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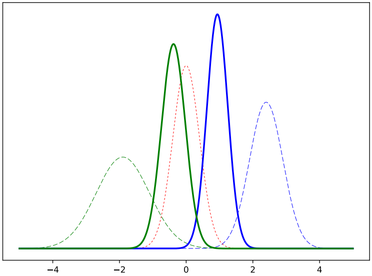
Horizon 2020
European Union funding
for Research & Innovation

Towards model-independent constraints on neutrino properties from cosmology

Including several concepts of Bayesian statistics

RWTH Aachen, Informal Cosmology Seminar, 11/02/2019



−4
−2
0
2
4

Probability

[Trotta, arxiv:0803.4089]

What is probability?

a frequency

"the number of times the event occurs over the total number of trials, in the limit of an infinite series of equiprobable repetitions"

another subtle point:
"randomness" of the trial series

what is really "random"?

do we properly know the initial state (and do not cheat)?

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Advantages:

- recovers frequentist on the long run;
- can be applied when frequentist cannot;
- no need to assume a distribution of possible data;
- deals effortlessly with nuisance parameters (*marginalization*);
- relies on *prior information*.

Bayes' theorem

how to deal with **Bayesian probability**?

given hypothesis H, data d, some information I (true):

$p(\theta)$

Posterior probability:

what we know after

Bayes theorem:

$$p(H|d,I) = \frac{p(d|H,I)\, p(H|I)}{p(d|I)}$$

$\pi(\theta)$

Prior probability:

what we knew before

Likelihood: $\mathcal{L}(\theta)$

sampling distribution of data, given that H is true

Marginal likelihood:

or "Bayesian evidence",

$$p(d|I) \equiv \sum_H p(d|H,I)\, p(H|I)$$

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$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

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parameter inference

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model comparison

(Bayesian) Parameter inference

Parameter inference = what we learn on the parameters, given:

data d → build $\mathcal{L}(\theta)$ from d & $\mathcal{M}_0$

model $\mathcal{M}_0$ → build $\mathcal{L}(\theta)$ from d & $\mathcal{M}_0$

model $\mathcal{M}_0$ → parameters θ (& priors)

(physical ϕ + nuisance ψ)

Full posterior:

$$p(\theta|d,\mathcal{M}_0) \propto \mathcal{L}(\theta) \times \pi(\theta|\mathcal{M}_0)$$

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Marginalize over nuisance to obtain posterior for physical:

$$p(\phi|d,\mathcal{M}_0) \propto \int_{\Omega_\psi} \mathcal{L}(\phi,\psi)\pi(\phi,\psi|\mathcal{M}_0)d\psi$$

marginalize over all the parameters except one (two)

→ 1D (2D) posterior

Credible intervals from the posterior

Credible interval α?

range of values within which an unobserved parameter value falls
with a particular subjective probability α

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Analogous to frequentist confidence intervals α

Bayesian credible interval:

- bounds as fixed;
- estimated parameter as a random variable.

Frequentist confidence interval:

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highest posterior density interval: narrowest interval, includes values of highest probability density

equal-tailed interval: same probability of being below or above the interval

interval for which the mean is the central point

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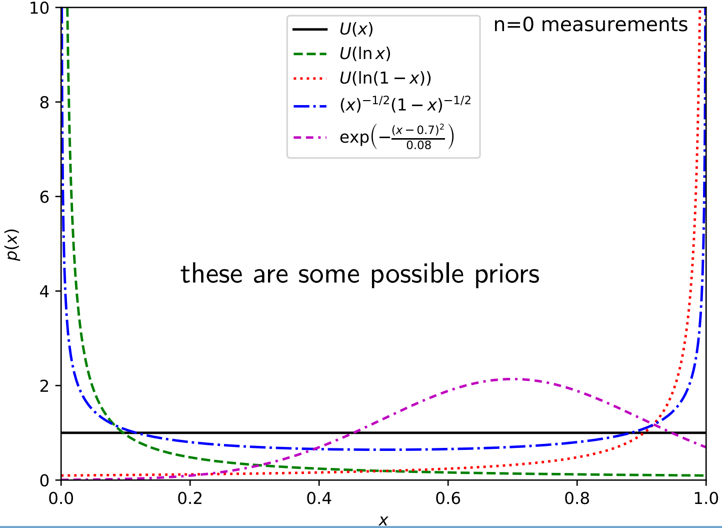
Prior dependence in parameter estimation - I

example: need to measure $0 < x < 1$

likelihood $\mathcal{L}(x) \propto \exp[-(x-0.2)^2/(2 \cdot 0.3^2)]$



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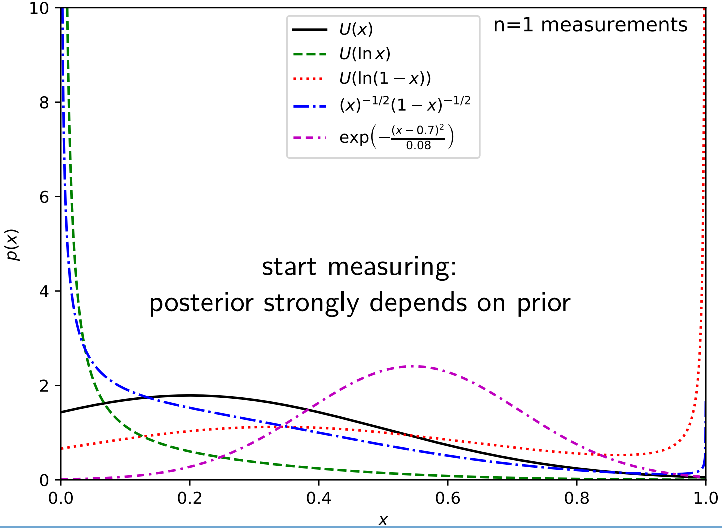

start measuring:
posterior strongly depends on prior

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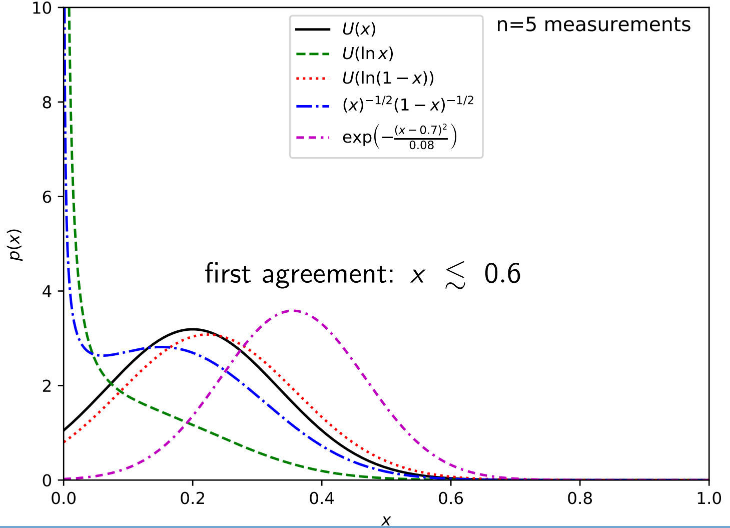

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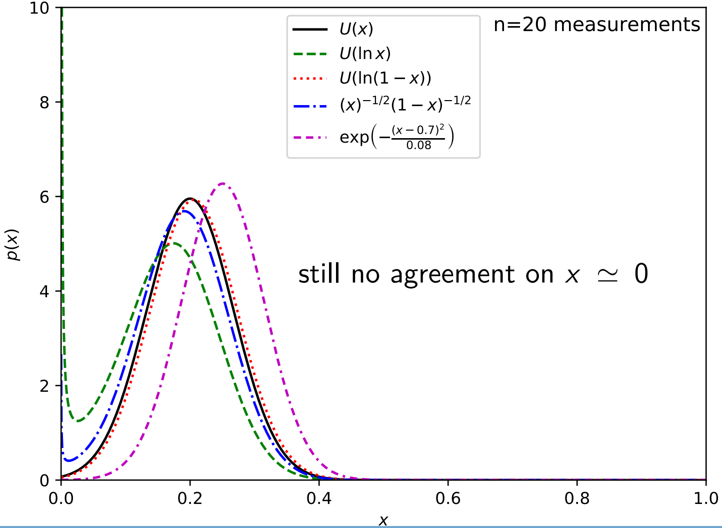

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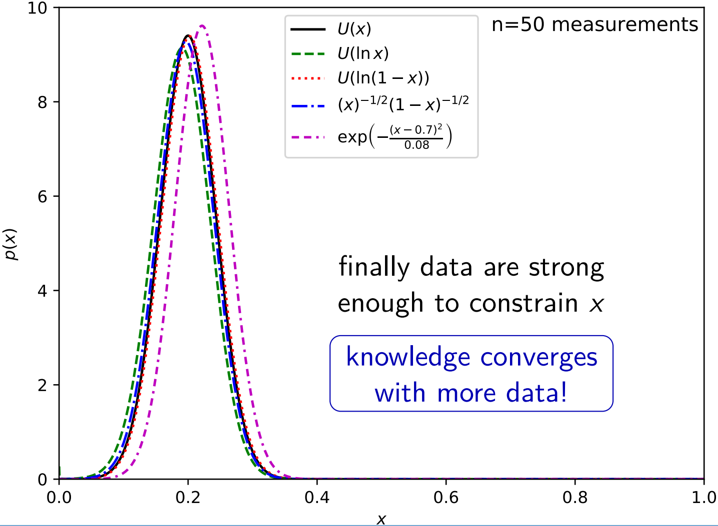


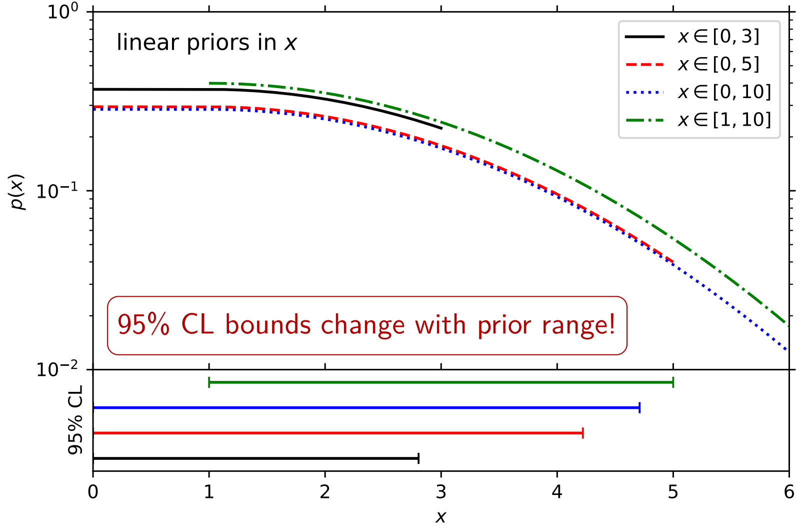
finally data are strong enough to constrain x

knowledge converges with more data!

Prior dependence in parameter estimation - II

other example: need to measure $x > 0$ (Σm_ν?)

likelihood $\mathcal{L}(x) \propto \exp[-(x-1)^2/(2\cdot 1^2)]$ for $x > 1$, constant otherwise



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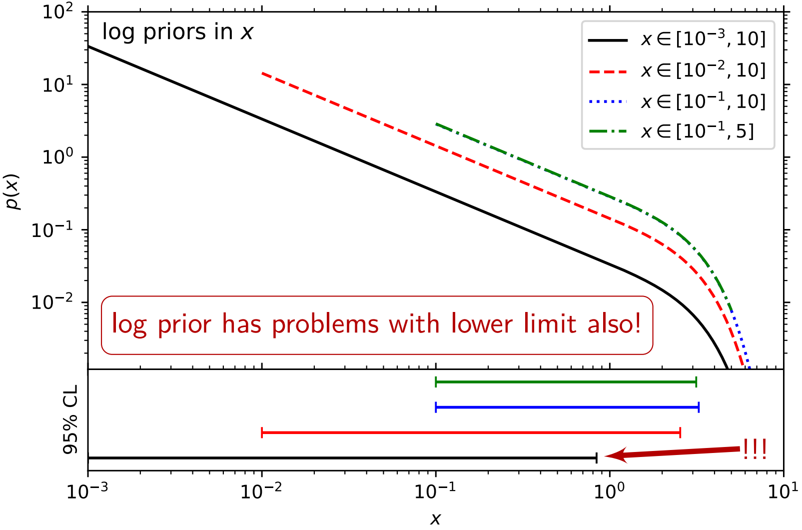

Bayesian evidence

"Bayesian evidence" or "Marginal likelihood"

$$p(d|\mathcal{M}) = Z = \int_{\Omega_\mathcal{M}} \mathcal{L}(d|\theta,\mathcal{M})\,\pi(\theta|\mathcal{M})\,d\theta$$

integrate over all possible (continuous) parameters of model $\mathcal{M}$ (given that $\mathcal{M}$ is true)

What if there are several possible models $\mathcal{M}_i$?

use Z_i to perform bayesian model comparison

Warning: compare models given the same data!

Model posterior:

$$p(\mathcal{M}_i|d) \propto \pi(\mathcal{M}_i)\,Z_i$$

← proportional to constant that depends only on data

given model prior $\pi(\mathcal{M}_i)$

Bayes factor

Posterior odds of $\mathcal{M}_1$ versus $\mathcal{M}_2$:

$$\frac{p(\mathcal{M}_1|d)}{p(\mathcal{M}_2|d)} = B_{1,2}\,\frac{\pi(\mathcal{M}_1)}{\pi(\mathcal{M}_2)}$$

Bayes factor:

$$B_{1,2} = \frac{Z_1}{Z_2} \quad \Rightarrow \quad \ln B_{1,2} = \ln Z_1 - \ln Z_2$$

if priors are the same $[\pi(\mathcal{M}_1) = \pi(\mathcal{M}_2)]$,
$B_{1,2}$ tells which model is preferred:

$B_{1,2} > 1$ ($\ln B_{1,2} > 0$)
$\mathcal{M}_1$ preferred

$B_{1,2} < 1$ ($\ln B_{1,2} < 0$)
$\mathcal{M}_2$ preferred

$\exp(|\ln B_{1,2}|)$ tells the odds in favor of preferred model

Occam's razor

what the Bayesian model comparison tells us?

