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Stokes drift for inertial particles transported by water waves

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Abstract – We study the effect of surface gravity waves on the motion of inertial particles in an incompressible fluid. We perform analytical calculations based on perturbation expansions which allow us to predict the dynamics of inertial particles in the deep-water regime. We find that the presence of inertia leads to a non-negligible correction to the well-known horizontal Stokes drift velocity. Moreover, we find that the vertical sedimentation velocity is also affected by a drift induced by waves. The latter result may have some relevant consequences on the rate of sedimentation of particles of finite size. We underline that the vertical drift would also be observed in the (hypothetical) absence of the gravitational force. Kinematic numerical simulations are performed and the results are found to be in excellent agreement with the analytical predictions, even for values of the parameters beyond the perturbative limit.

Introduction. – The study of the Stokes drift velocity is a problem of paramount importance both from a fundamental point of view [1] and in connection with applications, especially in the area of sediment transport [2–6]. As far as the first point is concerned, the Stokes drift velocity is, for instance, responsible of important fluid-mixing processes, including the mass and momentum transport near the free surface and the vertical-mixing enhancement owing to turbulent kinetic-energy production [7]. In the ocean, the Stokes drift is thought to be an important factor responsible for the Langmuir circulation [8]. In relation to applications, it is known that an accurate evaluation of the Stokes drift velocity is important for the correct representation of surface physics in ocean general circulation models and ocean models at smaller scales. Other relevant effects on the ocean circulation are discussed, *e.g.*, by [3].

Since the seminal paper [9], Stokes drift has been recognized as an important example that illustrates the difference between the Eulerian and the Lagrangian statistics [10]. It predicts that a fluid particle (*i.e.*, a tracer of negligible inertia) experiences a mean drift in the direction of wave propagation proportional to U^2/c, where U is the amplitude of the wave-induced velocity and c is the wave phase velocity. Since the Stokes drift arises from the average of the wave motion along a Lagrangian trajectory, it is relevant for all floating and suspended particles present in the water column, and not only for fluid particles considered in the original derivation. Inertia of finite-size particles with density different from the fluid modifies Lagrangian trajectories with respect to those of fluid particles. This has important consequences on particle dispersion in both laminar and turbulent flows [11–20], and therefore we expect that inertia might affect the Stokes drift experienced by inertial particles. Previous studies in the field have investigated the case of particles close to be neutrally buoyant in a velocity field generated by internal gravity waves [21] and small particles in deep water in the presence of surface gravity waves [22].

Our main aim here is to push forward the analysis performed by these previous studies and to investigate the role of inertia on the Stokes drift velocity for particles of arbitrary density in deep-water waves. As a result of our analysis, we show that inertia induces a correction to the horizontal Stokes drift velocity which is a second-order effect in particle inertia, and generates a vertical drift (at the first order in inertia) which modifies the sedimentation velocity in still fluid. We show that this vertical drift has a dynamical origin as it is active even in the absence of gravity, a remarkable result not pointed out in previous studies. The analytical results carried out by means of perturbative expansions are corroborated by a set of numerical simulations which extend the range of validity of our results beyond the perturbative regime.

Analytical study of inertial-particle motion. – The Stokes drift velocity is a second-order effect in the wave amplitude. Therefore, in order to be consistent in the perturbative expansion, one has to consider at least a second-order expansion for the wave motion, *i.e.*, a Stokes wave. We will consider the limit of deep water, $kh \to \infty$, for which the second-order terms in the velocity field expansion vanish and we have the two-dimensional, irrotational and incompressible velocity field $\boldsymbol{u} = (u, w)$ [23,24],

$$u(x,z,t) = U e^{kz}\cos(kx - \omega t), \qquad (1)$$
$$w(x,z,t) = U e^{kz}\sin(kx - \omega t), \qquad (2)$$

