Wave Packet Treatment of Neutrino Oscillations

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- ⇝ Standard Theory of Neutrino Oscillations
- ⇝ Critical Discussion of Standard Assumptions
- ⇝ The Necessity of a Wave Packet Approach
- ⇝ Neutrino Wave Packets in Quantum Field Theory
- ⇝ Derivation of Neutrino Oscillations in QFT

New Dimensions in Astroparticle Physics

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Neutrino Unbound

http://www.to.infn.it/~giunti/NU

Standard Theory of Neutrino Oscillations in Vacuum

[Bilenky & Pontecorvo, Phys. Rep. 41 (1978) 225]

Neutrino Production: $j_\rho^{\mathrm{CC}\dagger} = 2 \sum_{\alpha=e,\mu,\tau} \overline{\ell_{\alpha L}} \gamma_\rho \nu_{\alpha L}$

Mixing of Fields: $\nu_{\alpha L} = \sum_k U_{\alpha k} \nu_{kL} \qquad (\alpha = e, \mu, \tau)$

Mixing of States: $|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle \qquad (\alpha = e, \mu, \tau)$

$$\langle 0 | \nu_{\alpha L} | \nu_\beta \rangle = \sum_{k,j} U_{\alpha k} U_{\beta j}^* \underbrace{\langle 0 | \nu_{kL} | \nu_j \rangle}_{\propto \delta_{kj}} \propto \sum_k U_{\alpha k} U_{\beta k}^* = \delta_{\alpha\beta}$$

$$\mathcal{H} |\nu_k\rangle = E_k |\nu_k\rangle \quad \Rightarrow \quad |\nu_k(t)\rangle = e^{-iE_k t} |\nu_k\rangle$$

$$|\nu_\alpha(t)\rangle = \sum_k U_{\alpha k}^* e^{-iE_k t} |\nu_k\rangle$$

$$|\nu_k\rangle = \sum_{\beta=e,\mu,\tau} U_{\beta k} |\nu_\beta\rangle$$

$$|\nu_\alpha(t)\rangle = \sum_{\beta=e,\mu,\tau} \underbrace{\left(\sum_k U_{\alpha k}^* e^{-iE_k t} U_{\beta k} \right)}_{\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t)} |\nu_\beta\rangle$$

Transition Probability:

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \left| \mathcal{A}_{\nu_\alpha \to \nu_\beta}(t) \right|^2$$

$$P_{\nu_\alpha\to\nu_\beta}(t) = \left| \sum_k U_{\alpha k}^* e^{-iE_k t} U_{\beta k} \right|^2$$

$$= \left| \cancel{e^{-iE_1 t}} \sum_k U_{\alpha k}^* e^{-i(E_k - E_1)t} U_{\beta k} \right|^2$$

The transition probability depends only on the energy differences $E_k - E_1$

RELATIVISTIC APPROXIMATION

$$E_k = \sqrt{p^2 + m_k^2} \simeq p + \frac{m_k^2}{2p} = E + \frac{m_k^2}{2E}$$

ASSUMPTION

$$p_k = p = E$$

$$E_k - E_j \simeq \frac{\Delta m_{kj}^2}{2E}$$

$$\Delta m_{kj}^2 \equiv m_k^2 - m_j^2$$

$$(E_k - E_j)\, t \simeq \frac{\Delta m_{kj}^2 L}{2E}$$

APPROXIMATION

$$t \simeq L$$

$$P_{\nu_\alpha\to\nu_\beta}(L) = \left| \sum_k U_{\alpha k}^* U_{\beta k} \exp\left(-i \frac{\Delta m_{k1}^2 L}{2E} \right) \right|^2$$

$$P_{\nu_\alpha \to \nu_\beta}(L) = \left| \sum_k U_{\alpha k}^* U_{\beta k} \exp\left(-i \frac{\Delta m_{k1}^2 L}{2E} \right) \right|^2$$

Two Generations: $k = 1, 2$

$$U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \qquad \Delta m^2 \equiv \Delta m_{21}^2 \equiv m_2^2 - m_1^2$$

Transition $(\alpha \neq \beta)$: $$P_{\nu_\alpha \to \nu_\beta}(L) = \sin^2 2\vartheta \sin^2\left(\frac{\Delta m^2 L}{4E} \right)$$

Survival $(\alpha = \beta)$: $$P_{\nu_\alpha \to \nu_\alpha}(L) = 1 - P_{\nu_\alpha \to \nu_\beta}(L)$$

General formula for any number of generations

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E} \right)$$

MAIN ASSUMPTIONS OF STANDARD THEORY

(A1) Neutrinos are extremely relativistic particles

(A2) Neutrinos produced in CC weak interaction processes together with charged leptons α^+ are described by the flavor state

$$|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle \qquad (\alpha = e, \mu, \tau)$$

U is the mixing matrix of neutrino fields:

$$\nu_{\alpha L} = \sum_k U_{\alpha k} \nu_{kL} \qquad (\alpha = e, \mu, \tau)$$

(A3) The massive neutrino states $|\nu_k\rangle$ have the same momentum $p_k = p$ (“Equal Momentum Assumption”) but different energies:

$$E_k = \sqrt{p^2 + m_k^2} \simeq p + \frac{m_k^2}{2p} = E + \frac{m_k^2}{2E}$$

(A4) Massive neutrinos time evolution given by Schrödinger equation

(A5) Propagation Time $T \simeq L$ Source-Detector Distance

! detectable neutrinos are extremely relativistic !

Only ν's with $E \gtrsim 0.2\,\text{MeV}$ are detectable!

