Soliton Basis States in Shallow-Water Ocean Surface Waves

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The inverse scattering transform for the periodic Korteweg–de Vries equation is used to analyze surface-wave data obtained in the Adriatic Sea and a robust soliton spectrum is found. While the solitons are not observable in the data due to the presence of energetic radiation modes, a new nonlinear filtering technique renders the solitons visible. Numerical simulations support the existence of solitons in the measurements and suggest that, as a wave train propagates into shallow water, the solitons grow at the expense of the radiation.

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Zabusky and Kruskal [1] discovered the soliton in numerical simulations of the *periodic* Korteweg–de Vries (KdV) equation. The subsequent theoretical derivation of the *spectral solution* to KdV for *infinite-line boundary conditions* [2] hallmarked the development of a new mathematical method, the *inverse scattering transform* (IST), which solves certain nonlinear "integrable" (soliton) wave equations for selected boundary conditions [3]; IST may be viewed as a kind of *nonlinear Fourier analysis.* Nonlinear integrable systems are known in many fields including hydrodynamics, solid-state physics, optics, general relativity, molecular chemistry, and plasma physics [3–5].

The IST for KdV with *periodic or quasiperiodic boundary conditions* was discovered by Dubrovin, Matveev, and Novikov [6]. Periodic IST allows for the spectral decomposition of nonlinear wave motion into a *linear superposition of the nonlinearly interacting (hyperelliptic) oscillation modes or basis states* for KdV [6–10] and may be viewed as a *nonlinear generalization of Fourier series.* The goal of this paper is to use periodic-quasiperiodic IST to nonlinearly Fourier analyze ocean-wave data.

A major difficulty arises in the experimental search for solitons in physical systems and in the interpretation of experimental data, particularly when periodic or quasiperiodic boundary conditions are appropriate. It may not be possible to observe solitons in real space because there is no "asymptotic state" ($t\rightarrow\infty$), as with infinite-line boundary conditions, for which the solitons evolve into well-separated, rank-ordered pulses. In periodic systems, especially when (a) the soliton spatial-temporal density or energy is high or when (b) the radiation components obscure the solitons, no individual solitons are observable in configuration space [4,10]. It would be naive to conclude, however, that solitons are not present, or that their dynamics are not important. We view the nonlinear Fourier-analysis approach presented herein as providing a way to determine soliton behavior in complex systems of this type.

IST contains *all* the solutions to the periodic-quasiperiodic KdV equation. Wave forms such as sine waves, Stokes waves, solitary waves, cnoidal waves, narrow- and broad-banded wave trains, plus many other possibilities, together with nonlinear interactions, are automatically included. IST is essentially a kind of spectral analysis that determines which nonlinear modes are active in a system. Herein we analyze a time series *to allow the data to describe which particular nonlinear oscillation modes dominate the measured wave dynamics.*

We assume that shallow-water surface waves (of small but finite amplitude and little directional spreading) may, to second order in nonlinearity and dispersion, be described approximately by the (spacelike) KdV equation [11,12]:

$$\eta_t+c_0\eta_x+\alpha\eta\eta_x+\beta\eta_{xxx}=0\,, \tag{1}$$

where $\eta(x,t)$ is the wave amplitude as a function of space x and time t, $c_0=(gh)^{1/2}$, $\alpha=3c_0/2h$, and $\beta=c_0h^2/6$; (1) has the linearized dispersion relation $\omega=c_0k-\beta k^3$, g is the acceleration of gravity, c_0 is the linear phase speed, and h is the water depth. Subscripts with respect to x and t refer to partial derivatives. KdV solves the Cauchy problem: Given the wave train at $t=0$, $\eta(x,0)$, (1) determines the motion for all time thereafter, $\eta(x,t)$.

Data recorded as a function of time at a single spatial location imply the need to determine the scattering transform of a *time series,* $\eta(0,t)$. Hence we employ the *timelike* KdV equation (TKdV) [3]:

$$\eta_x+c_0'\eta_t+\alpha'\eta\eta_t+\beta'\eta_{ttt}=0\,, \tag{2}$$

where $c_0'=1/c_0$, $\alpha'=-\alpha/c_0^2$, and $\beta'=-\beta/c_0^4$; (2) has the linearized dispersion relation $k=\omega/c_0+\beta\omega^3/c_0^4$. TKdV solves a *boundary-value problem:* Given the temporal evolution $\eta(0,t)$ at some fixed spatial location $x=0$, (2) determines the wave motion over all space $\eta(x,t)$. We assume either periodic $[\eta(x,t)=\eta(x,t+T)]$ or quasiperiodic boundary conditions [there exists a $T(\varepsilon)$ such that $|\eta(x,t+T)-\eta(x,t)|<\varepsilon$ for all t].