where x and z are the horizontal and the vertical coordinates, k is the wave number and ω is the angular frequency related to k via the dispersion relation, $\omega = \sqrt{gk}$ (strictly speaking, one should include the nonlinear correction to the dispersion relation; however, this turns out to be inessential in our analysis). The phase velocity is $c = \omega/k$, and the maximum velocity U at the surface ($z = 0$) is related to the wave amplitude A by $U = \omega A$. We remark that the limit of deep water, which simplifies the following analysis, is already valid with good approximation for $kh \simeq 2$ [24]. Equations (1), (2) are obtained under the hypothesis of small steepness $\epsilon = kA$, which, because of the relation between U and A, is equivalent to the Froude number defined as $\mathrm{Fr} = U/c$. Note that in the above velocity field the mean flow that should appear at the same order as the second harmonic is not included because we are dealing with a monochromatic wave, and not with wave packets characterized by modulation length (see, *e.g.*, [23], p. 474, for a discussion).

The motion of a small inertial particle transported by the fluid flow $\boldsymbol{u}$ through the Stokes drag and subjected to gravity acceleration $\boldsymbol{g}$ is given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{V}, \qquad (3)$$
$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \frac{\boldsymbol{u}-\boldsymbol{V}}{\tau} + (1-\beta)\boldsymbol{g} + \beta\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}, \qquad (4)$$

where $\boldsymbol{x}(t)$ and $\boldsymbol{V}(t)$ represent the particle position and velocity. In (4) the Stokes response time is $\tau = a^2/(3\beta\nu)$ where a is the particle radius and ν the kinematic viscosity of the fluid, and the added-mass effect has been taken into account via the dimensionless number $\beta = 3\rho_\mathrm{f}/(\rho_\mathrm{f} + 2\rho_\mathrm{p})$, built from the fluid, ρ_f, and particle, ρ_p, densities [25,26].

In order to have an explicit expression for the particle velocity, we expand (4) perturbatively in τ [27], to obtain:

$$\boldsymbol{V} = \boldsymbol{u} + \tau(1-\beta)\left(\boldsymbol{g} - \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right) + \tau^2(1-\beta)\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2} + O(\tau^3). \qquad (5)$$

By introducing the dimensionless variables $\boldsymbol{x} \mapsto k\boldsymbol{x}$, $t \mapsto \omega t$ and $\boldsymbol{u} \mapsto \boldsymbol{u}/U$, we can expand $\boldsymbol{u}$ and its Lagrangian derivatives at the second order in ϵ and, by substitution in (3)–(5), we obtain for the particle motion:

$$\begin{aligned}\dot{x} &= \epsilon\left[u - \mathrm{St}\beta' w - \mathrm{St}^2\beta\beta' u\right] + \epsilon^2\mathrm{St}^2\beta\beta' e^{2z} \\ &\quad + O(\epsilon^3\mathrm{St}^2, \epsilon^2\mathrm{St}^3), \qquad (6)\\ \dot{z} &= -\mathrm{St}\beta' + \epsilon\left[w + \mathrm{St}\beta' u - \mathrm{St}^2\beta\beta' w\right] - \epsilon^2\mathrm{St}\beta' e^{2z} \\ &\quad + O(\epsilon^3\mathrm{St}^2, \epsilon^2\mathrm{St}^3), \qquad (7)\end{aligned}$$

where we have introduced $\beta' \equiv 1 - \beta$ and the Stokes number $\mathrm{St} \equiv \omega\tau$. We observe that the term $-\mathrm{St}\beta'$ represents the sedimentation velocity in still fluid.

By expanding perturbatively the coordinate as $\boldsymbol{x} = \boldsymbol{x}_0 + \epsilon\boldsymbol{x}_1 + \epsilon^2\boldsymbol{x}_2 + \dots$ and inserting it into (6), (7), we obtain a set of equations for the different orders in ϵ. At order ϵ^0 we simply have

$$\begin{aligned} x_0 &= x^*, \\ z_0 &= z^* - \mathrm{St}\beta' t, \end{aligned} \qquad (8)$$

where (x^*, z^*) represents the initial position of the tracer. At order ϵ, after integration and taking up to order St^2, we obtain