⇝ CC and NC Weak Processes: Threshold

$$\nu + A \to \sum_X X \quad \Rightarrow \quad s = 2Em_A + m_A^2 \geq (\sum_X m_X)^2$$

$$\Rightarrow \quad E_{\text{th}} = \frac{(\sum_X m_X)^2}{2m_A} - \frac{m_A}{2}$$

☼ $\nu_e + {}^{37}\text{Cl} \to {}^{37}\text{Ar} + e^-$ $\quad E_{\text{th}} = 0.81\,\text{MeV}$

☼ $\nu_e + {}^{71}\text{Ga} \to {}^{71}\text{Ge} + e^-$ $\quad E_{\text{th}} = 0.233\,\text{MeV}$

☼ $\nu + d \to p + n + \nu$ $\quad E_{\text{th}} = 2.2\,\text{MeV}$

♁ $\bar{\nu}_e + p \to n + e^+$ $\quad E_{\text{th}} = 1.8\,\text{MeV}$

♁ $\nu_\mu + n \to p + \mu^-$ $\quad E_{\text{th}} = 110\,\text{MeV}$

♁ $\nu_\mu + e^- \to \nu_e + \mu^-$ $\quad E_{\text{th}} \simeq \frac{m_\mu^2}{2m_e} = 10.9\,\text{GeV}$

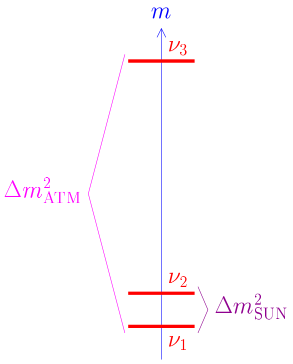
⇝ Elastic Scattering Processes: Cross Section $\propto$ Energy

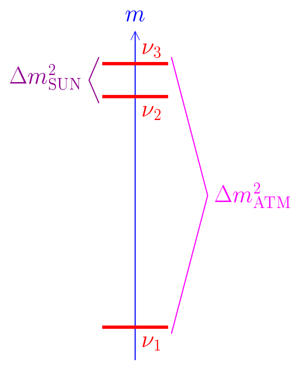
☼ $\nu + e^- \to \nu + e^-$ $\quad$ Background $\Rightarrow E_{\text{th}}^{\text{SK}} \simeq 5\,\text{MeV}$

$\sigma(E) \sim \sigma_0\, E/m_e \qquad \sigma_0 \sim 10^{-44}\,\text{cm}^2$

ABSOLUTE SCALE OF NEUTRINO MASSES

$$\Delta m^2_{\rm SUN} \sim 5 \times 10^{-5}\,{\rm eV}^2 \qquad \Delta m^2_{\rm ATM} \sim 2.5 \times 10^{-3}\,{\rm eV}^2$$

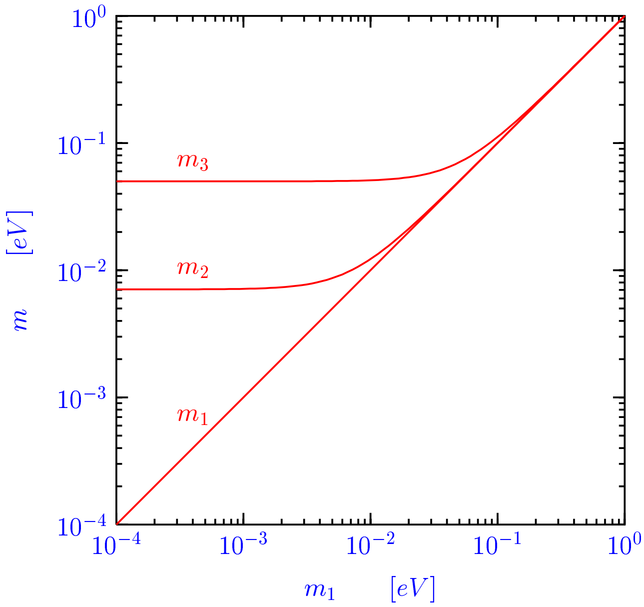


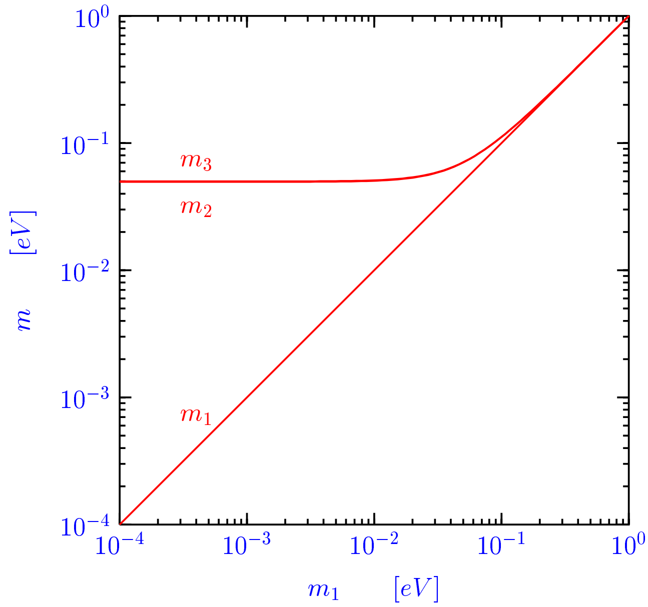


Tritium β-decay $\quad m_{\nu_e} \lesssim 2 - 3\,{\rm eV}$ (95% CL) [Mainz,Troitsk]

Cosmology $\quad \sum_{\text{light } \nu} m_\nu \lesssim 2 - 3\,{\rm eV}$ (95% CL)

[2dFGRS team, astro-ph/0204152] [Hannestad, astro-ph/0205223]





POSSIBLE: very small mixing of ν_e, ν_μ, ν_τ with heavy ν_k's

IN THIS CASE:

- heavy neutrino masses must be taken into account in calculation of production and detection rates
- oscillations due to large squared-mass differences are not observable $\Longrightarrow$ constant flavor-changing transition probability due to mixing $\Longrightarrow$ no coherence problem
- almost degenerate very heavy massive neutrinos (such that the corresponding squared-mass differences generate observable oscillations) seem very unlikely

IN THE FOLLOWING: we study oscillations due to light extremely relativistic massive neutrinos

RELATIVISTIC APPROXIMATION

$$E_k \simeq E + \xi \, \frac{m_k^2}{2E} \qquad\qquad p_k \simeq E - (1 - \xi) \, \frac{m_k^2}{2E}$$