Best model strikes optimum balance between

Quality of fit → Occam's razor ← Predictivity

the simplest theory that fits data is preferred

model with more parameters → better fit (usually)

→ are all the parameters needed?

Bayes factor penalizes unnecessarily complex models!

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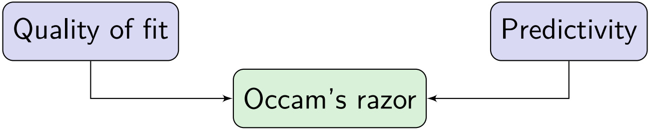

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Bayesian evidence depends on priors!

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Bayes factor DOES NOT penalize models with parameters that are unconstrained by the data

Prior dependence in the Bayesian evidence

Bayes factors depend on priors!

$$\text{likelihood: } \mathcal{L}(x) \propto \begin{cases} 1 & \text{for } x \leq 1 \\ \exp[-(x-1)^2/(2\cdot 1^2)] & \text{for } x > 1 \end{cases}$$

linear prior		log prior	
range	Z	range	Z
$0 \leq x \leq 3$	0.180	$10^{-3} \leq x \leq 10$	0.192
$0 \leq x \leq 5$	0.135	$10^{-2} \leq x \leq 10$	0.172
$0 \leq x \leq 10$	0.070	$10^{-1} \leq x \leq 10$	0.151
$1 \leq x \leq 10$	0.056	$10^{-1} \leq x \leq 5$	0.177

linear prior $x \in [a, b]$ is $\propto 1/(b-a)$

irrelevant for Bayes factor
if the compared models
have the parameter x in common

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towards Lindley's paradox:

use $\pi(x) \propto \exp[-x^2/(2\Sigma^2)]$,
$\mathcal{L}(x) \propto \exp[-(x - N\sigma_t)^2/(2\sigma^2)]$,
with $\sigma_t = \sqrt{\sigma^2 + \Sigma^2}$

$$Z = \exp(-N^2/2) \Big/ \left(\sqrt{2\pi}\,\sigma_t\right)$$

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max evidence for a given likelihood $\mathcal{L}(x)$?

Select a Dirac delta centered on the $\hat{x}$
that gives the maximum of the likelihood

useful estimate of the **max Bayes factor**, in particular for nested models

$\mathcal{M}_1$: free x

$\mathcal{M}_0$: $\mathcal{M}_1 | \; x = x_0$

$$B_{01} = \frac{\mathcal{L}(x_0)}{\int \mathrm{d}x\, \mathcal{L}(x)\, \pi(x)} \geq \frac{\mathcal{L}(x_0)}{\mathcal{L}(\hat{x})} = \frac{\mathcal{L}(x_0)}{\int \mathrm{d}x\, \mathcal{L}(x)\, \delta(x-\hat{x})}$$

maximum likelihood ratio

you will never find a prior that gives a better B_{01} than this!

useful for prior-independent estimates of B_{01}

Jeffreys' scale

[Trotta, arxiv:0803.4089]

odds in favor of the preferred model:

$$\exp(|\ln B_{1,2}|) : 1$$

strength of preference according to Jeffreys' scale:

$\lvert\ln B_{1,2}\rvert$	Odds	$N\sigma$	strength of evidence
< 1.0	$\lesssim 3:1$	< 1.1	inconclusive
$\in [1.0, 2.5]$	$(3-12):1$	$1.1-1.7$	weak
$\in [2.5, 5.0]$	$(12-150):1$	$1.7-2.7$	moderate
$\in [5.0, 10]$	$(150-2.2\times10^4):1$	$2.7-4.1$	strong
$\in [10, 15]$	$(2.2\times10^4-3.3\times10^6):1$	$4.1-5.1$	very strong
> 15	$> 3.3\times10^6:1$	> 5.1	decisive

odds & strength always valid

$N\sigma$ correspondence is valid only given equal model priors
and that only two models are possible
(see e.g. neutrino mass ordering: normal OR inverted)

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Can we extend to more than two (mutually exclusive) models?

How to compute the model posterior

[SG+, PRD 99 (2019) 021301]

Assume N models, equal model prior probabilities:

$\pi_i \equiv \pi(\mathcal{M}_i)$ $\pi_i = \pi_j \quad \forall i, j$ $\sum_i \pi_i = 1 \rightarrow \pi_i = 1/N$

Compute model posterior probabilities:

$p_i \equiv p(\mathcal{M}_i|d)$ $p_i = A\pi_i Z_i$ with A constant $\sum_i p_i = 1$

$$\sum_i^N p_i = A \sum_i^N \pi_i Z_i = 1 \quad \Longrightarrow \quad p_i = \pi_i Z_i \Big/ \sum_j^N \pi_j Z_j = \pi_i \Big/ \sum_j^N \pi_j B_{ji}$$

Selecting a generic $\mathcal{M}_0$ as a reference, we have:

$$p_0 = \left(\sum_i^N B_{i0} \right)^{-1}$$

the sum includes $B_{00} = 1$

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example 1: $N = 2$

$$p_0 = 1/(1 + B_{10})$$
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$p_1 = B_{10}/(1 + B_{10})$

example 2: $N = 8$

assume $B_{i0} \simeq e^{-5}$ $(i \neq 0)$ to get

$p_0 = 1/(1 + \sum_{i \neq 0} B_{i0}) \simeq 0.955$

strong? no, only 2σ!

Model posterior with many models

$$p_i = Z_i \bigg/ \sum_j^N Z_j = B_{i0} \bigg/ \sum_j^N B_{j0}$$

Do the result depend on N?

Does $p_0 \to 0$ when $N \to \infty$?

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ΛCDM ⟵ this will probably be the favorite one

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ΛCDM

$+1$ parameter

$+r \quad +\Sigma m_\nu \quad +N_{\rm eff} \quad +w \quad +\Omega_k \quad +Y_p \quad +A_{\rm lens} \quad +\ldots$

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+2 parameters

$+\Sigma m_\nu + N_{\rm eff}$ $+N_{\rm eff} + m_s^{\rm eff}$ $+w_0 + w_a$ $+\alpha_s + \beta_s$ $+Y_p + N_{\rm eff}$

$+r + \alpha_s$ $+A_{\rm lens} + \Sigma m_\nu$ $+\alpha_s + N_{\rm eff}$ $+\ldots$

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$+r + \alpha_s$ $+A_{\rm lens} + \Sigma m_\nu$ $+\alpha_s + N_{\rm eff}$ $+\ldots$

+3 parameters (and so on...)

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Do the result depend on N?

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in principle one should consider all the possible models,
starting from the simplest one under consideration
(e.g. ΛCDM in cosmology) and then extending it

ΛCDM

$+1$ parameter

$+r$ $\quad +\Sigma m_\nu$ $\quad +N_{\rm eff}$ $\quad +w$ $\quad +\ldots$

$+2$ parameters

$+\Sigma m_\nu + N_{\rm eff}$ $\quad +N_{\rm eff} + m_s^{\rm eff}$ $\quad +w_0 + w_a$ $\quad N_{\rm eff}$

$+r + \alpha_s$ $\quad +A_{\rm lens} + \Sigma m_\nu$ $\quad +\alpha_s + N_{\rm eff}$ $\quad +\ldots$

Complexity increases:
more and more
penalized by
Occam's razor

$+3$ parameters (and so on...)

Model posterior with many models

$$p_i = Z_i \Big/ \sum_j^N Z_j = B_{i0} \Big/ \sum_j^N B_{j0}$$

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ΛCDM

$+1$ parameter

the number of relevant models is not infinite!

$+\Sigma m_\nu + N_{\rm eff}$ $\quad +N_{\rm eff} + m_s$ $\quad +w_0 + w_a$ $\quad +\alpha_s + \beta_s$

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$+3$ parameters (and so on...)

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$+\sum m_\nu + N_{\rm eff}$ $\quad +N_{\rm eff} + m_s$ $\quad +w_0 + w_a$ $\quad +\alpha_s + \beta_s$ $\quad +r_p + N_{\rm eff}$

but for a complete analysis one should consider
also modified gravity models and so on...

$+3$ pa

The two ways of ~~the Force~~ Bayesianism

prior dependence is intrinsic of Bayesian statistics

two ways to deal with this

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Subjective "dark side"?

- priors depend on the researcher
- state your assumptions and present your results
- results *may* be different
- they will converge with more data

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- mathematics can help to minimize subjectivity
- priors from objective criteria (e.g. maximize information gain)
- *still, dependence on prior ranges may remain* (see later)

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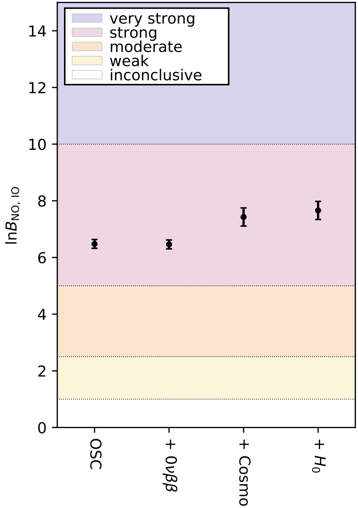
Balance is the way

sensitivity analysis: try different priors+ranges, see if results are stable


very strong
strong
moderate
weak
inconclusive
$\ln B_{\mathrm{NO,\,IO}}$
0
2
4
6
8
10
12
14
OSC
+ $0\nu\beta\beta$
+ Cosmo
+ H_0

Neutrinos and their masses

Normal ordering (NO)

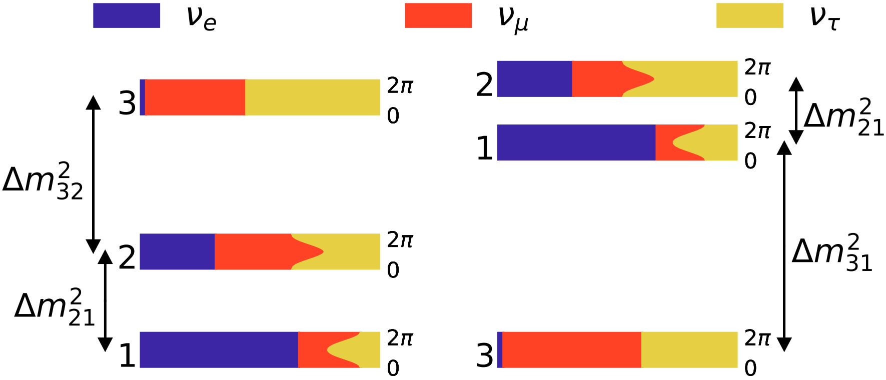
$m_1 < m_2 < m_3$

$\sum m_k \gtrsim 0.06$ eV

Inverted ordering (IO)

$m_3 < m_1 < m_2$

$\sum m_k \gtrsim 0.1$ eV



Absolute scale unknown!

Can we constrain the mass ordering using bounds on $\sum m_\nu$?

Neutrino masses from β decay

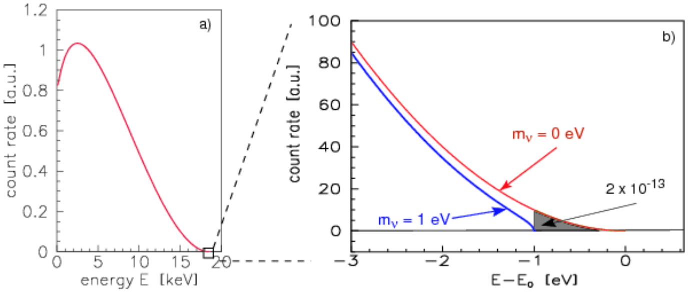
Must measure β decay endpoint → $m^2_{\nu_e} = \sum_k |U_{ek}|^2 m_k^2$

U_{ek} mixing matrix

Mainz/Troitsk limits, $m_{\nu_e} \lesssim 2$ eV

Katrin, (expected) $m_{\nu_e} \lesssim 0.2$ eV

[Katrin L.o.I., 2001]



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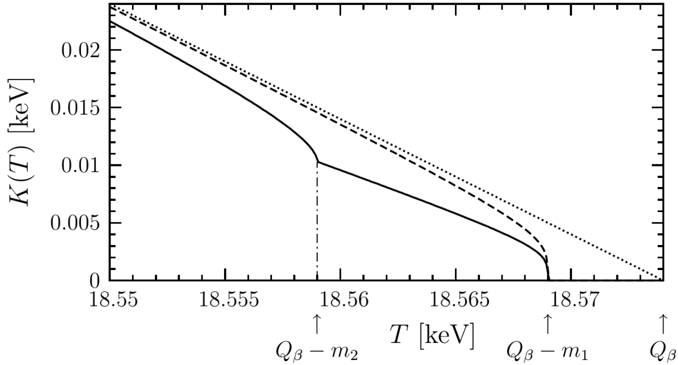
[Giunti&Kim, 2007]



Neutrino masses from neutrinoless double β decay

(if neutrino is Majorana)

[Schechter&Valle, 1982]

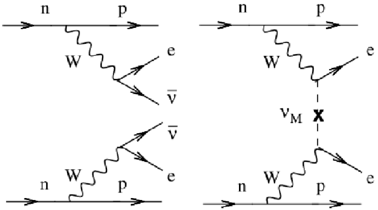
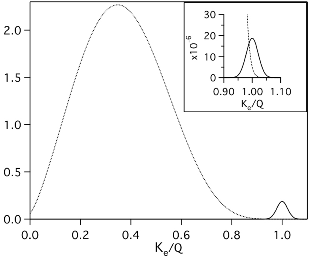
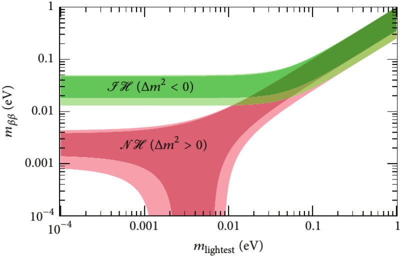


figure from [NEXT] webpage





[Dell'Oro et al., 2016]

Measure $T^{0\nu}_{1/2}$

m_e electron mass,
$G_{0\nu}$ phase space,
$\mathcal{M}'^{\nu}$ matrix element

convert into $m_{\beta\beta} = \dfrac{m_e}{\mathcal{M}'^{\nu}\sqrt{G_{0\nu}\,T^{0\nu}_{1/2}}}$

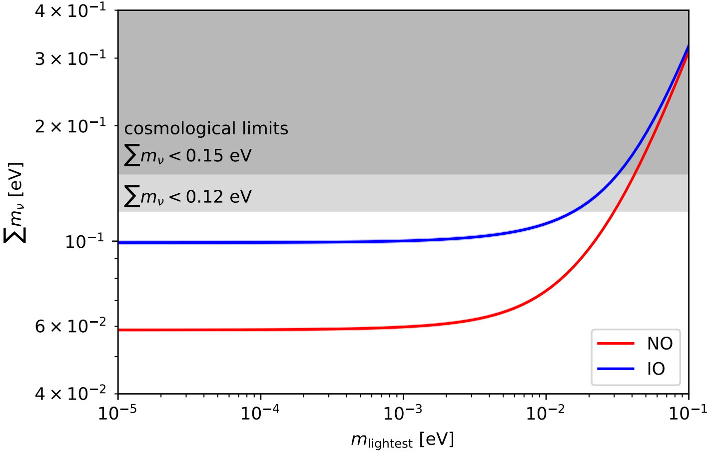
α_k Majorana phases

and then use $m_{\beta\beta} = \left|\sum_k e^{i\alpha_k}\, U^2_{ek}\, m_k\right|$

From cosmology...

