Appendice C

Espressione di operatori differenziali in diverse coordinate

Ricordiamo le espressioni degli operatori differenziali in un sistema di coordinate cartesiane $(x,\, y,\, z)$.

$$\nabla f \;=\; \vec{i}\frac{\partial}{\partial x} \;+\; \vec{j}\frac{\partial}{\partial y} \;+\; \vec{k}\frac{\partial}{\partial z} \tag{C.1}$$

$$\nabla \cdot \vec{u} \;=\; \frac{\partial u_x}{\partial x} \;+\; \frac{\partial u_y}{\partial y} \;+\frac{\partial u_z}{\partial z} \tag{C.2}$$

$$\nabla \times \vec{u} \;=\; \vec{i}\left(\frac{\partial u_z}{\partial y} \;-\; \frac{\partial u_y}{\partial z}\right) + \vec{j}\left(\frac{\partial u_x}{\partial z} \;-\; \frac{\partial u_z}{\partial x}\right) + \vec{k}\left(\frac{\partial u_y}{\partial x} \;-\; \frac{\partial u_x}{\partial y}\right) \tag{C.3}$$

$$\nabla^2 f \;=\; \nabla \cdot \nabla f \;=\; \frac{\partial^2 f}{\partial x^2} \;+\; \frac{\partial^2 f}{\partial y^2} \;+\; \frac{\partial^2 f}{\partial z^2} \tag{C.4}$$

$$\nabla^2 \vec{u} \;=\; \nabla^2 u_x\, \vec{i} \;+\; \nabla^2 u_y\, \vec{j} \;+\; \nabla^2 u_z\, \vec{k} \tag{C.5}$$

Capita assai comunemente che la simmetria di un particolare problema suggerisca l'uso di un sistema di coordinate curvilinee diverso da quello cartesiano. Per questo motivo riportiamo di seguito le espressioni degli operatori differenziali suindicati in coordinate polari sferiche e cilindriche.

In coordinate polari sferiche $(r,\, \theta,\, \varphi)$, con versori $\vec{i}_r$, $\vec{i}_\theta$ e $\vec{i}_\varphi$:

$$\nabla f \;=\; \vec{i}_r\frac{\partial f}{\partial r} \;+\; \vec{i}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} \;+\; \vec{i}_\varphi\frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi} \tag{C.6}$$

$$\begin{array}{lll} x = r\ \sin\theta\ \cos\varphi & & r = (x^2 + y^2 + z^2)^{1/2} \\ y = r\ \sin\theta\ \sin\varphi & \rightarrow & \theta = arccos\frac{z}{\sqrt{(x^2 + y^2 + z^2)}} \\ z = r\ \cos\theta & & \varphi = arctg\ y/x \end{array}$$

$$\nabla \cdot \vec{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u_\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} \tag{C.7}$$

$$\begin{aligned} \nabla \times \vec{u} = & \vec{i}_r \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\ u_\varphi) - \frac{\partial u_\theta}{\partial\varphi}\right] + \\ & \vec{i}_\theta \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial}{\partial r}(r\ u_\varphi)\right] + \\ & \vec{i}_\varphi \frac{1}{r}\left[\frac{\partial}{\partial r}(r\ u_\theta) - \frac{\partial u_r}{\partial\theta}\right] \end{aligned} \tag{C.8}$$

$$\begin{aligned} \nabla^2 f & = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} \\ & = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} \end{aligned} \tag{C.9}$$

$$\begin{aligned} \nabla^2\vec{u} = & \vec{i}_r\left[\nabla^2 u_r - \frac{2}{r^2}u_r - \frac{2}{r^2\sin^2\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\ u_\theta) + \frac{\partial u_\varphi}{\partial\varphi}\right)\right] + \\ & \vec{i}_\theta\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right] + \\ & \vec{i}_\varphi\left[\nabla^2 u_\varphi - \frac{u_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi}\right] \end{aligned} \tag{C.10}$$

In coordinate cilindriche $(r,\ \varphi,\ z)$, con versori $\vec{i}_r$, $\vec{i}_\varphi$ e $\vec{i}_z$:

$$\begin{array}{lll} x = r\ \cos\varphi & & r = (x^2 + y^2)^{1/2} \\ y = r\ \sin\varphi & \rightarrow & \varphi = arctg\ y/x \\ z = z & & z = z \end{array}$$

$$\nabla f = \vec{i}_r\frac{\partial f}{\partial r} + \vec{i}_\varphi\frac{1}{r}\frac{\partial f}{\partial\varphi} + \vec{i}_z\frac{\partial f}{\partial z} \tag{C.11}$$

$$\nabla \cdot \vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(r\ u_r) + \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{\partial u_z}{\partial z} \tag{C.12}$$

$$\nabla \times \vec{u} = \vec{i}_r \left(\frac{1}{r} \frac{\partial u_z}{\partial \varphi} - \frac{\partial u_\varphi}{\partial z} \right) + \vec{i}_\varphi \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) + \vec{i}_z \frac{1}{r} \left(\frac{\partial}{\partial r}(r \ u_\varphi) - \frac{\partial u_r}{\partial \varphi} \right) \tag{C.13}$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \tag{C.14}$$

$$\nabla^2 \vec{u} = \vec{i}_r \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\varphi}{\partial \varphi} \right) + \vec{i}_\varphi \left(\nabla^2 u_\varphi - \frac{u_\varphi}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \varphi} \right) + \vec{i}_z \nabla^2 u_z \tag{C.15}$$