the nonlinear particle motion during the passage of a modulated, small-amplitude cnoidal wave *in the absence of radiation stress*: the amplitude $a_0 = 2.8$ cm, wavelength $L = 5.0$ m in depth $h = 0.5$ m. The modulation envelope is a snoidal wave of modulus $m = 0.98$ and wavelength 100 m. The (a) vertical motion combines with the (b) horizontal motion to give the (c) orbit particle. The particle orbit shown is that over the first half-cycle of a single modulation period.

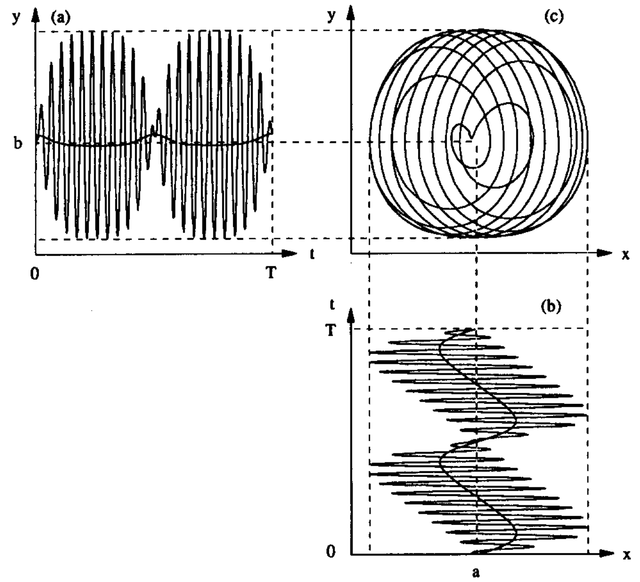

FIG. 8. The orthogonal decomposition of the nonlinear particle motion during the passage of the modulated, small-amplitude cnoidal wave of Fig. 7 *in the presence of radiation stress*: the amplitude $a_0 = 2.8$ cm, wavelength $L = 5.0$ m in depth $h = 0.5$ m. The modulation envelope is a snoidal wave of modulus $m = 0.98$ and wavelength 100 m. The (a) vertical motion combines with the (b) horizontal motion to give the (c) particle orbit. The radiation stress contributions to the particle motions are the low-frequency, small-amplitude waves shown beneath the rapidly oscillating wave trains in (a) and (b). The particle orbit shown is that over the first half-cycle of a single modulation period.

Fourier analysis of computer or experimentally measured data.[21,22,25–27,34–36] To obtain an appropriate perspective we first discuss how the nonlinear Fourier approach works for Eulerian data, where two basic procedures are available. The first corresponds to the classical *Cauchy problem* in which a wave field is given at some initial time $t = 0$, $\eta(x,0)$, and the subsequent evolution is followed for all time t thereafter. In this approach $\eta(x,0)$ is recorded experimentally as a *space series*. The alternative approach is a *boundary value problem* where the wave field is specified at some fixed values of the space coordinate, $x = 0$, for all time t, $\eta(0,t)$; the field is then evolved over all space.[5,29,37] This latter method has been exploited to nonlinearly Fourier analyze *time series* of numerical, laboratory, and field data[21,22,34–39]

In the study of Lagrangian particle motions one also has two different, but analogous approaches. For the LNLS equation (40) the *Cauchy problem* is well defined: one has the modulation field at $t = 0$, $\psi(a,0)$ and then solves for the field $\psi(a,t)$ for all time t thereafter. For the *boundary value problem* one specifies the temporal motion for some initial particle position, say $a = a_0$, so that $\psi(a_0,t)$ is well defined. The implication is that one experimentally records a time series of the field ψ for some initial particle position a_0, i.e., one "colors" a particle at $a = a_0$ and follows its motion in time thereafter. Unfortunately, given $\psi(a_0,t)$ the solution to (40) is not well posed. LNLS as given by (40), is what we refer to as a "space" evolution equation; this terminology is taken from the fact that there are spatial derivatives in the dispersive term. To alleviate the difficulty that (40) does not solve the boundary value problem associated with $\psi(a_0,t)$ we introduce a "time" evolution equation for LNLS, given by

$$i\frac{\partial\psi}{\partial a} + iC_g'\frac{\partial\psi}{\partial t} + \mu'\frac{\partial^2\psi}{\partial t^2} + \nu'|\psi|^2\psi = 0, \tag{74}$$

where

$$C_g' = 1/C_g, \tag{75}$$

$$\mu' = -\frac{1}{2C_g}\frac{\partial^2\omega_0}{\partial k_0^2} = \frac{c_0h^2k_0}{2C_g}, \tag{76}$$

$$\nu' = 9c_0/14h^4k_0C_g. \tag{77}$$

To obtain (74) we used the approach suggested by Ablowitz and Segur[5] and Karpman[37] and exploited by Osborne *et al.*[29] for the Lagrangian KdV equation. Spatial derivatives are evaluated in the second-order terms using the first-order approximation: $\partial\psi/\partial a \cong -(1/C_g)\partial\psi/\partial t$.