Warning: model dependent content!

How the limit change when considering extensions of the ΛCDM model?



Warning: $\sum m_\nu \lesssim 0.1$ eV at 95% CL
does not mean IO disfavored at 95% CL!

Can current data tell us the neutrino mass ordering?

1. [Hannestad, Schwetz, 2016]: extremely weak (2:1, 3:2) preference for NO (cosmology + [Bergstrom et al., 2015] neutrino oscillation fit) Bayesian approach;
2. [Gerbino et al, 2016]: extremely weak (up to 3:2) preference for NO (cosmology only), Bayesian approach;
3. [Simpson et al., 2017]: strong preference for NO (cosmological limits on $\sum m_\nu$ + constraints on Δm^2_{21} and $|\Delta m^2_{31}|$) Bayesian approach;
4. [Schwetz et al., 2017], "Comment on ..."[Simpson et al., 2017]: effect of prior?
5. [Capozzi et al., 2017]: 2σ preference for NO (cosmology + [Capozzi et al., 2016, updated 2017] neutrino oscillation fit) frequentist approach;
6. [Caldwell et al., 2017] very mild indication for NO (cosmology + neutrinoless double-beta decay + [Esteban et al., 2016] readapted oscillation results) Bayesian approach;
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Parameterizing neutrino masses

[SG+, JCAP 03 (2018) 11]

[Simpson et al, 2017]

using m_1, m_2, m_3 (A)

[Caldwell et al, 2017]

using $m_{\rm lightest}, \Delta m^2_{21}, |\Delta m^2_{31}|$ (B)

intuition says: (B) is closer to observable quantities! Better than (A)?

Should we use linear or logarithmic priors on m_k ($m_{\rm lightest}$)?

Can data help to select (A) or (B), linear or log?

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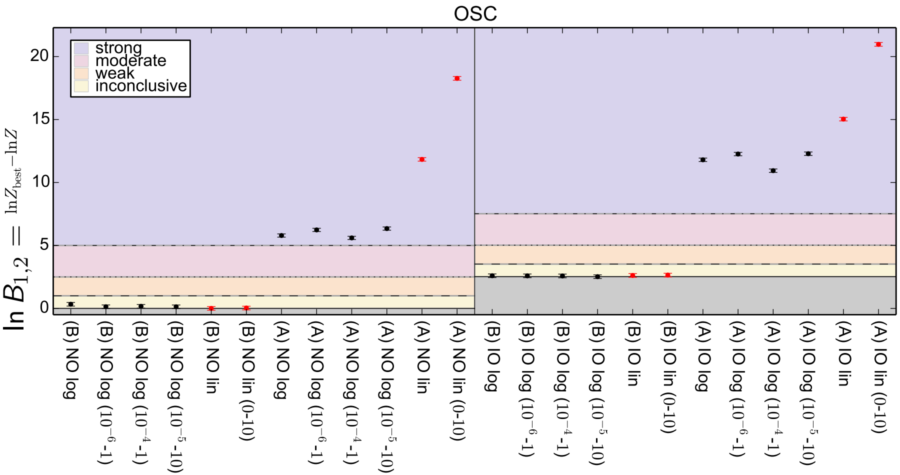
Can data help to select (A) or (B), linear or log?

Case A			Case B		
Parameter	Prior	Range	Parameter	Prior	Range
m_1/eV	linear log	$0 - 1$ $10^{-5} - 1$	$m_{\rm lightest}/\rm eV$	linear log	$0 - 1$ $10^{-5} - 1$
m_2/eV	linear log	$0 - 1$ $10^{-5} - 1$	$\Delta m^2_{21}/\rm eV^2$	linear	$5 \times 10^{-5} - 10^{-4}$
m_3/eV	linear log	$0 - 1$ $10^{-5} - 1$	$\lvert\Delta m^2_{31}\rvert/\rm eV^2$	linear	$1.5 \times 10^{-3} - 3.5 \times 10^{-3}$

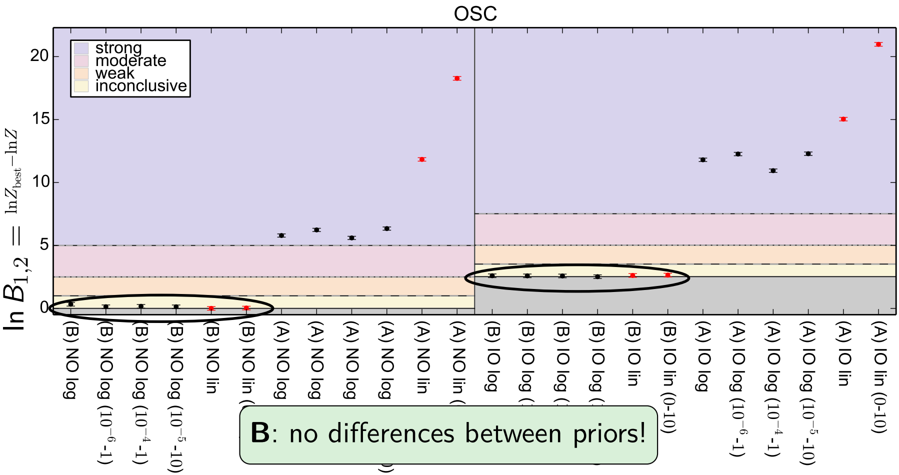
Comparing parameterizations/priors

[SG+, JCAP 03 (2018) 11]



Comparing parameterizations/priors

[SG+, JCAP 03 (2018) 11]

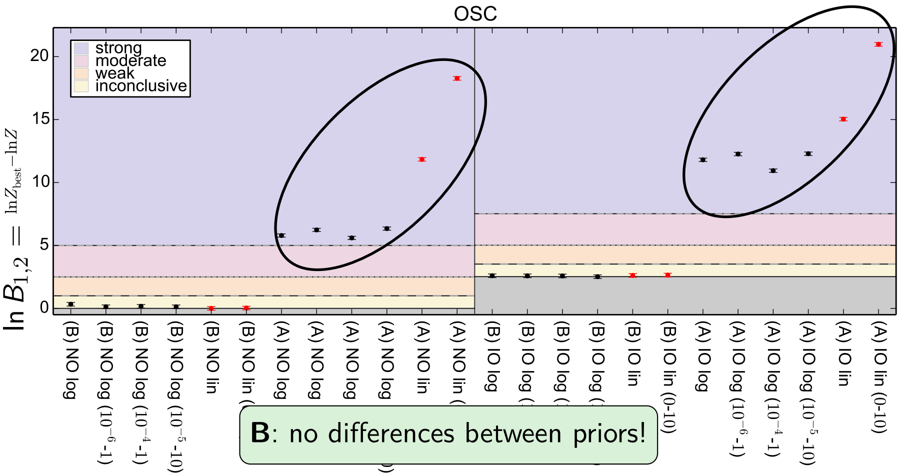


B: no differences between priors!

(no penalty for unconstrained parameter $m_{\rm lightest}$!)

Comparing parameterizations/priors

[SG+, JCAP 03 (2018) 11]



B: no differences between priors!

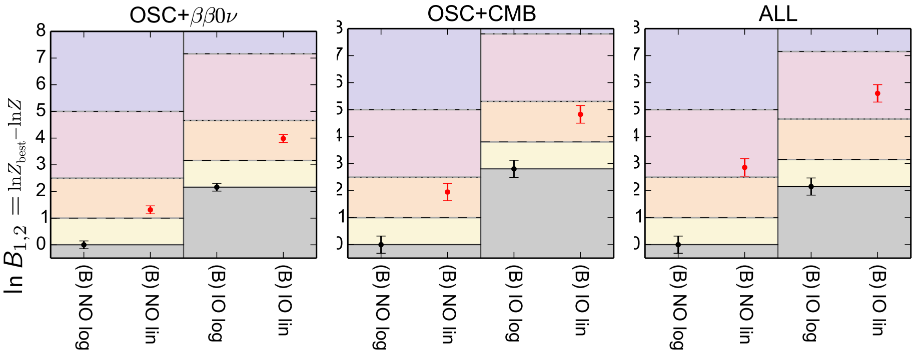
(no penalty for unconstrained parameter $m_{\rm lightest}$!)

A: always strongly disfavored!

(waste of parameter space, no unconstrained parameters due to Δm^2_{i1}!)

Comparing parameterizations/priors

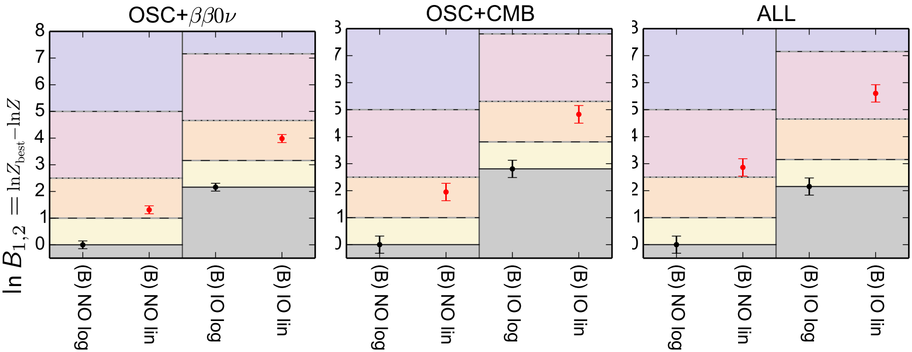
[SG+, JCAP 03 (2018) 11]



compare linear versus **logarithmic**

Comparing parameterizations/priors

[SG+, JCAP 03 (2018) 11]

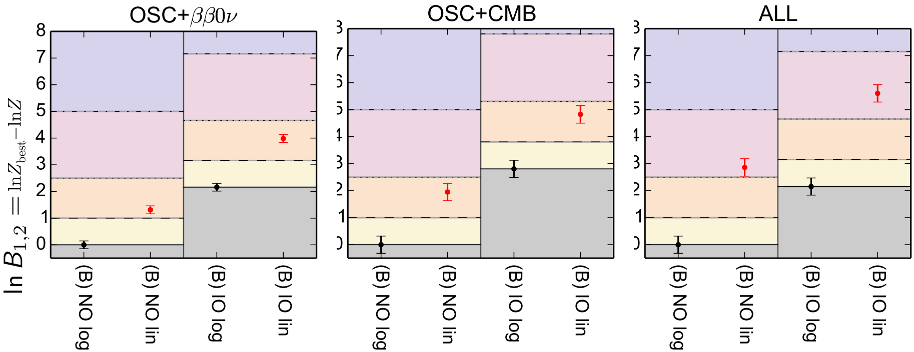


compare linear versus **logarithmic**

log priors are
weakly-to-moderately more efficient

Comparing parameterizations/priors

[SG+, JCAP 03 (2018) 11]

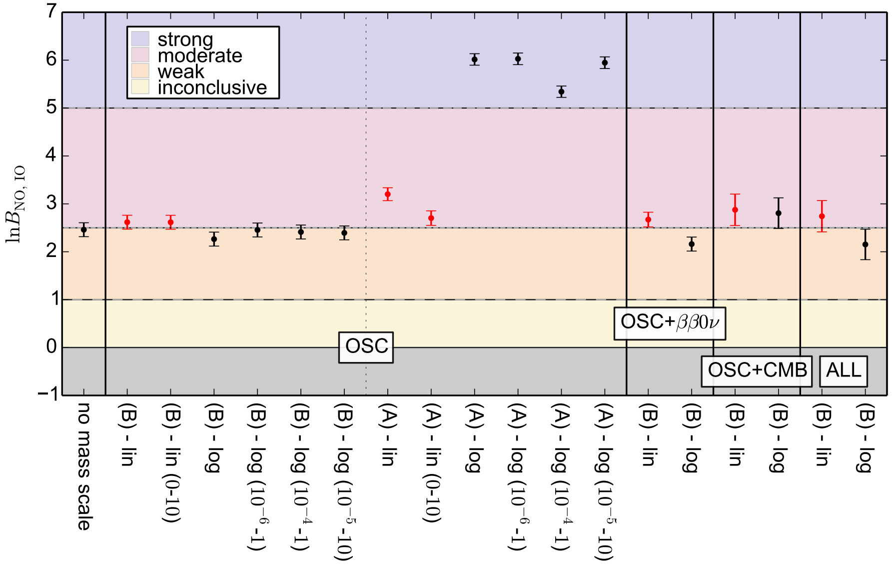


compare linear versus **logarithmic**

log priors are
weakly-to-moderately more efficient

summary: case B, log prior is better!