We now describe the IST of a periodic, broad-spectrum wave train $\eta(0,t)$ (simple modifications are required for quasiperiodic boundary conditions [9]). The IST *spectrum* is determined from the one-dimensional Schrödinger eigenvalue problem [6]:

$$\psi_{tt}+[\lambda\eta(0,t)+\Omega^2]\psi=0\,, \tag{3}$$

where $\lambda=\alpha c_0^2/6\beta$ and Ω is a complex frequency, with $\Omega^2=E$ real. One solves (3) to obtain the nonlinear Fourier spectrum of a measured time series. Bloch eigenfunctions solutions of (3) are periodic or antiperiodic on $0<t<T$. The trace $\Delta(E)$ of the monodromy matrix $\mathbf{M}$ [which maps solutions of (3) from t to $t+T$] is the Floquet discriminant, $\Delta(E)=\frac{1}{2}\,\mathrm{Tr}\mathbf{M}$. The solutions of $\Delta(E)=\pm 1$ determine the discrete eigenvalues E_j $(1\le j\le 2N+1)$ which are the *main spectrum* of the motion. Two adjacent eigenvalues define an *open band* (E_{2j}, E_{2j+1}) when $|\Delta(E)|>1$; when $E_{2j}=E_{2j+1}$ the band is *degenerate*. The *auxiliary spectrum* consists of hyperelliptic functions $\mu_j(x,t)$ which oscillate between the two eigenvalues of an open band according to nonlinear ordinary differential equations given elsewhere [6,8,9]. The width of an open band is the amplitude of an hyperelliptic oscillation mode, e.g., a "single degree of freedom," "spectral component," or "basis state" of KdV with amplitudes $a_j(f_j)=|\mu_j|=(E_{2j+1}-E_{2j})/2\lambda$ and associated frequencies $f_j=j/T$ $(f_j=\Omega_j/\pi)$ [9,10]. A *linear superposition* of these basis states is the solution to the KdV equation:

$$\lambda\eta(x,t)=-E_1+\sum_{j=1}^{N}[2\mu_j(x,t)-E_{2j}-E_{2j+1}]\,. \tag{4}$$

In the absence of interactions among nonlinear spectral components, the $\mu_j(x,t)$ degenerate to elliptic functions. For small-amplitude wave motion the μ_j reduce to sine waves and (4) becomes an ordinary Fourier series [7-9]. Linear Fourier analysis is the linear superposition of noninteracting sinusoidal waves; nonlinear Fourier analysis for KdV (4) is the linear superposition of the nonlinearly interacting, hyperelliptic oscillation modes.

The periodic KdV spectrum is often separable into soliton and radiation components [7,10]. Solitons are found on the extreme left of the Floquet discriminant where we compute the index $I_n=(E_{2n+1}-E_{2n})/(E_{2n+1}-E_{2n-1})$ for all n, $1\le n\le N$; I_n decreases from 1 as n increases and the value of $n=N$ at which $I_n\sim 0.99$ is used to estimate the *reference level*, $E_{\mathrm{ref}}=E_{2n+1}$, upon which the solitons propagate in configuration space [7]. The interval $1\ge I_n\ge I_{\mathrm{ref}}$ defines the soliton spectrum and $I_{\mathrm{ref}}>I_n$ defines the radiation. The N soliton amplitudes are then given by $\eta_n=2(E_{\mathrm{ref}}-E_{2n})/\lambda$ $(1\le n\le N)$ for $E\le E_{\mathrm{ref}}$. The radiation components consist of the amplitudes $a_j(f_j)$ for $E>E_{\mathrm{ref}}$.

We call the determination of the KdV spectrum from (3) [e.g., the soliton amplitudes η_n, and radiation modes $a_j(f_j)$] the *direct* scattering transform. We call the determination of the $\mu_j(x,t)$ and the solution to KdV by the linear superposition formula (4) the *inverse* scattering transform. Computer algorithms are given elsewhere [9,10,13,14].