ξ depends on production process

Easy Example of Neutrino Production

$$\pi^+ \to \mu^+ + \nu_\mu \qquad \pi^- \to \mu^- + \bar{\nu}_\mu$$

Two-Body Decay

$$\boxed{E_k^2 = p_k^2 + m_k^2}$$

Rest Frame of π

$$p_k^2 = \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 - \frac{m_k^2}{2}\left(1 + \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_k^4}{4\,m_\pi^2}$$

$$E_k^2 = \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 + \frac{m_k^2}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_k^4}{4\,m_\pi^2}$$

0^{th} order of Relativistic Approximation

$$m_k = 0 \Rightarrow p_k = E_k = E = \frac{m_\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 30\,\text{MeV}$$

1^{st} order of Relativistic Approximation

$$\boxed{E_k \simeq E + \xi\,\frac{m_k^2}{2E}} \qquad \boxed{p_k \simeq E - (1-\xi)\,\frac{m_k^2}{2E}}$$

$$\xi = \frac{1}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 0.2$$

FLAVOR STATES

Mixing of Fields: $\nu_{\alpha L} = \sum_k U_{\alpha k} \nu_{kL} \qquad (\alpha = e, \mu, \tau)$

Mixing of States: $|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle \qquad (\alpha = e, \mu, \tau)$

$$\langle 0|\nu_{\alpha L}|\nu_\beta\rangle = \sum_{k,j} U_{\alpha k} U_{\beta j}^* \underbrace{\langle 0|\nu_{kL}|\nu_j\rangle}_{\propto \delta_{kj}} \propto \sum_k U_{\alpha k} U_{\beta k}^* = \delta_{\beta\alpha}$$

NEUTRINO MASSES have been neglected!

$$\nu_k(x) = \int \mathrm{d}^3 p \sum_h \left[a_k(\vec{p}, h) u_k(\vec{p}, h) e^{-ip\cdot x} + b_k^\dagger(\vec{p}, h) v_k(\vec{p}, h) e^{ip\cdot x} \right]$$

$$|\nu_j(\vec{p}, h)\rangle = a_j^\dagger(\vec{p}, h)|0\rangle \qquad \left\{ a_k(\vec{p}, h)\, ,\, a_j^\dagger(\vec{p}\,', h') \right\} = \delta^3(\vec{p} - \vec{p}\,')\, \delta_{hh'}\, \delta_{kj}$$

$$\langle 0|\nu_{kL}(0)|\nu_j(\vec{p}, h)\rangle = u_{kL}(\vec{p}, h)\, \delta_{kj}$$

$$\langle 0|\nu_{\alpha L}(0)|\nu_\beta(\vec{p}, h)\rangle = \sum_k U_{\alpha k} U_{\beta k}^*\, u_{kL}(\vec{p}, h) \not\propto \delta_{\beta\alpha}$$

MIXING $\Rightarrow$ Flavor States are only approximations !

Crucial: EXTREMELY RELATIVISTIC NEUTRINOS!

States $|\nu_\alpha\rangle$ are not quanta of Field ν_α !

[Giunti & Kim & Lee, Phys. Rev. D 45 (1992) 2414]

EQUAL MOMENTUM?

(standard assumption)

$$p_k \simeq E - (1 - \xi)\,\frac{m_k^2}{2E}$$

SPECIAL CASE: $\xi = 1 \Rightarrow p_k = p_j = E$

in general it does not correspond to reality!

BUT

for extremely relativistic neutrinos

phase of $P_{\nu_\alpha \to \nu_\beta}$ is independent from ξ

$\Downarrow$

Equal Momentum Assumption gives

correct oscillation phase

DIFFERENT MOMENTUM CONTRIBUTIONS

$\Updownarrow$

LORENTZ INVARIANT

OSCILLATION PROBABILITY

$$P_{\nu_\alpha \to \nu_\beta}(L,T) = \left| \sum_k U_{\alpha k}^* \, e^{i p_k L - i E_k t} \, U_{\beta k} \right|^2$$

[Dolgov & Morozov & Okun & Schepkin, Nucl. Phys. B 502 (1997) 3, hep-ph/9703241]
[Dolgov, hep-ph/0004032]
[Giunti & Kim, Found. Phys. Lett. 14 (2001) 213, hep-ph/0011074]
[Bilenky & Giunti, Int. J. Mod. Phys. A 16 (2001) 3931, hep-ph/0102320]

DIFFERENT OBSERVERS MEASURE

THE SAME TRANSITION PROBABILITY !

FLAVOR IS LORENTZ INVARIANT !

EQUAL ENERGY?

[Grossman & Lipkin, Phys. Rev. D 55 (1997) 2760, hep-ph/9607201]

[Lipkin, Phys. Lett. B 477 (2000) 195, hep-ph/9907551]

$$E_k \simeq E + \xi \, \frac{m_k^2}{2E}$$

SPECIAL CASE: $\xi = 0 \Rightarrow E_k = E_j = E$

in general it does not correspond to reality!

BUT

for extremely relativistic neutrinos

phase of $P_{\nu_\alpha \to \nu_\beta}$ is independent from ξ

$\Downarrow$

also Equal Energy Assumption gives

correct oscillation phase

With the Equal Energy Assumption the approximation $T \simeq L$ is not needed:

$$\phi_{kj} = (p_k - p_j)\, L - (E_k - E_j)\, T = (p_k - p_j)\, L \simeq -\frac{\Delta m_{kj}^2 L}{2E}$$

Equal Energy Assumption AND Equal Momentum Assumption ARE INCOMPATIBLE WITH LORENTZ INVARIANCE !

[Giunti, Mod. Phys. Lett. A 16 (2001) 2363, hep-ph/0104148]

Assume for illustration that in a Lorentz frame S

$$E_k = E \quad \Rightarrow \quad p_k = \sqrt{E^2 - m_k^2} \simeq \underbrace{E}_{\text{Energy of massless } \nu} - \frac{m_k^2}{2E}$$

In another frame S' with velocity v along the neutrino path

$$E'_k = \gamma\,(E_k + v\,p_k) = \underbrace{\gamma\,(1+v)\,E}_{E'} - \boxed{\gamma v\,\frac{m_k^2}{2E}}$$

ENERGIES ARE DIFFERENT!

$$\boxed{\Delta E'_{kj} = -\frac{v}{1-v}\,\frac{m_k^2}{2E'}} \qquad \boxed{\Delta p'_{kj} = -\frac{1}{1-v}\,\frac{m_k^2}{2E'}}$$

$\Delta E'_{kj} \sim \Delta p'_{kj}$ for relativistic velocities!