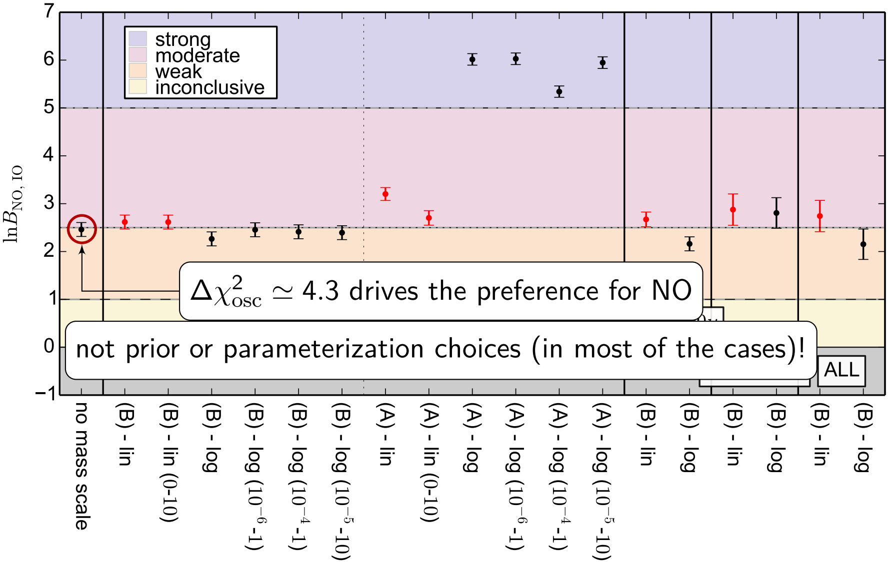
Comparing the mass orderings



Note: only oscillation data until the end of 2017 are included!

Comparing the mass orderings

[SG+, JCAP 03 (2018) 11]



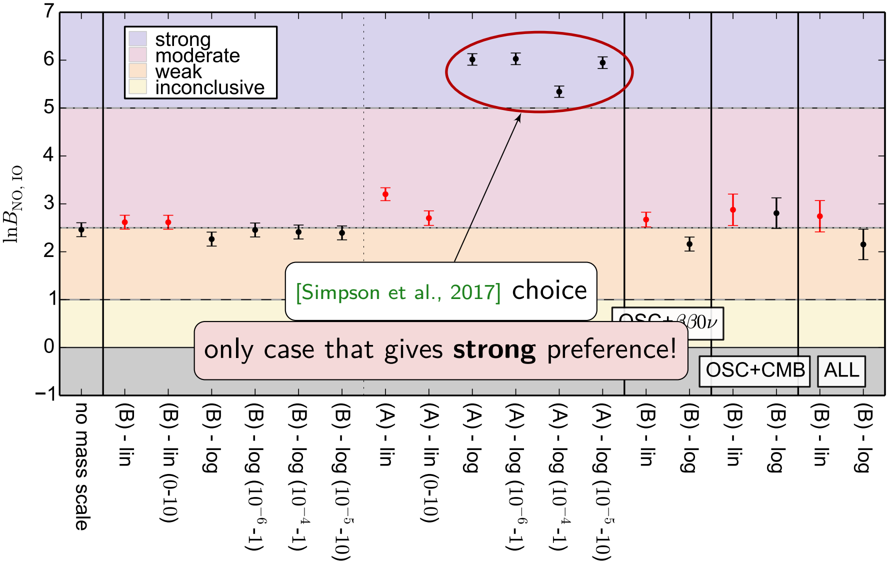
$\Delta\chi^2_{\rm osc} \simeq 4.3$ drives the preference for NO

not prior or parameterization choices (in most of the cases)!

Note: only oscillation data until the end of 2017 are included!

Comparing the mass orderings

[SG+, JCAP 03 (2018) 11]



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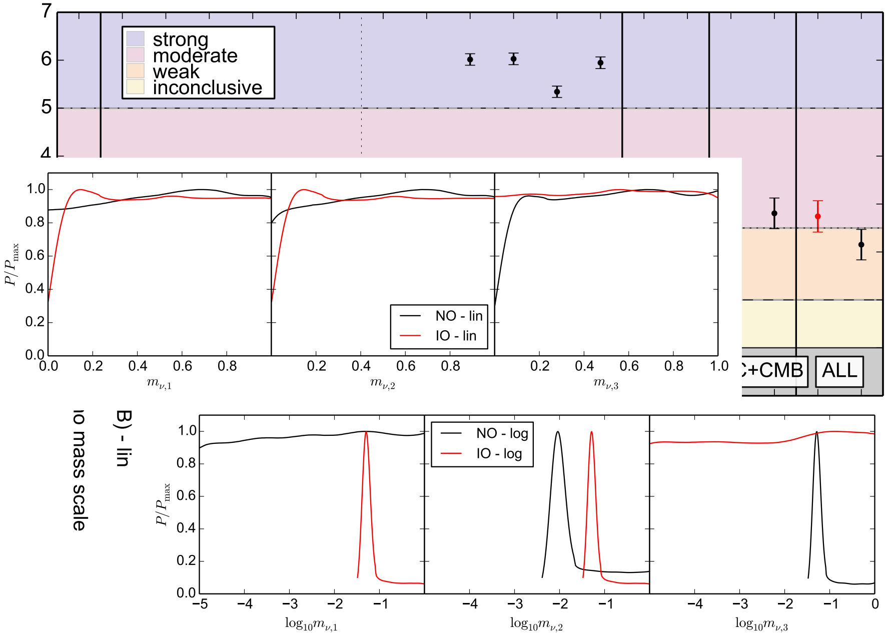
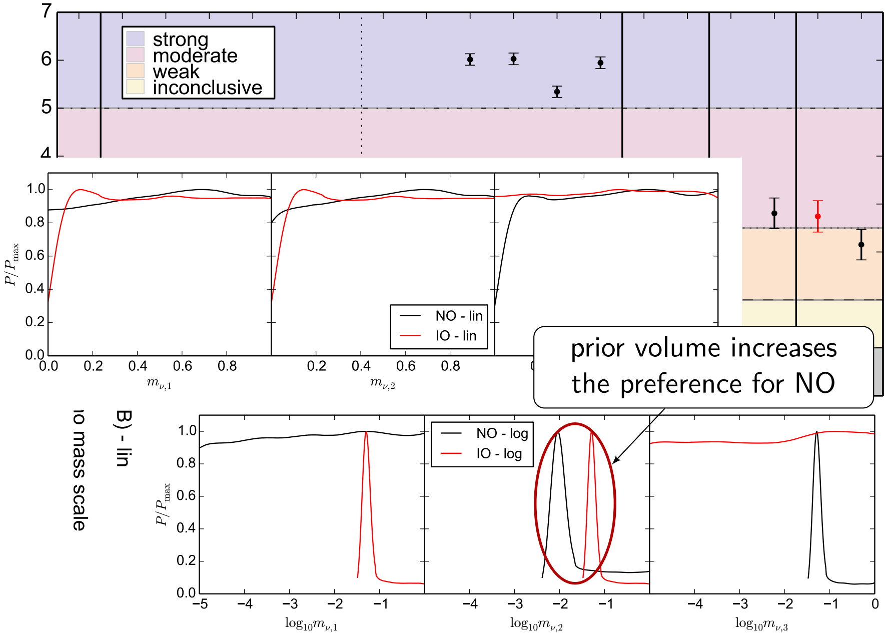
[SG+, JCAP 03 (2018) 11]



Comparing the mass orderings

[SG+, JCAP 03 (2018) 11]



Results in 2018

[de Salas+, Frontiers 5 (2018) 36]
http://globalfit.astroparticles.es/

Bayes theorem for models:

$$p(\mathcal{M}|d) \propto Z_{\mathcal{M}}\pi(\mathcal{M})$$

Bayesian evidence:

$$Z_{\mathcal{M}} = \int_{\Omega_{\mathcal{M}}} \mathcal{L}(\theta)\,\pi(\theta)\,d\theta$$

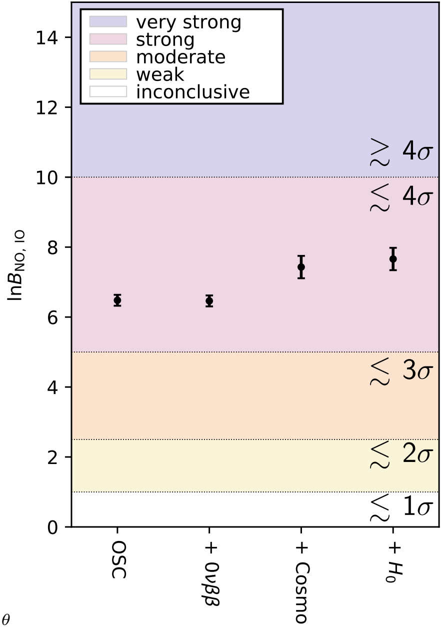
Bayes factor NO vs IO:

$$B_{\mathrm{NO,IO}} = Z_{\mathrm{NO}}/Z_{\mathrm{IO}}$$

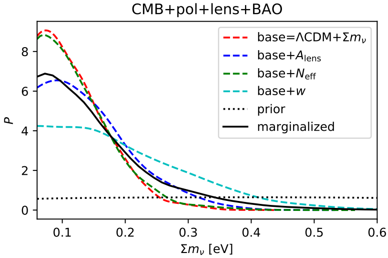
Posterior probability:

$$\begin{aligned} P_{\mathrm{NO}} &= B_{\mathrm{NO,IO}}/(B_{\mathrm{NO,IO}}+1) \\ P_{\mathrm{IO}} &= 1/(B_{\mathrm{NO,IO}}+1) \end{aligned}$$

$$N\sigma \text{ from } P_{\mathrm{NO}} = \mathrm{erf}(N/\sqrt{2})$$



$\pi(\mathcal{M})$ model prior $\mathcal{L}(\theta)$ likelihood
$p(\mathcal{M}|d)$ model posterior $\Omega_{\mathcal{M}}$ parameter space, for parameters θ


CMB+pol+lens+BAO
base=ΛCDM+Σm_ν
base+A_{lens}
base+N_{eff}
base+w
prior
marginalized
P
0
2
4
6
8
0.1
0.2
0.3
0.4
0.5
0.6
Σm_ν [eV]

Playing with priors

Bayes theorem:

$$p(\theta|d, \mathcal{M}) = \mathcal{L}(\theta)\frac{\pi(\theta|\mathcal{M})}{Z_{\mathcal{M}}}$$

posterior depends on prior!

parameters θ, model $\mathcal{M}$, data d $\pi(\theta|\mathcal{M})$ prior $p(\theta|d, \mathcal{M})$ posterior $\mathcal{L}(\theta)$ likelihood $Z_{\mathcal{M}}$ Bayesian evidence

Playing with priors

Bayes theorem:

$$p(\theta|d,\mathcal{M}) = \mathcal{L}(\theta)\frac{\pi(\theta|\mathcal{M})}{Z_\mathcal{M}}$$

posterior depends on prior!

[Planck 2018]: prior
$0 < \Sigma m_\nu < \mathcal{O}(1)$ eV

strongest upper limit (95%):
$\Sigma m_\nu < 113$ meV
(CMB+lens+BAO+SN)

corresponding to
$\Sigma m_\nu < 53.6$ meV (68%)

below minimum for NO!
does it make sense?

parameters θ, model $\mathcal{M}$, data d $\pi(\theta|\mathcal{M})$ prior $p(\theta|d,\mathcal{M})$ posterior $\mathcal{L}(\theta)$ likelihood $Z_\mathcal{M}$ Bayesian evidence

Playing with priors

Bayes theorem:

$$p(\theta|d,\mathcal{M}) = \mathcal{L}(\theta)\frac{\pi(\theta|\mathcal{M})}{Z_\mathcal{M}}$$

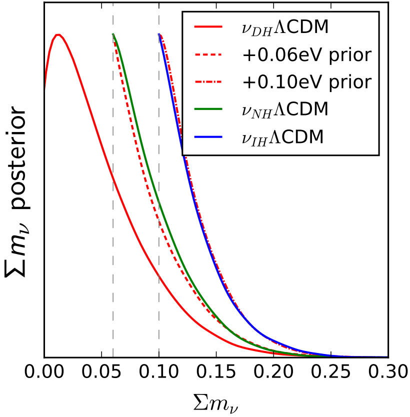
posterior depends on prior!

Different limits if you consider simply $\Sigma m_\nu > 0$ or you take into account oscillation results...

[Wang+, 2017]

degenerate (DH) vs normal (NH) vs inverted (IH) hierarchy

(i.e. change the prior lower bound)



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Playing with priors

Bayes theorem:

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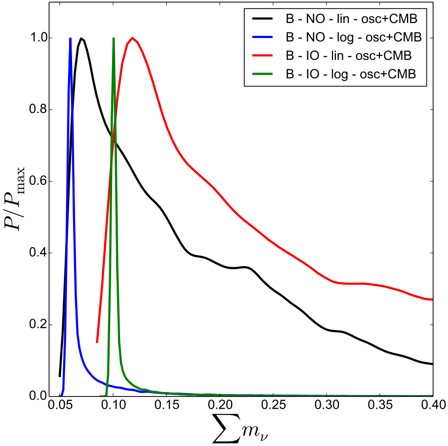
posterior depends on prior!

You can artificially tighten the bounds on Σm_ν with different priors...