$$\begin{aligned} x_1 &= -[1 - \mathrm{St}^2\beta\beta']e^{-\mathrm{St}\beta' t}\left[w^* + \mathrm{St}\beta' u^*\right] \\ &\quad - \mathrm{St}\beta' e^{-\mathrm{St}\beta' t}\left[u^* - \mathrm{St}\beta' w^*\right], \qquad (9)\\ z_1 &= [1 - \mathrm{St}^2\beta\beta']e^{-\mathrm{St}\beta' t}\left[u^* - \mathrm{St}\beta' w^*\right] \\ &\quad - \mathrm{St}\beta' e^{-\mathrm{St}\beta' t}\left[w^* + \mathrm{St}\beta' u^*\right], \qquad (10)\end{aligned}$$

where $\boldsymbol{u}^* = \boldsymbol{u}(\boldsymbol{x}^*, t)$. As in the original derivation for fluid particles, no drift velocity appears at the first order, and we need to go to the next order ϵ^2. By substituting (8)–(10) into (6), (7) and finally going back to the original dimensional variables we obtain

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= U e^{kz_0(t)}\left[(1 - \mathrm{St}^2\beta\beta')\cos\phi^*(t) - \mathrm{St}\beta'\sin\phi^*(t)\right] \\ &\quad + \frac{U^2}{c}e^{2kz_0(t)}\left[1 - \mathrm{St}^2\beta\beta'\right], \qquad (11)\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= U e^{kz_0(t)}\left[(1 - \mathrm{St}^2\beta\beta')\sin\phi^*(t) + \mathrm{St}\beta'\cos\phi^*(t)\right] \\ &\quad - c\mathrm{St}\beta'\left[1 + 2\frac{U^2}{c^2}e^{2kz_0(t)}\right], \qquad (12)\end{aligned}$$

where $\phi^*(t) \equiv kx^* - \omega t$ is the Eulerian phase at the initial position and $z_0(t) \equiv z^* - \mathrm{St}\beta' t$. Now we assume that the bare sedimentation velocity $\mathrm{St}\beta'$ is small with respect to the wave motion, which is consistent with the expansion leading to (5), so that we can take e^{kz_0} constant over a wave period. Under this assumption, the drift velocities are simply given by the last terms in (11), (12), which do not vanish when averaged over one period

$$u_\mathrm{d} = \frac{U^2}{c}\left[1 - \beta(1-\beta)\mathrm{St}^2\right]e^{2k[z^* - (1-\beta)g\tau t]}, \qquad (13)$$
$$w_\mathrm{d} = -(1-\beta)g\tau - 2(1-\beta)\mathrm{St}\frac{U^2}{c}e^{2k[z^* - (1-\beta)g\tau t]}. \qquad (14)$$

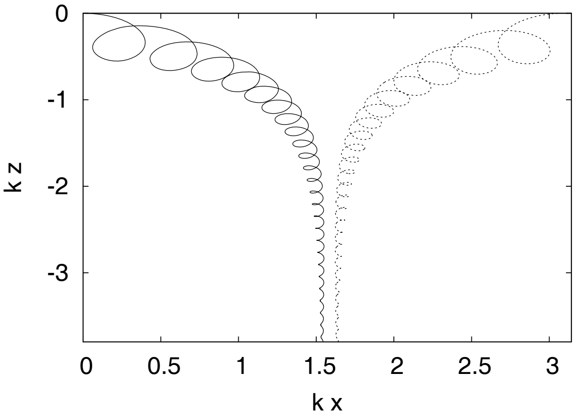

Fig. 1: Two examples of trajectories of slightly heavy ($\beta = 0.99$, continuous line) and light ($\beta = 1.01$, dotted line) particles obtained by numerical integration of (3), (4) in the velocity field (1), (2) with $\epsilon = \mathrm{Fr} = 0.33$ and $\mathrm{St} = 0.5$. The initial position for particles are $x^* = 0$, $z^* = 0$ (heavy) and $x^* = 0.13$, $z^* = -0.3$ (light). Wave parameters: $A = 0.026$, $k = 4\pi$, $g = 9.8$.