VII. HAMILTONIAN FORMULATION

The Lagrangian nonlinear Schrödinger equation (40) can also be derived from the following Lagrangian density:

$$L = (i/2)(\psi^*\psi_t - \psi_t^*\psi) + (iC_g/2)(\psi^*\psi_a - \psi_a^*\psi) - \mu\psi_a\psi_a^* + (\nu/2)\psi^2\psi^{*2}. \tag{78}$$

To obtain the associated Hamiltonian density it is convenient to change the field $\psi(a,t)$ to two real fields (N,Φ), defined by

$$\psi = \sqrt{N}e^{i\Phi(a,t)}. \tag{79}$$

Then the Lagrangian density becomes

$$L = N\Phi_t + C_gN\Phi_a + \frac{\mu}{4}\frac{N_a^2}{N} + \mu N\Phi_a^2 - \frac{\nu}{2}N^2 \tag{80}$$

and one can see that the field $N(a,t)$ is the momentum conjugate to the coordinate $\Phi(a,t)$; then the Hamiltonian density is given by

$$H = -C_gN\Phi_a - \frac{\mu}{4}\frac{N_a^2}{N} - \mu N\Phi_a^2 + \frac{\nu}{2}N^2 \tag{81}$$

and the equations of motion, in terms of $N(a,t)$ and $\Phi(a,t)$, are

$$\Phi_t = \frac{\partial H}{\partial N} - \frac{\partial}{\partial a}\frac{\partial H}{\partial N_a}, \tag{82}$$

$$N_t = \frac{\partial}{\partial a}\frac{\partial H}{\partial \Phi_a}. \tag{83}$$

The explicit coupled evolution equations are given by

$$N_t + C_gN_a + 2\mu(NV)_a = 0, \tag{84}$$

$$V_t + C_gV_a + 2\mu VV_a - \frac{1}{2}\mu\left(\frac{N_a}{N}\right)^3 + \mu\frac{N_aN_{aa}}{N^2} - \frac{\mu}{2}\frac{N_{aaa}}{N} - \nu N_a = 0, \tag{85}$$

where $V(a,t) = \Phi_a(a,t)$ and $N(a,t) = A^2(a,t)$ [compare (79) with (42); note that $\Phi(a,t) = -\omega't + \phi(a,t)$]. Equations (84) and (85) may also be derived directly from (40) with (79). The orthogonally decomposed particle motions, in terms of solutions to (84) and (85), are given by (59) and (60).

VIII. SUMMARY AND CONCLUSIONS

We have derived the Lagrangian form of the shallow-water nonlinear Schrödinger equation from the LKdV equation. This brings to a total of four the number of nonlinear, Lagrangian wave equations known to be integrable by the inverse scattering transform: the Boussinesq, Korteweg–de Vries, potential Korteweg–de Vries, and nonlinear Schrödinger equations. Orthogonal decomposition of the horizontal and vertical particle motions into independently evolving nonlinear trajectories is a characteristic property of the theory. Radiation stress has been included in a natural way and has profound effects on the particle motions: (i) the horizontal component of the motion of fluid parcels displays a quite skewed modulation envelope for increasing nonlinearity and (ii) the (approximately elliptical) particle orbits are seen to vary their orientation periodically during a single modulation period. Both of these predictions should be experimentally accessible.

As a result of this and previous research,[29] it now appears that for every nonlinear, integrable wave equation in Eulerian coordinates it is conceivable that there is a corresponding wave equation in Lagrangian coordinates that is also integrable by IST. In the results obtained so far there is a simple transformation of coordinates from the Eulerian (x,t) to the Lagrangian (a,t) equations. In general, determination of this transformation evidently requires a detailed multiscale analysis of the primitive Lagrangian equations of motion and of the associated particle trajectories.

It is interesting to consider the range of validity of the present results. Formally speaking we use a multiscale averaging procedure to derive LNLS from LKdV when the wave motion is small in amplitude and the Fourier spectrum is narrow banded. Thus we are considering a subclass of possible solutions to LKdV, which we formulate in terms of LNLS. For wave motion sufficiently small in amplitude, with a narrow-banded spectrum, the two formulations must be equivalent. In a separate paper[24] we conduct numerical simulations and examine the range of Ursell numbers for which the two theories are approximately equal; we find that U must be less than about 0.268.

In the spirit of future work it seems plausible that the Lagrangian formulation given herein can be extended to the three-dimensional (3-D) case in which the motion is governed by the Kadomtsev–Petviashvilli[40] (KP) equation. Here KP is essentially a three-dimensional generalization of the KdV equation. Formally speaking, an extension of the present results might allow the shallow-water limit of the Davey–Stewartson[41] (DS) equations (a three-dimensional generalization of NLS) to be derived directly from KP. This seems entirely plausible, in view of the results of Freeman and Davey,[42] who show that, in Eulerian coordinates, both KP and DS may be derived from the 3-D Euler equations in the limit that the wavenumber $k \to 0$. In the present context it remains to formulate a multiscale averaging procedure that would allow the shallow-water DS equations to be derived directly from KP. We would expect to find the Lagrangian form of the DS equations and we would also anticipate the possibility of asymptotically summing the carrier to improve estimates of the particle motions.

ACKNOWLEDGMENTS

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