COMMON IN PRACTICE!

EXAMPLE : $\pi^+ \to \mu^+ + \nu_\mu$

Assume for sake of illustration S with $E_k = E$ is

rest frame of π

Many experiments measure oscillations of ν's produced in

π decay in flight

(short and long baseline, atmospheric ν experiments)

$$E_\pi \sim 100\,\mathrm{MeV} - 100\,\mathrm{GeV}$$

Example : $E_\pi \simeq 200\,\mathrm{MeV} \quad \Rightarrow \quad v \simeq 0.71$

$$\frac{v}{1-v} \simeq 2.4 \qquad\qquad \frac{1}{1-v} \simeq 3.4$$

$$\Delta E'_{kj} = -\frac{v}{1-v}\,\frac{m_k^2}{2E'} \quad \sim \quad \Delta p'_{kj} = -\frac{1}{1-v}\,\frac{m_k^2}{2E'}$$

SAME ORDER OF MAGNITUDE!

For higher energies $\Delta E'_{kj} \sim \Delta p'_{kj}$!

LORENTZ INVARIANCE

$\Downarrow$

EQUAL ENERGY OR MOMENTUM ASSUMPTION

CANNOT BE VALID IN ALL

NEUTRINO OSCILLATION EXPERIMENTS

ACTUALLY, IT CANNOT BE VALID EVEN IN ONE EXPERIMENT IF π HAVE A SPECTRUM OF ENERGY (IN PRACTICE ALWAYS)

CONCLUSION:

FORGET

EQUAL ENERGY OR MOMENTUM ASSUMPTION

Lorentz invariant probability of $\nu_\alpha \to \nu_\beta$ transitions:

$$P_{\nu_\alpha\to\nu_\beta}(L,T) = \left| \sum_k U_{\alpha k}^* \, e^{ip_k L - iE_k T} \, U_{\beta k} \right|^2$$

$$= \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2$$

$$+2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \, e^{i(p_k - p_j)L - i(E_k - E_j)T}$$

RELATIVISTIC APPROXIMATION

$$p_k - p_j \simeq -(1-\xi)\,\frac{\Delta m_{kj}^2}{2E} \qquad\qquad E_k - E_j \simeq \xi\,\frac{\Delta m_{kj}^2}{2E}$$

$$\Downarrow$$

$$P_{\nu_\alpha\to\nu_\beta}(L,T) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2$$

$$+ 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* e^{-i(1-\xi)\frac{\Delta m_{kj}^2}{2E} L - i\xi \frac{\Delta m_{kj}^2}{2E} T}$$

OSCILLATIONS IN SPACE AND TIME!

Real Experiments: $\begin{cases} T \text{ not measured} \\ L \text{ (}\pm\text{ approximately) known} \end{cases}$

$\Downarrow$

OSCILLATIONS IN SPACE!

$\Downarrow$

T must be expressed in terms of L

$\Downarrow$

WAVE PACKETS

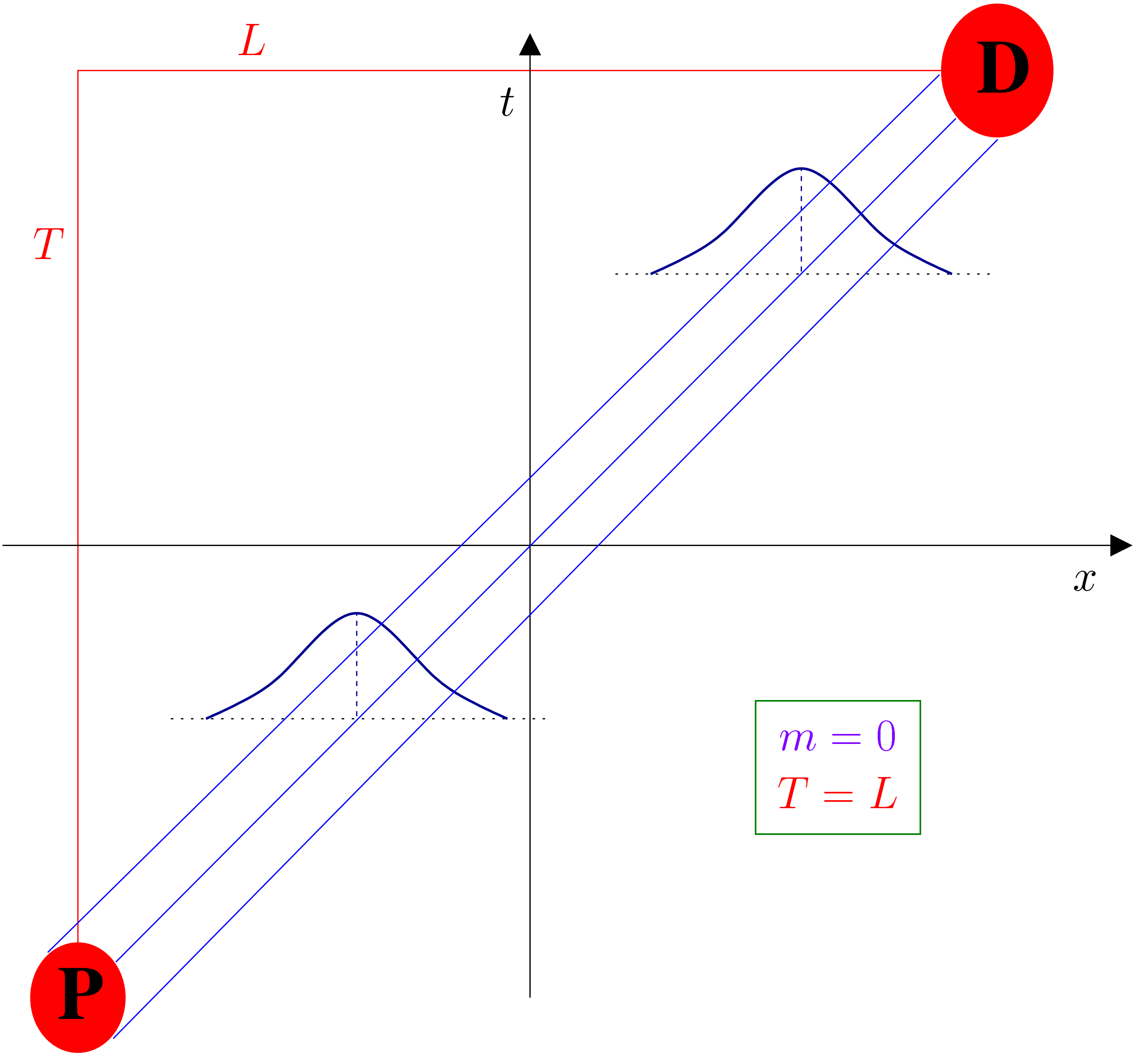
[Kayser, Phys. Rev. D 24 (1981) 110]