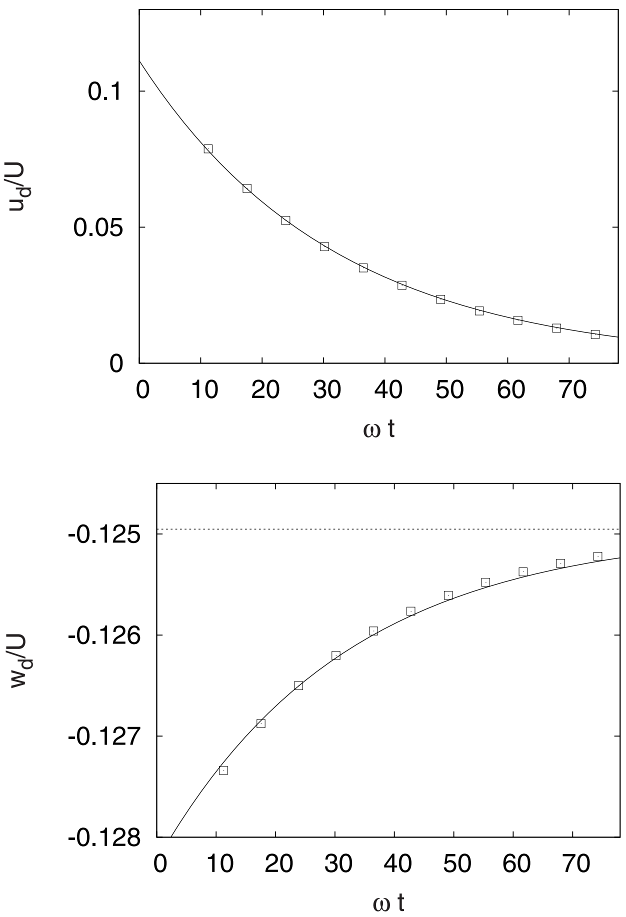

Fig. 2: Numerical (squares) *vs.* theoretical (solid line, eqs. (13), (14)) drift velocity: horizontal (upper panel) and vertical (lower panel) components. The dashed line represents the settling velocity in the absence of wave motion $w_\mathrm{d} = -(1-\beta)g\tau$. Parameters: $\epsilon = \mathrm{Fr} = 0.125$, $\mathrm{St} = 0.157$, $\beta = 0.9$, $A = 0.02$, $k = 2\pi$.

In the limit of tracers, $\mathrm{St} = 0$, and/or neutrally-buoyant particles, $\beta = 1$, the above expressions recover the velocities derived by Stokes: $u_\mathrm{d} = e^{2kz_0}U^2/c$, $w_\mathrm{d} = 0$. Inertia induces a correction of order St^2 to this horizontal drift velocity. The interesting result is that inertia produces a drift velocity also in the vertical direction, which corrects the bare sedimentation velocity. The correction has the same sign of the velocity induced by gravity, *i.e.*, negative (positive) for heavy (light) particles. This vertical mean motion is produced by the combined effect of vertical symmetry breaking due to the z-dependence of the velocity field and of the delayed dynamics induced by inertia. To our knowledge, the correction to sedimentation velocity induced by water waves on inertial particles has never been discussed before.

It is interesting to observe that the combination of inertia and wave motion induces a vertical velocity also in the unrealistic case of $g = 0$. Of course, we can consider the case $g = 0$ because we are considering the kinematic problem of particle motion in a given velocity field, and not the generation of the velocity itself. For this reason in this case we will consider k and ω independent. This limit is of course irrelevant for gravity waves (because $g = 0$ would trivially imply no wave motion) but could find applications in other physical contexts, *e.g.* capillary waves. By repeating the calculation with $g = 0$ we obtain

$$u_\mathrm{d} = \frac{U^2}{c}\left[1-(1-\beta)\mathrm{St}^2\right]e^{2kz^*}, \qquad (15)$$

$$w_\mathrm{d} = -(1-\beta)\mathrm{St}\frac{U^2}{c}e^{2kz^*}. \qquad (16)$$

A simple, and physically relevant, prediction one can derive from (13), (14) is the net displacement of particles from the point of release. For heavy particles ($\beta < 1$) released at the surface ($z_0 = 0$), a time integration of (13) from 0 to ∞ gives the following total displacement:

$$\Delta x = \frac{\mathrm{Fr}^2}{2k}\frac{1-\beta(1-\beta)\mathrm{St}^2}{(1-\beta)\mathrm{St}}. \qquad (17)$$

Of course this expression can be valid only for small St, for which we have $\Delta x > 0$.