[SG+, 2018]

logarithmic vs linear prior on $m_{\rm lightest}$



parameters θ, model $\mathcal{M}$, data d $\pi(\theta|\mathcal{M})$ prior $p(\theta|d,\mathcal{M})$ posterior $\mathcal{L}(\theta)$ likelihood $Z_\mathcal{M}$ Bayesian evidence

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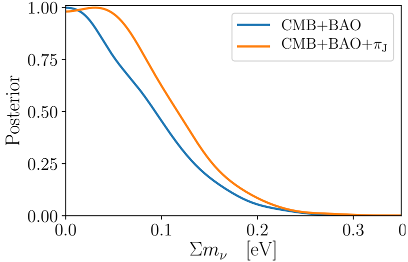
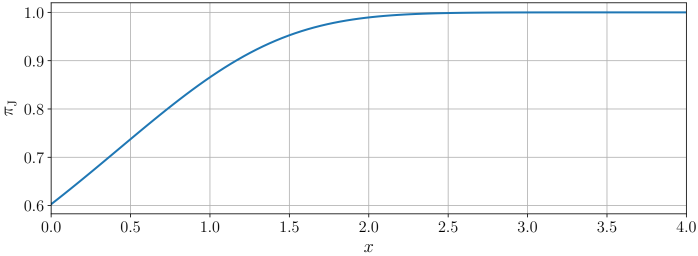


[Hannestad+, 2017]

Jeffreys prior (π_J) for Σm_ν

π_J makes the posterior maximally sensitive to data
for constrained parameter, compensate border effect



parameters θ, model $\mathcal{M}$, data d $\pi(\theta|\mathcal{M})$ prior $p(\theta|d,\mathcal{M})$ posterior $\mathcal{L}(\theta)$ likelihood $Z_\mathcal{M}$ Bayesian evidence

Playing with the baseline model

what if we release the assumption of the ΛCDM model?

CMB TT + lens		CMB TT + lens + BAO
CMB TT,TE,EE		**CMB TT,TE,EE + BAO**
	[Planck 2015]	
$\Sigma m_\nu < 0.68$ eV	ΛCDM	$\Sigma m_\nu < 0.25$ eV
$\Sigma m_\nu < 0.49$ eV		$\Sigma m_\nu < 0.17$ eV

Playing with the baseline model

what if we release the assumption of the ΛCDM model?

CMB TT + lens **CMB TT,TE,EE**		CMB TT + lens + BAO **CMB TT,TE,EE + BAO**
$\Sigma m_\nu < 0.68$ eV $\Sigma m_\nu < 0.49$ eV	[Planck 2015] ΛCDM	$\Sigma m_\nu < 0.25$ eV $\Sigma m_\nu < 0.17$ eV
-	wCDM	$\Sigma m_\nu < 0.37$ eV [Planck 2015] $\Sigma m_\nu < 0.27$ eV [Wang+, 2016]

free dark energy equation of state $w \neq -1$

Playing with the baseline model

what if we release the assumption of the ΛCDM model?

	CMB TT + lens **CMB TT,TE,EE**		CMB TT + lens + BAO **CMB TT,TE,EE + BAO**
[Planck 2015] ΛCDM	$\Sigma m_\nu < 0.68$ eV $\Sigma m_\nu < 0.49$ eV		$\Sigma m_\nu < 0.25$ eV $\Sigma m_\nu < 0.17$ eV
wCDM (free dark energy equation of state $w \neq -1$)	-		$\Sigma m_\nu < 0.37$ eV [Planck 2015] $\Sigma m_\nu < 0.27$ eV [Wang+, 2016]
[Planck 2015] ΛCDM+$A_{\rm lens}$ (free phenomenological lensing amplitude $A_{\rm lens} \neq -1$)	-		$\Sigma m_\nu < 0.41$ eV

Playing with the baseline model

what if we release the assumption of the ΛCDM model?

CMB TT + lens / **CMB TT,TE,EE**	Model	CMB TT + lens + BAO / **CMB TT,TE,EE + BAO**
$\Sigma m_\nu < 0.68$ eV $\Sigma m_\nu < 0.49$ eV	[Planck 2015] ΛCDM	$\Sigma m_\nu < 0.25$ eV $\Sigma m_\nu < 0.17$ eV
-	wCDM free dark energy equation of state $w \neq -1$	$\Sigma m_\nu < 0.37$ eV [Planck 2015] $\Sigma m_\nu < 0.27$ eV [Wang+, 2016]
-	[Planck 2015] ΛCDM+$A_{\rm lens}$ free phenomenological lensing amplitude $A_{\rm lens} \neq -1$	$\Sigma m_\nu < 0.41$ eV
$\Sigma m_\nu < 0.96$ eV	[Di Valentino+, 2015] eCDM 12-parameters cosmological model, ΛCDM based	$\Sigma m_\nu < 0.53$ eV

Marginalize over models?

[SG+, PRD 99 (2019) 021301]

We usually marginalize over parameters:

$$p(\theta|d, \mathcal{M}_0) \propto \int \mathcal{L}(\theta, \psi)\pi(\theta, \psi|\mathcal{M}_0)d\psi$$

Can we marginalize over models?

Marginalize over models?

[SG+, PRD 99 (2019) 021301]

We usually marginalize over parameters:

$$p(\theta|d,\mathcal{M}_0) \propto \int \mathcal{L}(\theta,\psi)\pi(\theta,\psi|\mathcal{M}_0)d\psi$$

Can we marginalize over models?

Yes, if we know the model posteriors:

$$p(\theta|d) = \sum_i^N p(\theta|d,\mathcal{M}_i)\, p_i$$

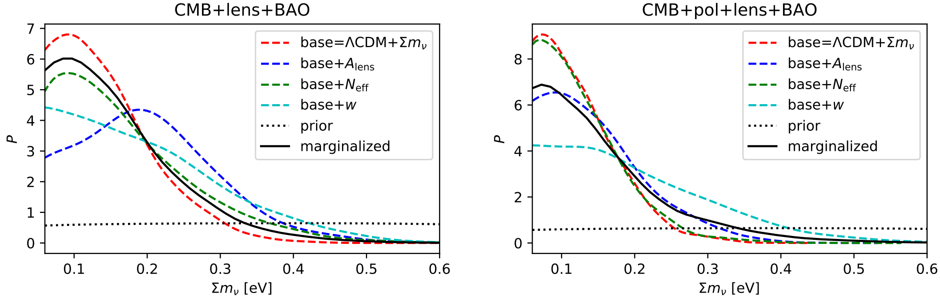
Select a model $\mathcal{M}_0$ and use $p_i = Z_i/(\sum Z_j) = B_{i0}/(\sum B_{j0})$:

$$p(\theta|d) = \sum_i^N p(\theta|d,\mathcal{M}_i) Z_i \Bigg/ \sum_j^N Z_j$$

$p(\theta|d)$ is a model-marginalized posterior for θ, given the data d

Model-marginalization applied to Σm_ν

[SG+, PRD 99 (2019) 021301]



model	CMB+lens+BAO		CMB+pol+lens+BAO	
	$\ln B_{i0}$	Σm_ν [eV]	$\ln B_{i0}$	Σm_ν [eV]
base=ΛCDM+Σm_ν	0.0	< 0.28	0.0	< 0.23
base+$A_{\rm lens}$	-2.6	< 0.38	-2.4	< 0.29
base+$N_{\rm eff}$	-1.5	< 0.37	-2.3	< 0.25
base+w	-1.4	< 0.42	-0.1	< 0.42
marginalized	–	< 0.33	–	< 0.35
p_0	0.65		0.48	

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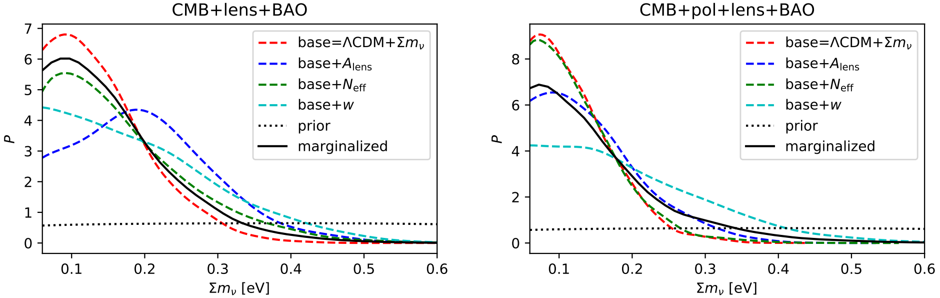
[SG+, PRD 99 (2019) 021301]



model	CMB+lens+BAO $\ln B_{i0}$	CMB+lens+BAO Σm_ν [eV]	CMB+pol+lens+BAO $\ln B_{i0}$	CMB+pol+lens+BAO Σm_ν [eV]
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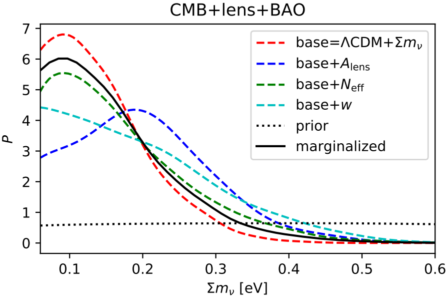
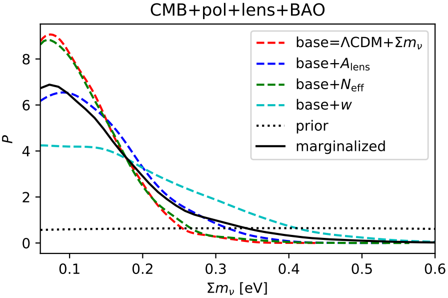
[SG+, PRD 99 (2019) 021301]





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Prior-independent Bayesian parameter constraints - I

Bayes theorem (again!): $p(\theta|d,\mathcal{M}_i) = \pi(\theta|\mathcal{M}_i)\,\mathcal{L}_{\mathcal{M}_i}(\theta)/\,Z_i$

⟶ function of x

We usually present 1-dim marginalized posterior distributions:

⟶ over params ψ

$$p(x|d,\mathcal{M}_i) = \int_{\Omega_\psi} d\psi\, p(x,\psi|\mathcal{M}_i,d)$$

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$$p(x|d,\mathcal{M}_i) = \int_{\Omega_\psi} d\psi\, p(x,\psi|\mathcal{M}_i,d)$$

Assume that prior is separable: $\pi(\theta|\mathcal{M}_i) = \pi(x|\mathcal{M}_i)\cdot\pi(\psi|\mathcal{M}_i)$
and that $\pi(x) \equiv \pi(x|\mathcal{M}_i)$ does not depend on $\mathcal{M}_i$

$$p(x|d,\mathcal{M}_i) = \frac{\pi(x)}{Z_i}\int_{\Omega_\psi} d\psi\,\pi(\psi|\mathcal{M}_i)\mathcal{L}_{\mathcal{M}_i}(x,\psi)$$

$\equiv Z_i^x$ Bayesian evidence of model $\mathcal{M}_i|_{\text{fixed}\,x}$
independent of $\pi(x)$ but not of x

Prior-independent Bayesian parameter constraints - I

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[SG+, PRD 99 (2019) 021301]

Model marginalization: $p(x|d) = \sum_i p(x|\mathcal{M}_i,d)\,Z_i \Big/ \sum_j Z_j$

Replace $p(x|\mathcal{M}_i,d)$: $p(x|d) = \pi(x)\underbrace{\sum_i Z_i^x \Big/ \sum_j Z_j}_{\text{independent of } \pi(x)?}$

Prior-independent Bayesian parameter constraints - I

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Replace $p(x|\mathcal{M}_i,d)$: $p(x|d) = \pi(x)\sum_i Z_i^x \Big/ \sum_j Z_j$

independent of $\pi(x)$!

[Astone, 1999]
[D'Agostini, 2000]
relative belief updating ratio

$$\mathcal{R}(x,x_0|d) \equiv \frac{p(x|d)/\pi(x)}{p(x_0|d)/\pi(x_0)} = \frac{\sum_i Z_i^x}{\sum_j Z_j^{x_0}}$$

Prior-independent Bayesian parameter constraints - II

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Rewrite in a more familiar form:

see $\frac{p(\mathcal{M}_i|d)}{p(\mathcal{M}_j|d)} = B_{ij}\frac{\pi(\mathcal{M}_i)}{\pi(\mathcal{M}_j)}$

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x_0 is limit to which data are insensitive to x, e.g. $x_0 = 0$ (if x is Σm_ν)

$\mathcal{R}(x, x_0|d)$ describes how **data** update our initial beliefs on x

→ $\mathcal{R} \simeq 1$ ($x \to x_0$): data are insensitive to x

→ $\mathcal{R} \to 0$ ($x \gg x_0$): data disfavor x, regardless of prior

Prior-independent Bayesian parameter constraints - II

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- → $\mathcal{R} \simeq 1$ $(x \to x_0)$: data are insensitive to x
- → $\mathcal{R} \to 0$ $(x \gg x_0)$: data disfavor x, regardless of prior

we can use $\mathcal{R}$ to derive a (non-probabilistic) "sensitivity bound x_s"

$x > x_s$ disfavored because $\mathcal{R}(x, x_0|d) < s$, with $s = 5\%$ or 1%

x_s is a hedge "which separates the region in which we are, and where we see nothing, from the the region we cannot see" [D'Agostini, 2000]

An example with Planck 2018

$$\mathcal{R}(x, x_0|d) \equiv \frac{p(x|d)/\pi(x)}{p(x_0|d)/\pi(x_0)}$$

Numerically easy to compute: fix $\pi(x)$, get $p(x|d)$ normally and divide

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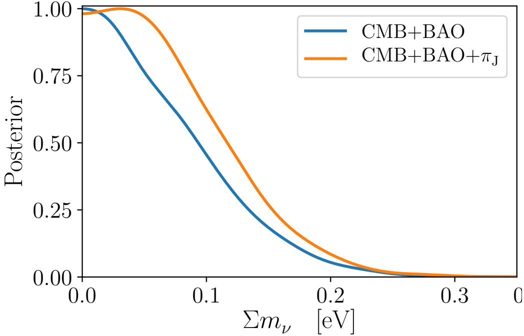
Note: 1D plots in cosmology are already close to show $\mathcal{R}$
as for linear priors, the shape of $\mathcal{R}(x, x_0|d)$ is equal to the one of $p(x|d)$!