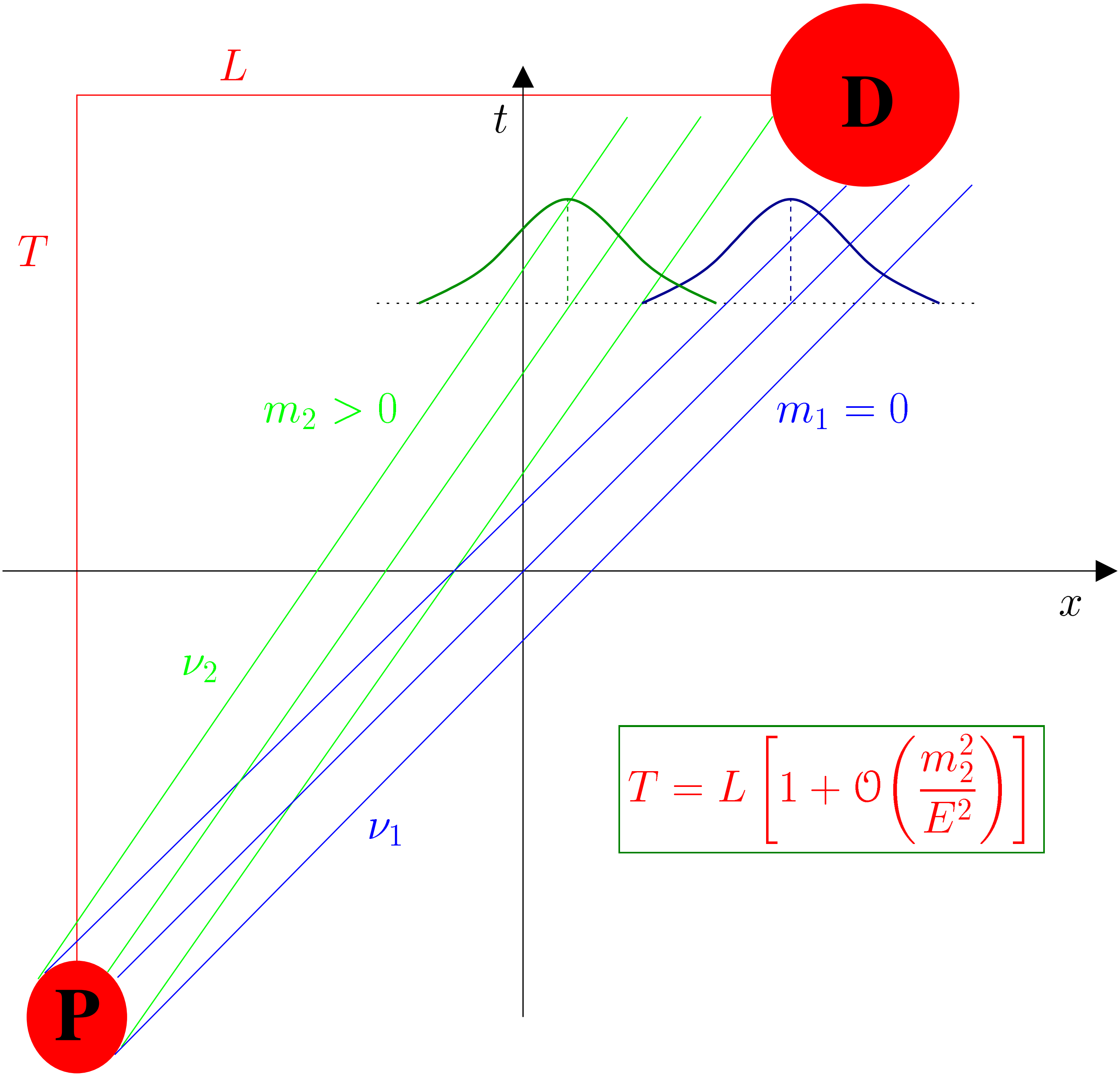
standard least order approximation: $T = L$

$\Downarrow$

$$P_{\nu_\alpha \to \nu_\beta}(L,T) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* e^{-i \frac{\Delta m_{kj}^2}{2E} L}$$

ξ has magically disappeared!


L
t
D
T
x
P
$m = 0$
$T = L$



L
T
D
P
t
x
$m_2 > 0$
$m_1 = 0$
ν_2
ν_1
$T = L\left[1 + \mathcal{O}\left(\frac{m_2^2}{E^2}\right)\right]$

MASSIVE NEUTRINO WAVE PACKETS

Size of neutrino wave packets is determined by coherence size of the Production Process:

$$\sigma_x \sim \delta t_{\mathrm{P}}$$

($\delta t_{\mathrm{P}} \gtrsim \delta x_{\mathrm{P}}$, coherence region must be causally connected)

Velocity of neutrino wave packets: $v_k = \frac{p_k}{E_k} \simeq 1 - \frac{m_k^2}{2E^2}$