Consistent with the assumptions implicit in the derivation of the KdV equation [11] we have selected a 500-point time series from the Adriatic Sea measurement program [15] [Fig. 1(a)] for which the wave amplitude is small but finite with respect to the depth ($\sim$16.5 m) and for which there is little directional spreading (only 2% of the energy in the particle velocity spectrum is transverse to the dominant wave direction). We first remove the mean of the time series. This fixes the mean level in the absence of waves as being equal to the water depth; the reference level E_{ref} upon which the solitons propagate then lies below the mean level. To estimate the degree of nonlinearity we have computed the (timelike) Ursell number, $\mathrm{Ur}=3gH_sT_d^2/4h^2$; H_s is the significant wave height (average of the highest one-third waves) and T_d is the dominant period (corresponding to the largest linear Fourier component). For the time series of Fig. 1(a), $\mathrm{Ur}\sim 8$. This may be compared to the following somewhat arbitrary classification: $\mathrm{Ur}\le 1$ describes linear motion, $1<\mathrm{Ur}\le 10$ is moderately nonlinear, and $\mathrm{Ur}\ge 10$ is strongly nonlinear. The linear Fourier spectrum is shown in Fig. 1(b). The *Floquet discriminant* of the data is given in Fig. 1(c); the radiation spectrum of the time series is rather large and there are nine soliton components. We graph the *nonlinear spectrum* as a function of the frequency f $(=\Omega/\pi)$ such that the radiation lies to the right of the reference level f_{ref} $[=(E_{\mathrm{ref}})^{1/2}/\pi]$, and the solitons to the left of f_{ref} in Fig. 1(d). The vertical arrows denote the nine solitons in the spectrum; the length of the arrows corresponds to the soliton amplitudes. A magnified view of the *soliton* part of the Floquet discriminant is given in Fig. 1(e). The zero crossings to the left of the vertical dashed line (E_{ref}) correspond to the soliton eigenvalues. Note that E_{ref} lies to the right of the mean level in spectral space [Fig. 1(e)] and below the mean level in configuration space [Fig. 1(f)], thus ensuring a zero mean for (4). In Figs. 1(b) and 1(d) we show the "KdV cutoff frequency" $f_{\mathrm{KdV}}\sim c_0/2\pi h$; for spectral components to the left of f_{KdV} the second-order term in the dispersion relation, relative to the first, is small (<0.2) and KdV is a good approximation to the wave motion. This cutoff is represented in Fig. 1(c) by $E_{\mathrm{KdV}}=(\pi f_{\mathrm{KdV}})^2$.

Finally we use IST to *filter* the radiation spectrum from the measured time series so that *only* the soliton spectrum is represented in real space [16]; this corresponds to summing (4) over only the soliton components of the spectrum [9]. The resultant filtered wave train is a long-period, low-amplitude wave field consisting of nine interacting solitons [Fig. 1(f)]; the solitons are also shown in Fig. 1(a) at the same amplitude scale as the data. Each of the maxima in Fig. 1(f) corresponds to a single soliton interacting with its nearest neighbors. It is clear that the solitons significantly contribute to the energetics