e.g. [Hannestad+, 2017]



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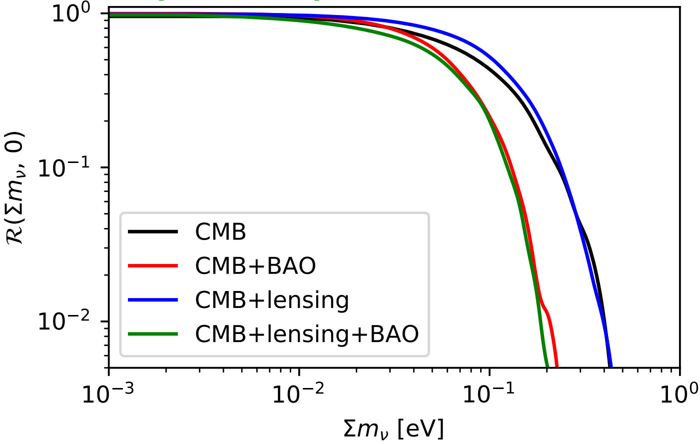
Example with [Planck 2018] chains from PLA, ΛCDM+Σm_ν



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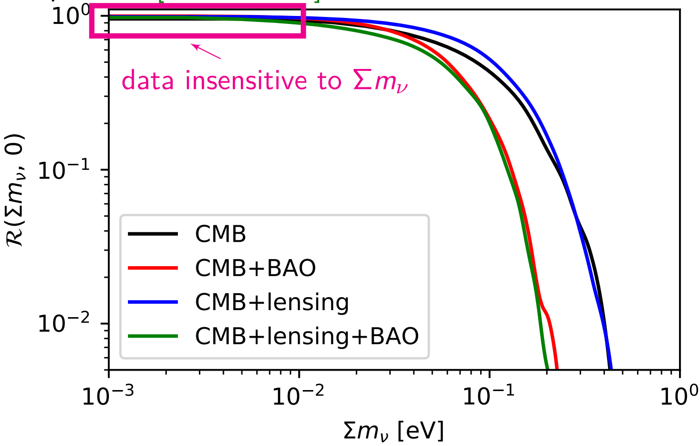

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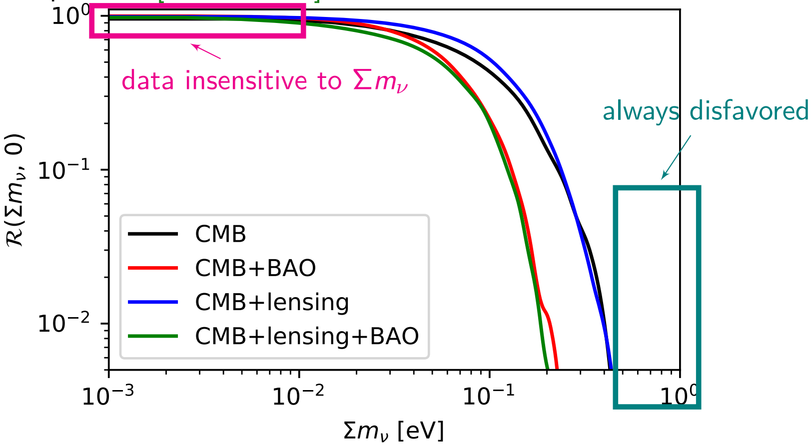

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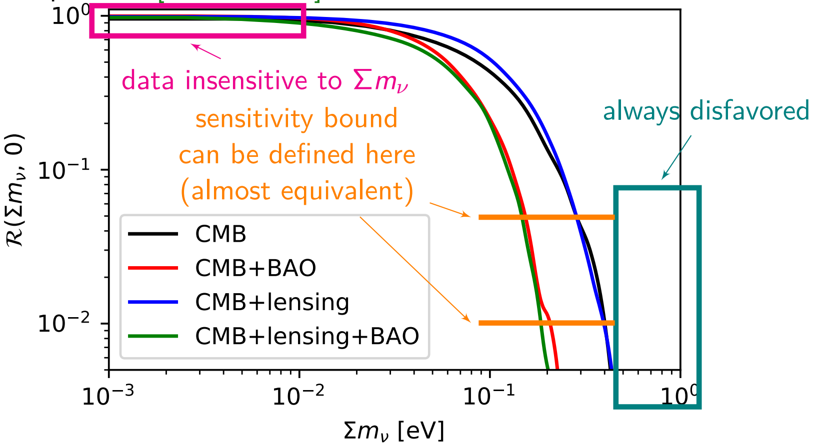
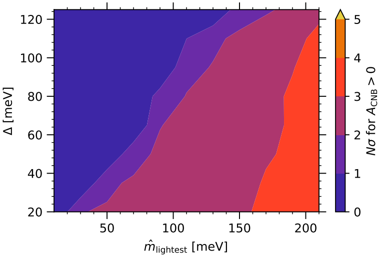
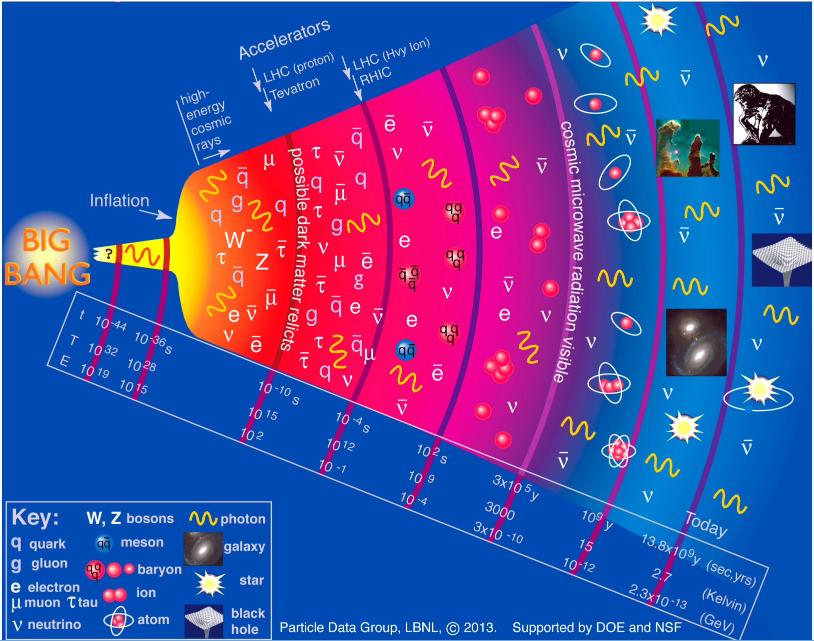




120
100
80
60
40
20
Δ [meV]
50
100
150
200
$\hat{m}_{\rm lightest}$ [meV]
0
1
2
3
4
5
$N\sigma$ for $A_{\rm CNB} > 0$

History of the universe



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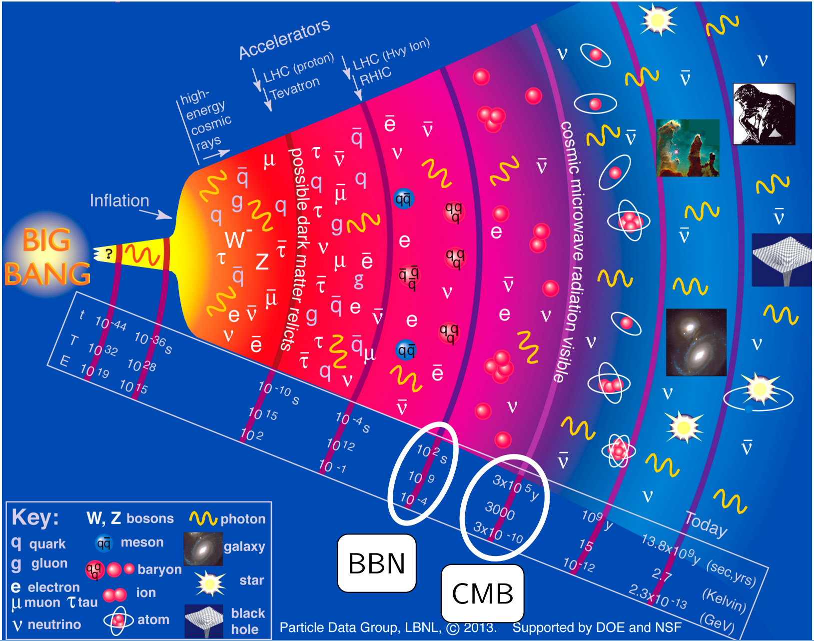

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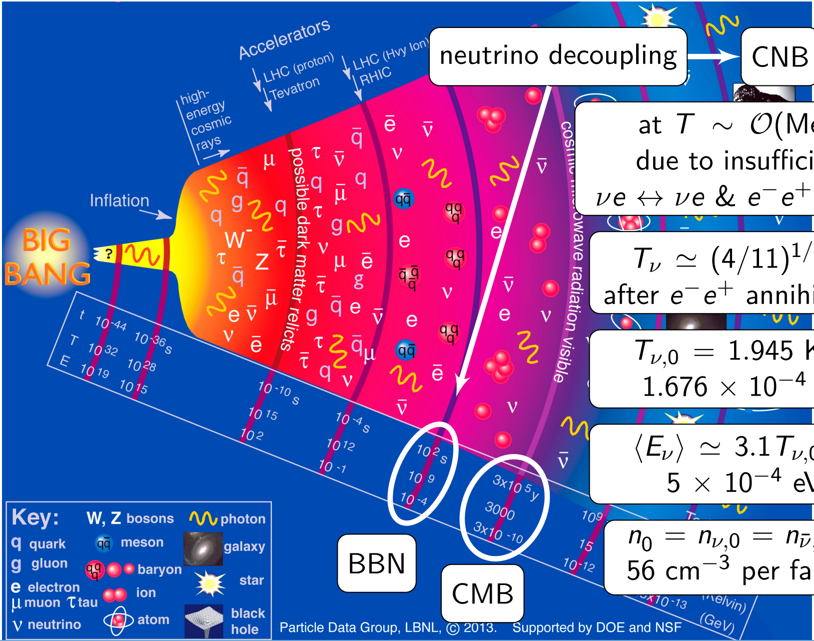


neutrino decoupling → CNB

at $T \sim \mathcal{O}(\text{MeV})$ due to insufficient $\nu e \leftrightarrow \nu e$ & $e^- e^+ \leftrightarrow \nu\bar\nu$

$T_\nu \simeq (4/11)^{1/3} T_\gamma$ after $e^- e^+$ annihilation

$T_{\nu,0} = 1.945$ K $\simeq 1.676 \times 10^{-4}$ eV

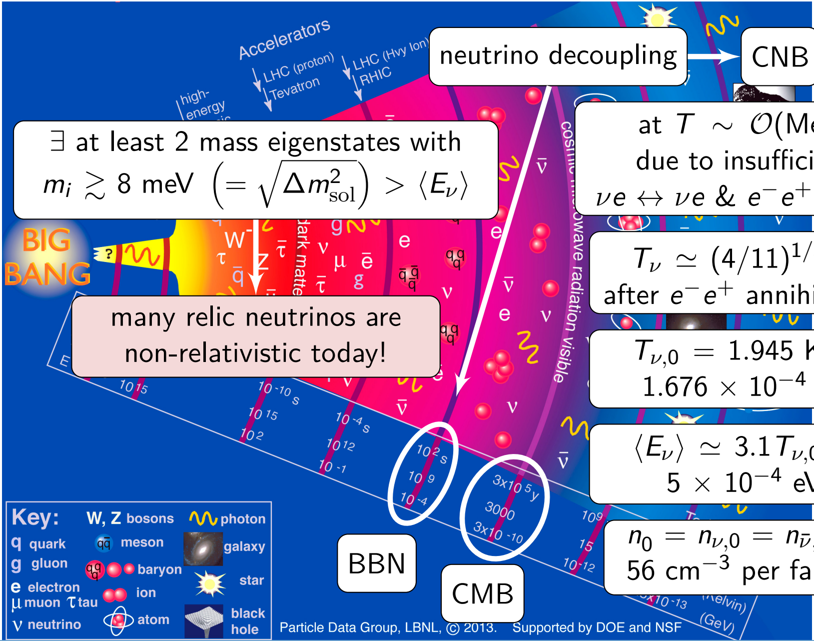
$\langle E_\nu \rangle \simeq 3.1\, T_{\nu,0} \simeq 5 \times 10^{-4}$ eV

$n_0 = n_{\nu,0} = n_{\bar\nu,0} \simeq 56$ cm^{-3} per family

BBN

CMB

History of the universe



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5×10^{-4} eV

$n_0 = n_{\nu,0} = n_{\bar{\nu},0} \simeq$
56 cm^{-3} per family

$\exists$ at least 2 mass eigenstates with
$m_i \gtrsim 8$ meV $\left(= \sqrt{\Delta m^2_{\rm sol}}\right) > \langle E_\nu \rangle$

many relic neutrinos are
non-relativistic today!

BBN

CMB

A viable detection method

[Long et al., JCAP 08 (2014) 038]

How to directly detect non-relativistic neutrinos?