Wave packets arrive at Detection Process at different times:

$$t_k = \frac{L}{v_k} \simeq L\left(1 + \frac{m_k^2}{2E^2}\right) \qquad m_k > m_j \Rightarrow v_k < v_j \Rightarrow t_k > t_j$$

$$\delta t_{kj} \equiv t_k - t_j \simeq \frac{\Delta m_{kj}^2}{2E^2}\,L \qquad \text{time separation}$$

$$\frac{\Sigma t_{kj}}{2} \equiv \frac{t_k + t_j}{2} \simeq L + \frac{1}{2}\,\frac{\Sigma m_{kj}^2}{2E^2}\,L \qquad \text{average time}$$

$$\Delta m_{kj}^2 \equiv m_k^2 - m_j^2 \qquad \Sigma m_{kj}^2 \equiv m_k^2 + m_j^2$$

$$\text{Range of T} \simeq [t_j, t_k] = \frac{1}{2}(\Sigma t_{kj} \pm \delta t_{kj})$$

$$\simeq L + \frac{\Sigma m_{kj}^2 \pm \Delta m_{kj}^2}{4E^2} L$$

Phase of oscillations:

$$\Phi_{kj} = -(1-\xi)\frac{\Delta m_{kj}^2}{2E} L - \xi \frac{\Delta m_{kj}^2}{2E} T$$

$$\simeq -(1-\xi)\frac{\Delta m_{kj}^2 L}{2E} - \xi \frac{\Delta m_{kj}^2 L}{2E} + \ldots$$

$$= -\underbrace{\frac{\Delta m_{kj}^2 L}{2E}}_{\text{leading}} - \underbrace{\frac{\Delta m_{kj}^2 L}{2E}\,\xi\,\frac{\Sigma m_{kj}^2 \pm \Delta m_{kj}^2}{4E^2}}_{\text{correction}}$$

flux energy spectrum
\+
detector energy resolution
\+
distance uncertainty

$\Rightarrow$ oscillations observable only if $\phi_{kj} \sim 1$

$$\frac{\Delta m_{kj}^2 L}{2E} \sim 1 \Rightarrow \underbrace{\frac{\Delta m_{kj}^2 L}{2E}\,\xi\,\frac{\Sigma m_{kj}^2 \pm \Delta m_{kj}^2}{4E^2}}_{\text{negligible}} \ll 1$$

Phase practically constant in time interval $[t_j, t_k]$:

$$\boxed{\Phi_{kj} \simeq -\frac{\Delta m_{kj}^2}{2E} L}$$

NEUTRINO WAVE PACKETS

$\Downarrow$

$T = L$ is correct approximation

$\Downarrow$

Phase : $$\Phi_{kj} \simeq -\frac{\Delta m^2_{kj}}{2E} L$$

VERY IMPORTANT: ξ has magically disappeared $\Rightarrow$ ν oscillations are independent from the specific details of the production process (crucial: relativistic neutrinos)

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} e^{-i \frac{\Delta m^2_{kj} L}{2E}}$$

$$\phi_{kj} = 2\pi \Rightarrow L^{\text{osc}}_{kj} = \frac{4\pi E}{\Delta m^2_{kj}}$$ Oscillation Length

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} e^{-2\pi i \frac{L}{L^{\text{osc}}_{kj}}}$$

COHERENCE LENGTH

[Nussinov, Phys. Lett. B 63 (1976) 201]

[Kiers & Nussinov & Weiss, Phys. Rev. D 53 (1996) 537, hep-ph/9506271]

Wave Packets have different velocity and separate

Different massive neutrinos can interfere

if

wave packets arrive with $\delta t_{kj} < \delta t_{\text{D}}$

This happens for $L \lesssim L^{\text{coh}}_{kj}$

Separation : $|\delta t_{kj}| \simeq |\delta x_{kj}| = |v_k - v_j|\, T \simeq \dfrac{|\Delta m^2_{kj}|}{2E^2} L$

Size of wave packets : $\sigma_x \sim \delta t_{\text{P}}$

Maximal separation for interference:

$$|\delta x|^2_{\text{max}} \simeq |\delta t|^2_{\text{max}} \sim \sigma_x^2 + \delta t_{\text{D}}^2 \sim \delta t_{\text{P}}^2 + \delta t_{\text{D}}^2$$

$$|\delta x_{kj}| \sim |\delta x|_{\text{max}} \quad \Rightarrow \quad \boxed{L^{\text{coh}}_{kj} \sim \frac{2E^2}{|\Delta m^2_{kj}|} |\delta x|_{\text{max}}}$$

Transition from $L \ll L_{kj}^{\text{coh}}$ to $L \gg L_{kj}^{\text{coh}}$?

$\Updownarrow$

WAVE PACKET TREATMENT

Effects of Production and Detection Processes ?

$\Updownarrow$

QUANTUM FIELD THEORY

Quantum Field Theory of Neutrino Oscillations with external particles in Production and Detection processes described by wave packets and intermediate virtual neutrino