Numerical simulations. – In this section we report the numerical results obtained from the integration of (3) and (4), with the aim of verifying the analytical predictions of the previous Section, and also to check the robustness of these predictions with respect to the expansion parameters.

Figure 1 shows two typical examples of trajectories of slightly heavy and light particles induced by linear waves in deep water. From these trajectories the drift velocity is obtained by computing the horizontal and vertical displacement of the position over one Lagrangian period

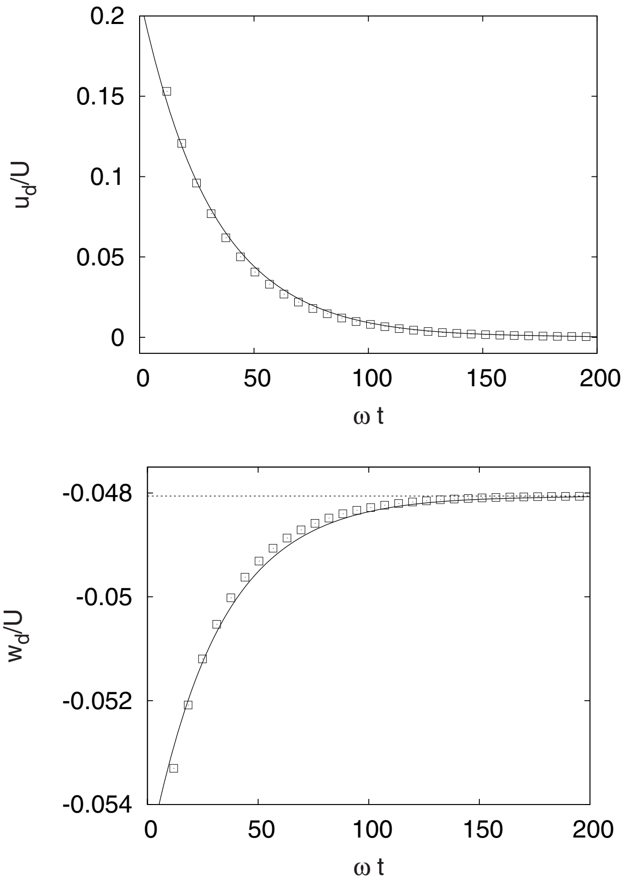

Fig. 3: Numerical (squares) *vs.* theoretical (solid line) drift velocity: horizontal (upper panel) and vertical (lower panel) components. The dashed line represents the settling velocity in the absence of wave motion $w_{\rm d} = -(1-\beta)g\tau$. Parameters: $\epsilon = {\rm Fr} = 0.33$, ${\rm St} = 0.157$, $\beta = 0.9$, $A = 0.052$, $k = 2\pi$.

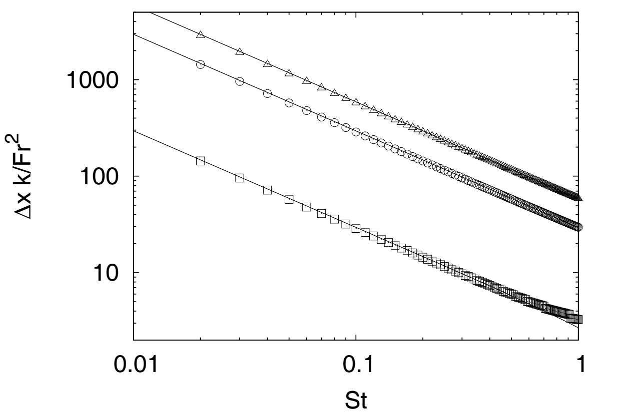

Fig. 4: Total horizontal displacement of heavy particles with $\beta = 0.9$ (squares), $\beta = 0.99$ (circles) and $\beta = 0.995$ (triangles), settling beneath a linear wave with $\epsilon = {\rm Fr} = 0.33$, as a function of St. Solid lines represent theoretical predictions (17).