Remember that $\langle E_\nu \rangle \simeq \mathcal{O}(10^{-4})$ eV today $\longrightarrow$ a process without energy threshold is necessary

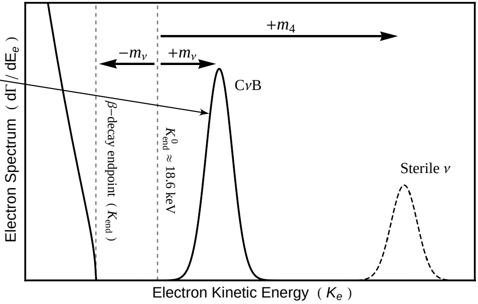
[Weinberg, 1962]: neutrino capture in β–decaying nuclei $\nu + n \to p + e^-$

Main background: β decay $n \to p + e^- + \bar{\nu}$!

signal is a peak at $2m_\nu$ above β–decay endpoint

only with a lot of material

need a very good energy resolution



PTOLEMY

[Long et al., JCAP 08 (2014) 038]
[Betts et al., arxiv:1307.4738]
[PTOLEMY LoI, arxiv:1808.01892]

PonTecorvo Observatory for Light, Early-universe, Massive-neutrino Yield (PTOLEMY)

expected resolution $\Delta \simeq 0.1$ eV? 0.05 eV?

built mainly for CNB

can probe $m_\nu \simeq 1.4\Delta \simeq 0.1$ eV

$M_T = 100$ g of atomic ^{3}H

$$\Gamma_{\rm CNB} = \sum_{i=1}^{3} |U_{ei}|^2 [n_i(\nu_{h_R}) + n_i(\nu_{h_L})]\, N_T\, \bar\sigma \quad \sim \mathcal{O}(10)\ \text{yr}^{-1}$$

N_T number of ^{3}H nuclei in a sample of mass M_T $\quad \bar\sigma \simeq 3.834 \times 10^{-45}$ cm^2 $\quad n_i$ number density of neutrino i

(without clustering)

PTOLEMY

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ehnancement from ν clustering in the galaxy?

$$\Gamma_{\text{CNB}} = \sum_{i=1}^{3} |U_{ei}|^2 [n_i(\nu_{h_R}) + n_i(\nu_{h_L})] \, N_T \, \bar{\sigma} \quad \sim \mathcal{O}(10) \text{ yr}^{-1}$$

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ν clustering with N-one-body simulations

[JCAP 09 (2017) 034]

Milky Way (MW) matter attracts neutrinos!

clustering →

$$\Gamma_{\rm CNB} = \sum_{i=1}^{3} |U_{ei}|^2 f_c(m_i) \left[n_{i,0}(\nu_{h_R}) + n_{i,0}(\nu_{h_L})\right] N_T \bar{\sigma}$$

$f_c(m_i) = n_i/n_{i,0}$ clustering factor ⟶ How to compute it?

Idea from [Ringwald & Wong, 2004] ⟶ N-one-body= N × single ν simulations

→ each ν evolved from initial conditions at $z = 3$

→ spherical symmetry, coordinates $(r,\ \theta,\ p_r,\ l)$

→ need $\rho_{\rm matter}(z) = \rho_{\rm DM}(z) + \rho_{\rm baryon}(z)$

Assumptions:

- νs are independent
- only gravitational interactions
- νs do not influence matter evolution ($\rho_\nu \ll \rho_{\rm DM}$)

how many νs is "N"?

→ must sample all possible r, p_r, l

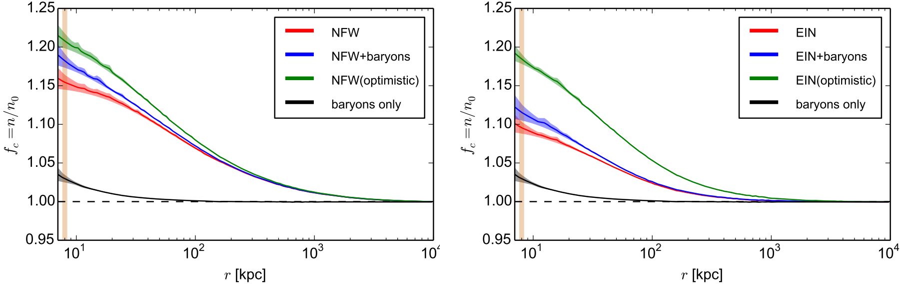
→ must include all possible νs that reach the MW (fastest ones may come from several (up to $\mathcal{O}(100)$) Mpc!)

given N ν:

→ weigh each neutrinos

→ reconstruct final density profile with kernel method from [Merritt&Tremblay, 1994]

Overdensity when $m_{\text{heaviest}} \simeq 60$ meV



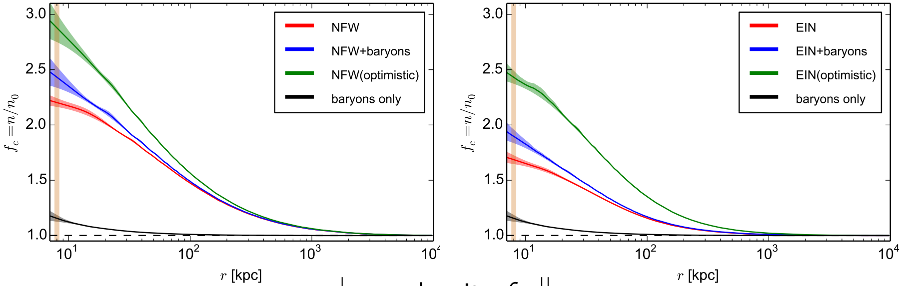
masses	ordering	matter halo	overdensity f_c: $f_1 \simeq f_2$	f_3	Γ_{tot} (yr^{-1})
any	any	any	no clustering		4.06
$m_3 = 60$ meV	NO	NFW(+bar)	$\sim$1	1.15 (1.18)	4.07 (4.08)
		NFW optimistic		1.21	4.08
		EIN(+bar)		1.09 (1.12)	4.07 (4.07)
		EIN optimistic		1.18	4.08
$m_1 \simeq m_2 = 60$ meV	IO	NFW(+bar)	1.15 (1.18)	$\sim$1	4.66 (4.78)
		NFW optimistic	1.21		4.89
		EIN(+bar)	1.09 (1.12)		4.42 (4.54)
		EIN optimistic	1.18		4.78

ordering dependence from $\Gamma_{\text{CNB}} = \sum_{i=1}^{3} |U_{ei}|^2 f_i \left[n_i(\nu_{h_R}) + n_i(\nu_{h_L})\right] N_T \bar{\sigma}$

Overdensity when $m_\nu \simeq 150$ meV

[JCAP 09 (2017) 034]

$\Longrightarrow$ minimal mass detectable by PTOLEMY if $\Delta \simeq 100$–150 meV



matter halo	overdensity f_c $f_1 \simeq f_2 \simeq f_3$	$\Gamma_{\rm tot}$ (yr^{-1})
any	no clustering	4.06
NFW(+bar)	2.18 (2.44)	8.8 (9.9)
NFW optimistic	2.88	11.7
EIN(+bar)	1.68 (1.87)	6.8 (7.6)
EIN optimistic	2.43	9.9

no ordering dependence: $m_1 \simeq m_2 \simeq m_3 \Longrightarrow f_1 \simeq f_2 \simeq f_3$

Additional clustering due to other galaxies

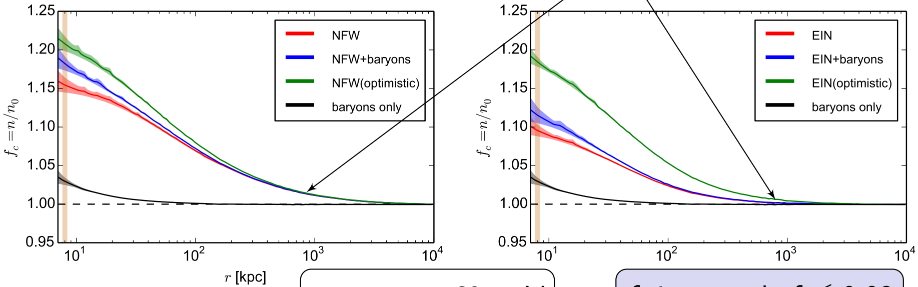
nearest galaxies: various MW satellites

with $M_{\rm sat} \ll M_{\rm MW}$ → negligibly small ν halo

nearest big galaxy:

Andromeda

$$M_{\rm Andromeda} = M_{\rm MW} \times \mathcal{O}(1) \;\text{—}\; d_{\rm Andromeda} \simeq 800 \text{ kpc}$$



$m_{\rm heaviest} \simeq 60$ meV → f_c increased of $\lesssim 0.03$

Additional clustering due to other galaxies

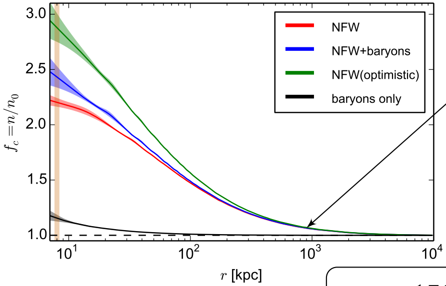
nearest galaxies: various MW satellites

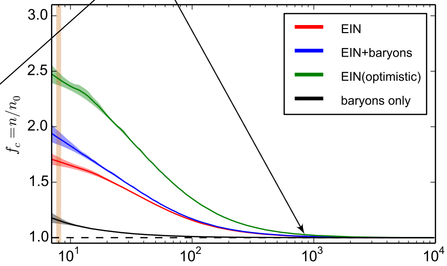
with $M_{\rm sat} \ll M_{\rm MW}$ ⟶ negligibly small ν halo

nearest big galaxy:

Andromeda

$$M_{\rm Andromeda} = M_{\rm MW} \times \mathcal{O}(1) \text{ — } d_{\rm Andromeda} \simeq 800 \text{ kpc}$$





$m_\nu \simeq 150$ meV ⟶ f_c increased of $\lesssim 0.1$

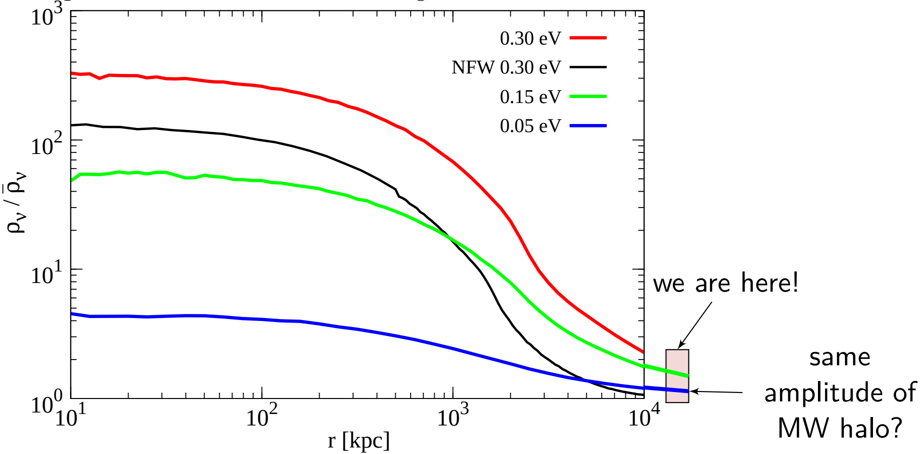
(halo is less diffuse for higher ν masses)

Additional clustering due to Virgo cluster

nearest galaxy cluster:

Virgo cluster → very wide ν halo, may reach Earth

$M_{\rm Virgo} = M_{\rm MW} \times \mathcal{O}(10^3)$ — $d_{\rm Virgo} \simeq 16$ Mpc



[Villaescusa-Navarro et al., JCAP 1106 (2011) 027]

Simulations - I

[PTOLEMY, in preparation]

Events in **bin** i, centered at E_i:

$$N^i_\beta = T\int_{E_i-\Delta/2}^{E_i+\Delta/2}\frac{d\widetilde{\Gamma}_\beta}{dE_e}dE_e \qquad N^i_{\rm CNB} = T\int_{E_i-\Delta/2}^{E_i+\Delta/2}\frac{d\widetilde{\Gamma}_{\rm CNB}}{dE_e}dE_e$$

fiducial number of events: $\hat{N}^i = N^i_\beta(\hat{E}_{\rm end},\hat{m}_i,\hat{U}) + N^i_{\rm CNB}(\hat{E}_{\rm end},\hat{m}_i,\hat{U})$

add **background** $\hat{N}_b = \hat{\Gamma}_b T$ with $\hat{\Gamma}_b \simeq 10^{-5}$ Hz $\longrightarrow$ $N^i_t = \hat{N}^i + \hat{N}_b$

simulated experimental spectrum: $N^i_{\rm exp}(\hat{E}_{\rm end},\hat{m}_i,\hat{U}) = N^i_t \pm \sqrt{N^i_t}$

repeat for theory spectrum, free **amplitudes** and **endpoint position**:

$$N^i_{\rm th}(\boldsymbol{\theta}) = \boldsymbol{A_\beta} N^i_\beta(\hat{E}_{end} + \boldsymbol{\Delta E_{end}}, m_i, U) + \boldsymbol{A_{\rm CNB}} N^i_{\rm CNB}(\hat{E}_{end} + \boldsymbol{\Delta E_{end}}, m_i, U) + N_b$$

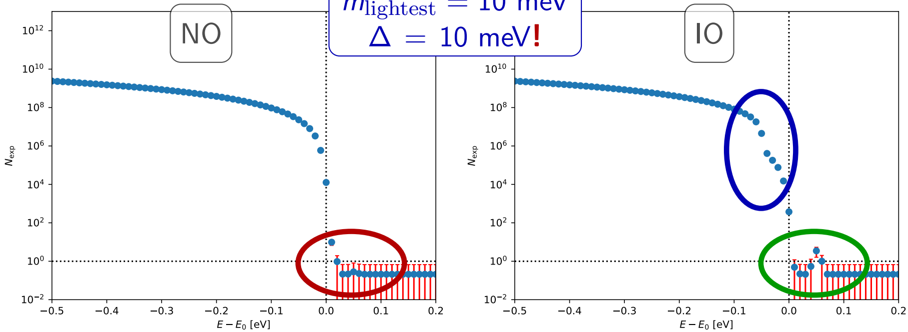
fit $\longrightarrow$ $\chi^2(\boldsymbol{\theta}) = \sum_i \left(\frac{N^i_{\rm exp}(\hat{E}_{\rm end},\hat{m}_i,\hat{U}) - N^i_{\rm th}(\boldsymbol{\theta})}{\sqrt{N^i_t}}\right)^2$ or $\log\mathcal{L} = -\frac{\chi^2}{2}$

T exposure time – $(\hat{E}_{\rm end}, \hat{m}_i, \hat{U})$ fiducial endpoint energy, masses, mixing matrix – $\boldsymbol{\theta} = (A_\beta, N_b, \Delta E_{end}, A_{\rm CNB}, m_i, U)$

Simulations - II

no random noise?