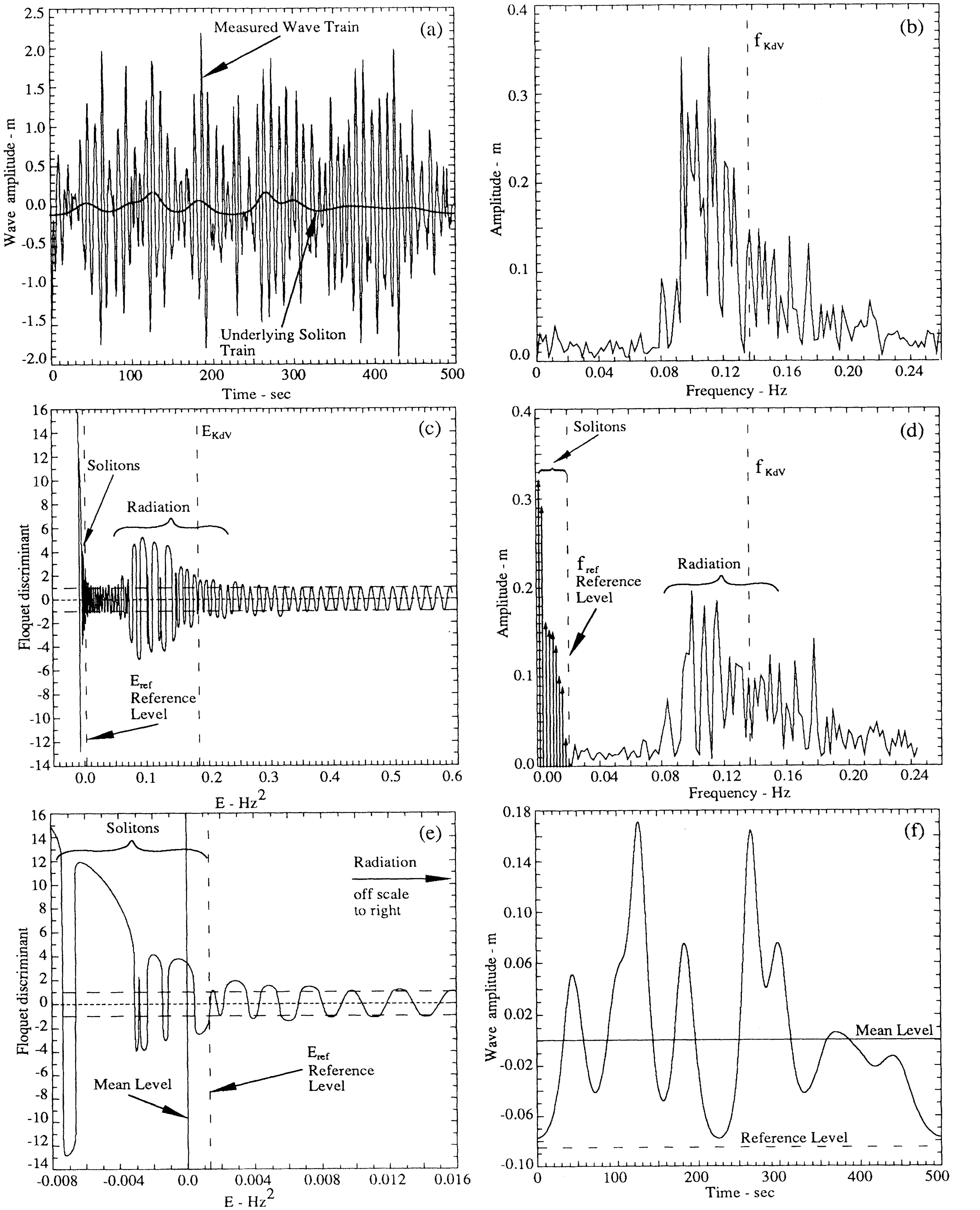

FIG. 1. (a) Measured times series with $H_s = 3.0$ m, $T_d = 9.1$ sec, and $h = 16.5$ m. (b) Fourier spectrum of (a). (c) Floquet discriminant of (a); the soliton and radiation parts of spectrum are well separated. (d) Scattering transform of (a). (e) Floquet discriminant in the *soliton part* of the spectrum of (a). (f) Soliton train obtained by filtering out the radiation spectrum and inverting the scattering transform of (d); solitons are also graphed in (a) at the same scale as the data.

of the surface wave field.

Are the observed solitons "spurious" in some sense, or perhaps an artifact of the data processing procedure? In order to investigate these possibilities we have conducted, in a more extensive analysis [16], both laboratory experiments and numerical simulations of a (spectrally broadbanded) Boussinesq model [17], using the numerically integrated KdV equation as a control. We find that solitons also occur in these systems (for both narrow- and broadbanded wave trains) and they behave qualitatively in the same way as those discovered in the Adriatic Sea. This suggests that the solitons observed herein are real physical phenomena which play a natural role in shallow-water wave dynamics.

An interesting result of our data analysis is that the solitons tend to be found beneath the maxima in the envelope of the wave train, e.g., they often lie under packets [Fig. 1(a)], but are rarely found between packets. In our field, laboratory, and numerical simulations [16] the majority of the observed solitons obey this rule. This suggests a possible phase-locking mechanism in which the solitons often remain beneath and are uniquely associated with a particular packet as it propagates shoreward. Our numerical simulations, together with IST, indicate that the solitons begin forming far offshore (they are infinitesimally small here) and then slowly change their form adiabatically [18] while growing at the expense of the radiation as the wave train propagates into shallow water. For sufficiently small bottom slope the soliton fission process is not evident; for large bottom slope the solitons may actually fission. Miles [19] has predicted the formation of solitons in shorewardly propagating wave trains, and our experimental results agree with many of his conclusions; future work may provide more quantitative comparisons.

The presence of solitons in the data contrasts with the traditionally held beliefs (1) that nonlinear wave trains of this type can be fully described perturbatively to second order (including the spectrum and the bispectrum) and (2) that the major nonlinear statistical effect is the non-Gaussian behavior of the amplitude probability density function. Note that we did not *a priori* need to include solitons as a descriptor of the wave motion. They have been found to occur naturally within the nonlinear spectral structure of the KdV equation.

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