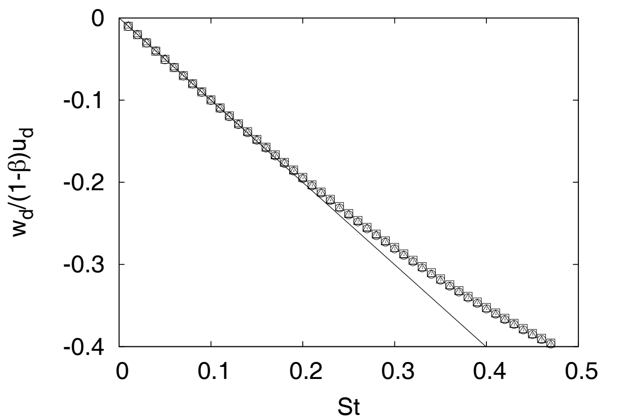

Fig. 5: Ratio of vertical to horizontal drift velocity $w_{\rm d}/(1-\beta)u_{\rm d}$ as a function of St for heavy particles in the absence of gravity ($g = 0$). Parameters: $\epsilon = 0.33$, $\beta = 0.9$ (squares), $\beta = 0.99$ (circles), $\beta = 0.995$ (triangles). The solid line represents the prediction $w_{\rm d}/(1-\beta)u_{\rm d} = -{\rm St}/[1-(2-\beta)(1-\beta){\rm St}^2]$.

(defined from the point at which the Lagrangian horizontal velocity changes from negative to positive) divided by the period. An example of the resulting velocity components is shown in fig. 2 for a wave with $\epsilon = 0.125$. For this case at moderate steepness the agreement with the theoretical prediction (13) and (14) is very good. From the plot of the vertical velocity $w_{\rm d}$ we see that the relative correction induced by waves to the bare sedimentation velocity at the initial time is $2{\rm Fr}^2 \approx 0.03$.

For steeper waves, as the example shown in fig. 3 at $\epsilon = 0.33$, the agreement between numerical simulations and perturbative predictions worsens, nonetheless (13) and (14) still give a good approximation of the numerical data, with a correction to the bare sedimentation velocity at initial time of about 0.2. Of course for even larger steepness other effects such as wave breaking, clearly not included in the theory or simulation, can take place.

The total horizontal displacement of heavy particles released at the surface of deep water for different values of the parameters is shown in fig. 4, together with the predictions given by (17). The agreement is very good not only in the perturbative regime of small St in which (17) is derived. Deviations are observable only for ${\rm St} = O(1)$.

As discussed in the previous Section, the perturbative analysis shows that inertial particles have a vertical drift even in the absence of gravity. This drift acts in the same direction of sedimentation as it has the same sign of the gravitational term, independently of β. By comparing (15) and (16), we see that for $g = 0$ the mean motion is along a straight line, with slope given by $w_{\rm d}/u_{\rm d} = -(1-\beta){\rm St}/[1-(2-\beta)(1-\beta){\rm St}^2]$. We observe that in this limit the total displacement Δx of a heavy particle in deep water diverges. Figure 5 shows the slope of the mean motion as a function of St for different values of β. Again, for small and moderate values of the parameter, the agreement with the analytical prediction is very good.

Conclusions. – We have considered the problem of Stokes drift induced by water waves (in the deep-water limit) on small inertial particles with two complementary perspectives. On the one hand, our results give the correction to the horizontal Stokes drift induced by inertia. This correction is found to be second order in the particle Stokes number, with a sign which depends on the particle density relative to water. On the other hand, we also obtain a vertical drift velocity, which therefore represents a correction to the sedimentation velocity induced by wave motion on the surface. This effect, which results to be at the first order in the Stokes number, has never been discussed before and is of possible relevance, *e.g.*, in the field of sediment transport in coastal regions.