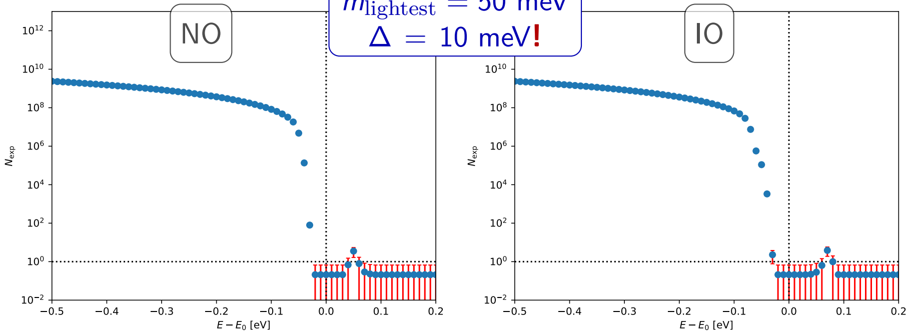
$m_{\text{lightest}} = 10$ meV
$\Delta = 10$ meV!



Simulations - II

no random noise?

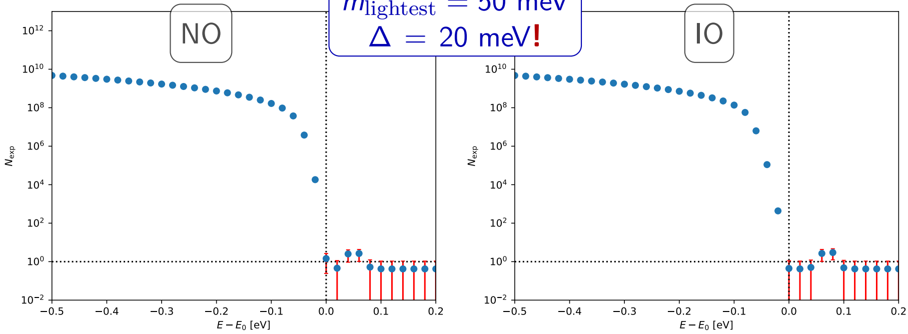
$m_{\text{lightest}} = 50$ meV
$\Delta = 10$ meV!



Simulations - II

no random noise?

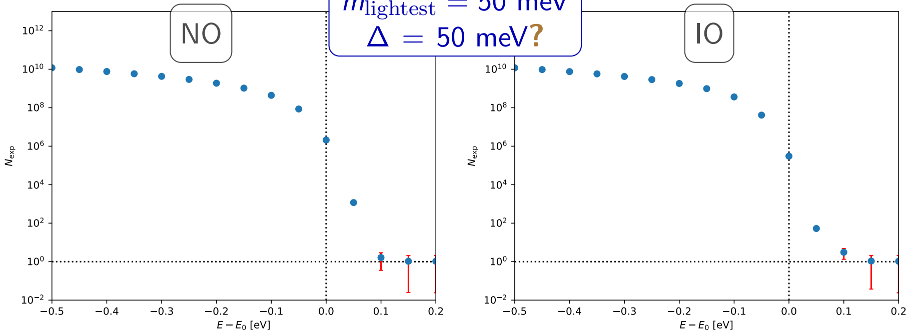
$m_{\rm lightest} = 50$ meV
$\Delta = 20$ meV!



Simulations - II

no random noise?

$m_{\text{lightest}} = 50$ meV
$\Delta = 50$ meV?



Simulations - II

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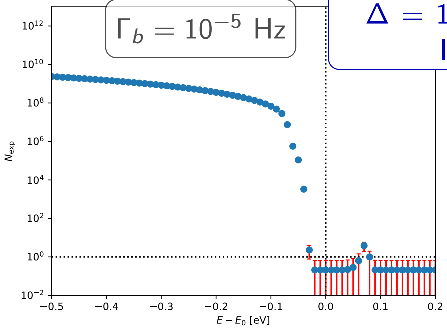
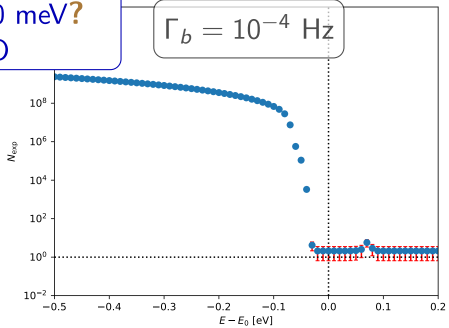
$m_{\text{lightest}} = 50$ meV
$\Delta = 10$ meV?
IO

$\Gamma_b = 10^{-5}$ Hz

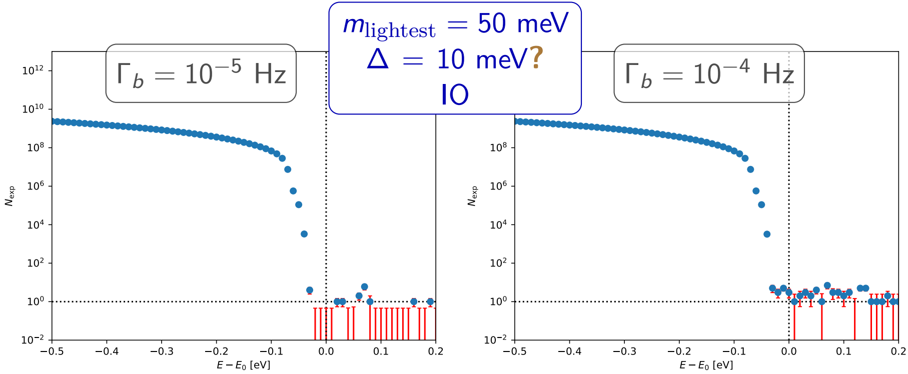


$\Gamma_b = 10^{-4}$ Hz



Simulations - II

with random noise!



things are more complicated in this way...low background needed!

Perspectives for the mass determination

[PTOLEMY, in preparation]

statistical only!

relative error on $m_{\rm lightest}$
as a function of $\hat{m}_{\rm lightest}$, Δ

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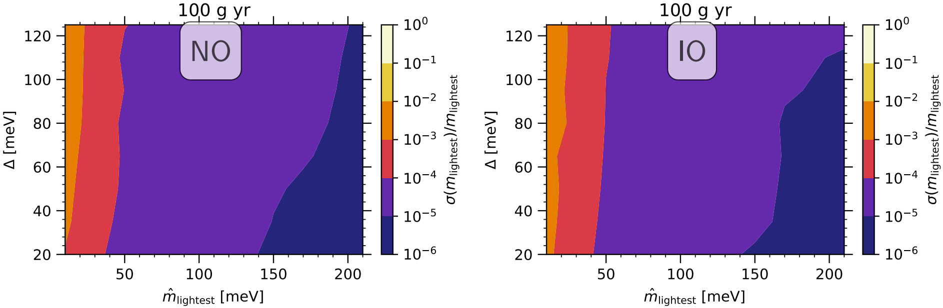

wonderful precision in determining the neutrino mass

(well, yes, with 100 g of tritium...)

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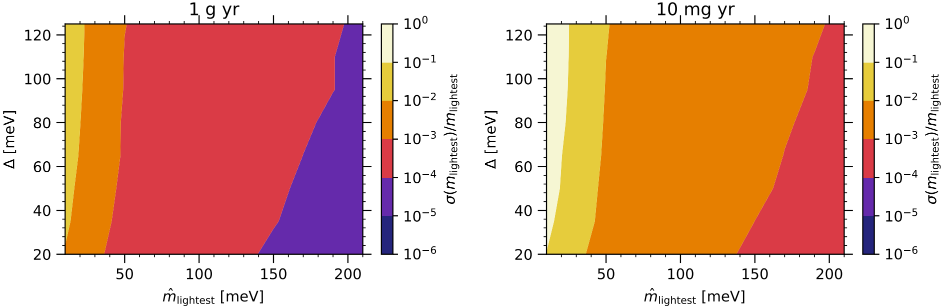

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(mass detection already with 10 mg of tritium!)

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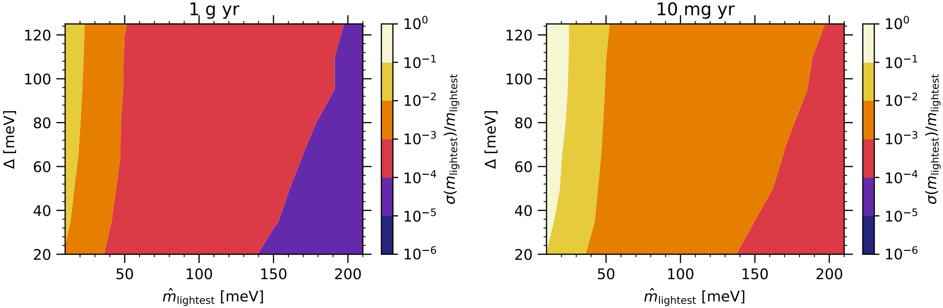


wonderful precision in determining the neutrino mass

(mass detection already with 10 mg of tritium!)

Δ has almost no impact

Detection of the relic neutrinos

[PTOLEMY, in preparation]

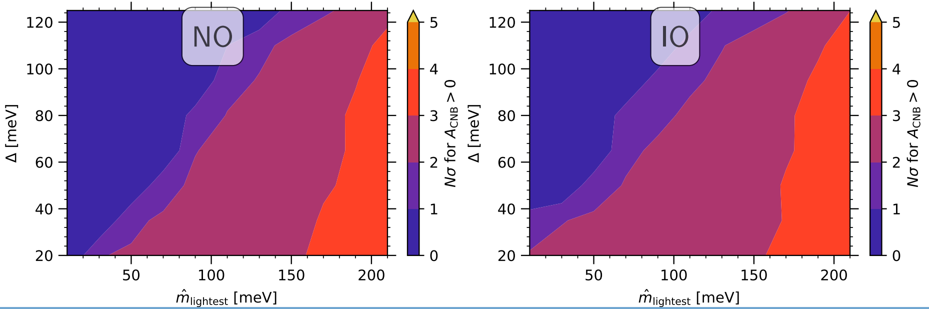
using the definition:

$$N^i_{\rm th}(\boldsymbol{\theta}) = A_\beta N^i_\beta(\hat{E}_{end} + \Delta E_{end}, m_i, U) + \boldsymbol{A}_{\rm CNB} N^i_{\rm CNB}(\hat{E}_{end} + \Delta E_{end}, m_i, U) + N_b$$

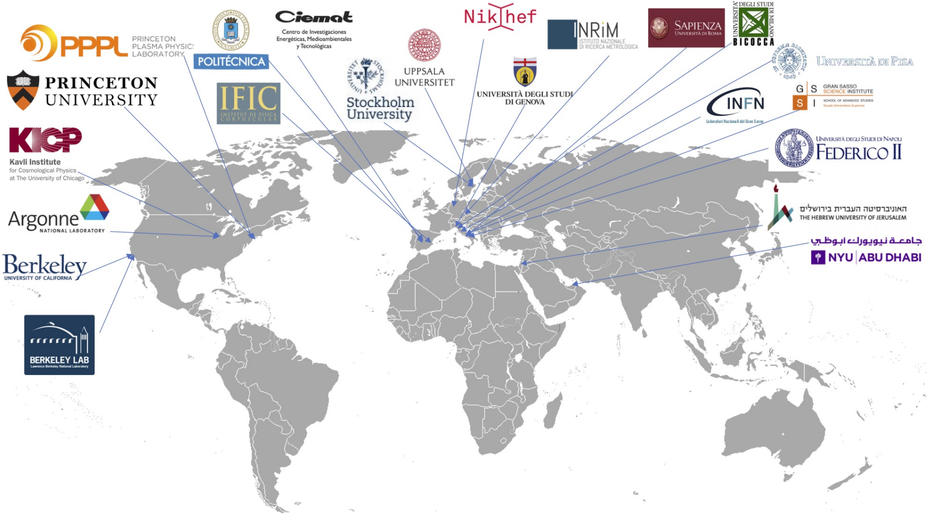
if $\boldsymbol{A}_{\rm CNB} > 0$ at $N\sigma$, direct detection of CNB accomplished at $N\sigma$

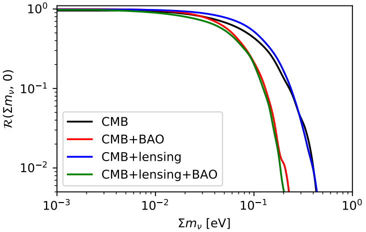
statistical only!

significance on $A_{\rm CNB} > 0$
as a function of $\hat{m}_{\rm lightest}$, Δ



PTOLEMY collaboration




$\mathcal{R}(\Sigma m_\nu, 0)$
Σm_ν [eV]
CMB
CMB+BAO
CMB+lensing
CMB+lensing+BAO

Conclusions

1

Be **careful** when you play
with **priors in Bayesian analysis!**
(and always declare your model completely)

2

Bayesian techniques allow
to **marginalize over different models/priors**
and to present
(nearly) model- and prior-independent results!

3

For the (far) future:
model independent neutrino properties (and more!)
from **direct detection of relic neutrinos**

Conclusions

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Thank you for the attention!