We conclude by observing that the present analysis is performed in the ideal world of linear two-dimensional water waves in infinite depth. One can speculate whether our main findings will survive in more complex and realistic situations. Because the results are based on a kinematic model, we conjecture that the corrections induced by finite Stokes number will survive to more complex velocity field, at least at a qualitative level. It would be therefore extremely interesting to study the drift velocity of inertial particles, and its effect on sedimentation, in more realistic simulations of wave motion and in laboratory experiments, where a precise determination of the mean horizontal and falling velocities is possible.

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REFERENCES

[1] Van den Broeck C., *EPL*, **46** (1999) 1.
[2] Longuet-Higgins M., *Philos. Trans. R. Soc. London, Ser. A*, **245** (1953) 535.
[3] Nielsen P., *Coastal Bottom Boundary Layers and Sediment Transport*, Vol. **4** (World Scientific) 1992.
[4] Vittori G. and Blondeaux P., *J. Fluid Mech.*, **314** (1996) 247.
[5] Blondeaux P., Brocchini M. and Vittori G., *Proc. R. Soc. London, Ser. A*, **458** (2002) 2053.
[6] Blondeaux P., Vittori G., Bruschi A., Lalli F. and Pesarino V., *J. Fluid Mech.*, **697** (2012) 115.
[7] Kantha L., Wittmann P., Sclavo M. and Carniel S., *Geophys. Res. Lett.*, **36** (2009) L02605.
[8] McWilliams J., Sullivan P. and Moeng C., *J. Fluid Mech.*, **334** (1997) 1.
[9] Stokes G., *Trans. Cambridge Philos. Soc.*, **8** (1847) 441.
[10] Longuet-Higgins M., *J. Fluid Mech.*, **173** (1986) 9823.
[11] Squires K. and Eaton J., *J. Fluid Mech.*, **226** (1991) 1.
[12] Wang L., Maxey M., Burton T. and Stock D., *Phys. Fluids A*, **4** (1992) 1789.
[13] Falkovich G. and Pumir A., *Phys. Fluids*, **16** (2004) L47.
[14] Boffetta G., De Lillo F. and Gamba A., *Phys. Fluids*, **16** (2004) .
[15] Bec J., Biferale L., Boffetta G., Celani A., Cencini M., Lanotte A., Musacchio S. and Toschi F., *J. Fluid Mech.*, **550** (2006) 349.
[16] Martins Afonso M., *J. Phys. A: Math. Theor.*, **41** (2008) 385501.
[17] Martins Afonso M., Mazzino A. and Olla P., *J. Phys. A: Math. Theor.*, **42** (2009) 275502.
[18] Bec J., Biferale L., Lanotte A., Scagliarini A., Toschi F. *et al.*, *J. Fluid Mech*, **645** (2010) 497.
[19] Martins Afonso M., Mazzino A. and Muratore-Ginanneschi P., *J. Fluid Mech.*, **694** (2012) 426.
[20] Martins Afonso M. and Mazzino A., *Geophys. Astrophys. Fluid Dyn.*, **105** (2011) 553.
[21] Grinshpun S., Redcoborody Y., Kravchuk S., Zadorozhnii V. and Zhdanov V., *Int. J. Multiphase Flow*, **26** (2000) 1305.
[22] Eames I., *J. Phys. Oceanogr.*, **38** (2008) 2846.
[23] Whitham G., *Linear and Nonlinear Waves*, Vol. **42** (John Wiley) 1974.
[24] Kundu P. and Cohen I., *Fluid Mechanics* (Academic Press) 1990.
[25] Maxey M. and Riley J., *Phys. Fluids*, **26** (1983) 883.
[26] Gatignol R., *J. Méc. Théor. Appl.*, **1** (1983) 143.
[27] Balkovsky E., Falkovich G. and Fouxon A., *Phys. Rev. Lett.*, **86** (2001) 2790.