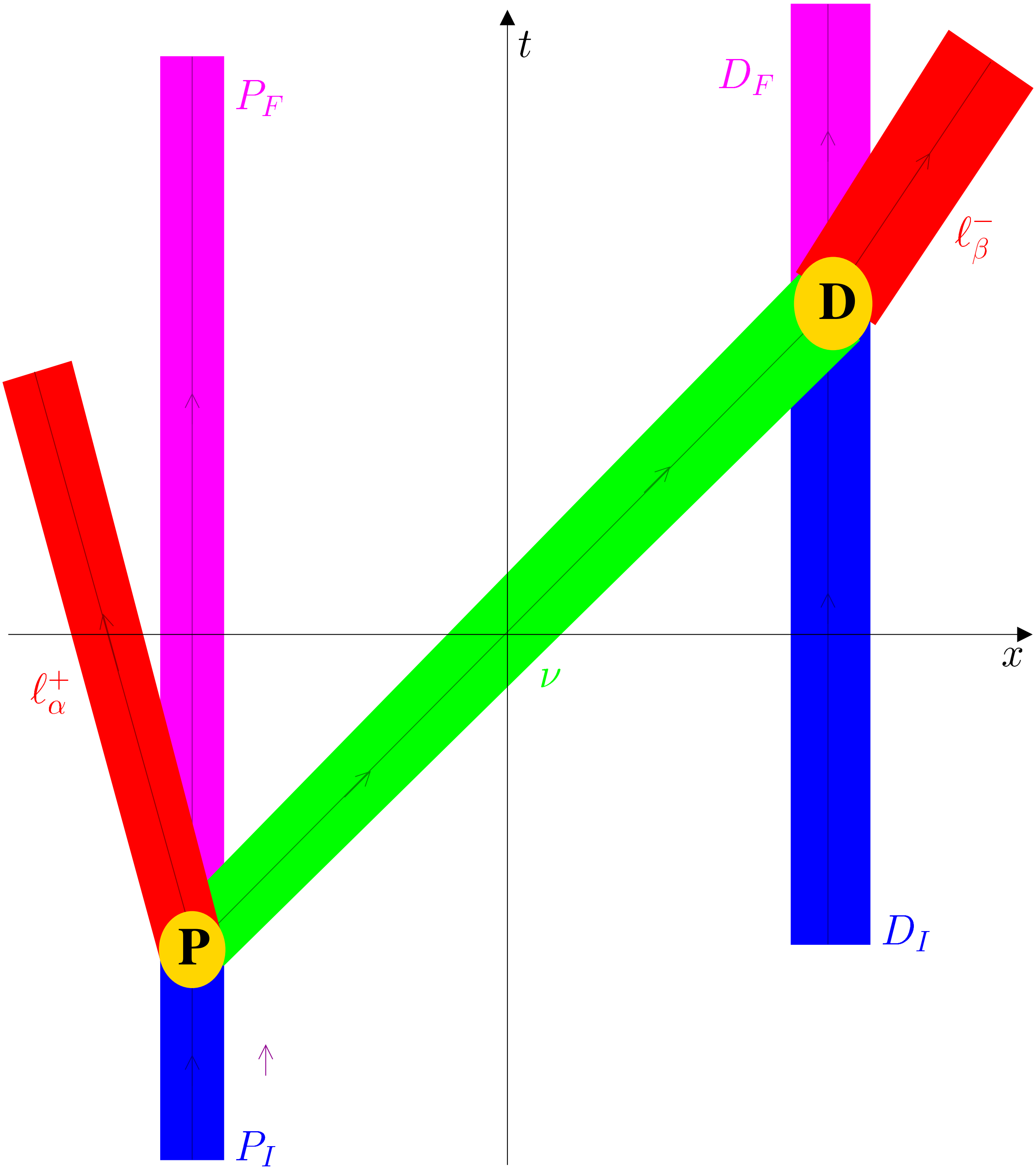
[Giunti & Kim & Lee & Lee, Phys. Rev. D 48 (1993) 4310, hep-ph/9305276]
[Giunti & Kim & Lee, Phys. Lett. B 421 (1998) 237, hep-ph/9709494]

confirm standard oscillation length and existence of coherence length

problem: neutrino has no properties !

in oscillation experiments neutrinos propagate as free particles over macroscopically large distance, sometimes astronomical distances

it must be possible to describe neutrinos in oscillation experiments with appropriate state



$$P_I \to P_F + \ell_\alpha^+ + \nu_\alpha$$

$$\downarrow$$

$$\nu_\beta + D_I \to D_F + \ell_\beta^-$$

Neutrino Wave Packets in Quantum Field Theory

[Giunti, hep-ph/0205014]

In Quantum Field Theory

$$|f\rangle \propto (\mathcal{S} - \mathbf{1})|i\rangle \simeq -i \int \mathrm{d}^4x \, \mathcal{H}_I(x) \, |i\rangle$$

Production Process: $P_I \to P_F + \ell_\alpha^+ + \nu_\alpha$

Entangled Final State:

$$|\widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle \propto -i \int \mathrm{d}^4x \, \mathcal{H}_I^P(x) \, |P_I\rangle$$

Disentangled by Interaction with Surrounding Medium (Measurement):

$$\begin{aligned} |\nu_\alpha\rangle &\propto \langle P_F, \ell_\alpha^+ | \widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle \\ &\propto \langle P_F, \ell_\alpha^+ | -i \int \mathrm{d}^4x \, \mathcal{H}_I^P(x) \, |P_I\rangle \end{aligned}$$

Effective Interaction Hamiltonian:

$$\begin{aligned} \mathcal{H}_I^P(x) &= \frac{G_F}{\sqrt{2}} \, \bar{\nu}_\alpha(x) \, \gamma^\rho \, (1 - \gamma_5) \, \ell_\alpha(x) \, J_\rho^P(x) \\ &= \frac{G_F}{\sqrt{2}} \sum_k U_{\alpha k}^* \, \bar{\nu}_k(x) \, \gamma^\rho \, (1 - \gamma_5) \, \ell_\alpha(x) \, J_\rho^P(x) \end{aligned}$$

Production Process Localization:

$$|\chi\rangle = \int \mathrm{d}^3p\, \psi_\chi(\vec{p};\vec{p}_\chi,\sigma_{p\chi})|\chi(\vec{p})\rangle \qquad (\chi = P_I, P_F, \ell_\alpha^+)$$

Gaussian momentum distribution:

$$\psi_\chi(\vec{p};\vec{p}_\chi,\sigma_{p\chi}) = \left(2\pi\sigma_{p\chi}^2\right)^{-3/4} \exp\left[-\frac{(\vec{p}-\vec{p}_\chi)^2}{4\sigma_{p\chi}^2}\right]$$

Wave function:

$$\psi_\chi(\vec{x},t;\vec{p}_\chi,\sigma_{p\chi}) = \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\, \psi_\chi(\vec{p};\vec{p}_\chi,\sigma_{p\chi})\, e^{-iE_\chi(\vec{p})t+i\vec{p}\vec{x}}$$

Energy:

$$E_\chi(\vec{p}) = \sqrt{\vec{p}^2 + m_\chi^2} \simeq E_\chi + \vec{v}_\chi\left(\vec{p}-\vec{p}_\chi\right)$$

Average Energy:

$$E_\chi \equiv E_\chi(\vec{p}_\chi) = \sqrt{\vec{p}_\chi^2 + m_\chi^2}$$

Group Velocity:

$$\vec{v}_\chi \equiv \left.\frac{\partial E_\chi}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_\chi} = \frac{\vec{p}_\chi}{E_\chi}$$

From Gaussian integration over d^3p:

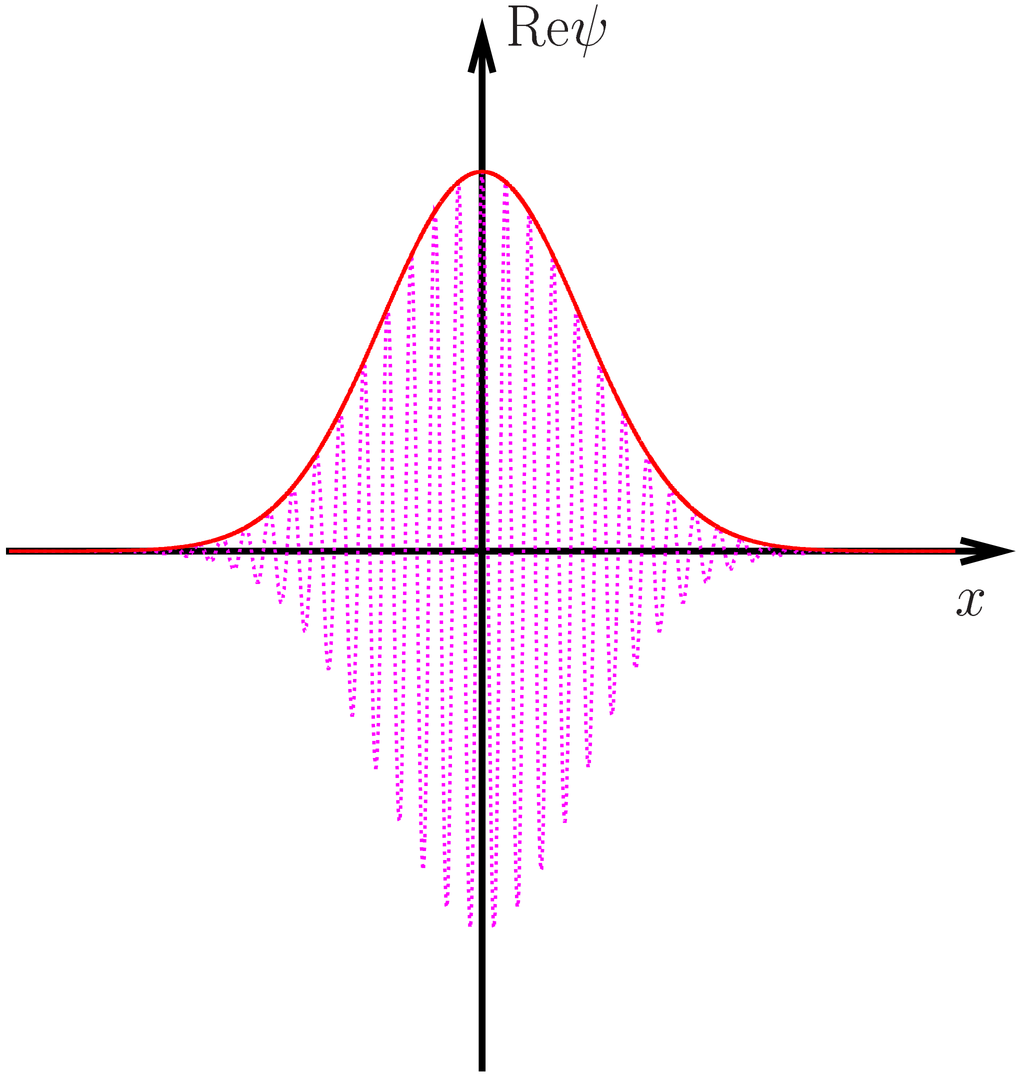
$$\psi_\chi(\vec{x}, t; \vec{p}_\chi, \sigma_{p\chi}) \simeq \left(2\pi\sigma_{x\chi}^2\right)^{-3/4} \exp\left[-iE_\chi t + i\vec{p}_\chi \cdot \vec{x} - \frac{(\vec{x} - \vec{v}_\chi t)^2}{4\sigma_{x\chi}^2}\right]$$

Space and Momentum uncertainties:

$$\sigma_{x\chi}\,\sigma_{p\chi} = \frac{1}{2}$$

Squared Energy uncertainty:

$$\langle(\delta E)^2\rangle_\chi = \langle\chi|(\widehat{E} - E_\chi)^2|\chi\rangle = \vec{v}_\chi^{\,2}\,\sigma_{p\chi}^2$$ [Beuthe]



Neutrino state:

$$|\nu_\alpha\rangle \propto \sum_k U_{\alpha k}^* \int d^3p$$
$$\times \int d^3p'_{P_F} \, \psi^*_{P_F}(\vec{p}'_{P_F}; \vec{p}_{P_F}, \sigma_{pP_F})$$
$$\times \int d^3p'_{\ell_\alpha^+} \, \psi^*_{\ell_\alpha^+}(\vec{p}'_{\ell_\alpha^+}; \vec{p}_{\ell_\alpha^+}, \sigma_{\ell_\alpha^+})$$
$$\times \int d^3p'_{P_I} \, \psi_{P_I}(\vec{p}'_{P_I}; \vec{p}_{P_I}, \sigma_{pP_I})$$
$$\times \sum_h \bar{u}_{\nu_k}(\vec{p}, h)\gamma^\rho (1-\gamma_5) \, v_{\ell_\alpha^+}(\vec{p}'_{\ell_\alpha^+}, h_{\ell_\alpha^+}) \, J^P_\rho(\vec{p}'_{P_F}, \vec{p}'_{P_I})$$
$$\times \int d^4x \, e^{i(p+p'_{\ell_\alpha^+}+p'_{P_F}-p'_{P_I})x} \, |\nu_k(\vec{p}, h)\rangle$$

From integrations over $d^3p'_{P_F}$, $d^3p'_{\ell_\alpha^+}$, $d^3p'_{P_I}$, d^4x:

$$|\nu_\alpha\rangle = N_\alpha \sum_k U_{\alpha k}^* \int d^3p \, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p}, h) \, |\nu_k(\vec{p}, h)\rangle$$

Amplitudes:

$$\mathcal{A}_k^P(\vec{p}, h) = \bar{u}_{\nu_k}(\vec{p}, h) \, \gamma^\rho (1-\gamma_5) \, v_{\ell_\alpha^+}(\vec{p}_{\ell_\alpha^+}, h_{\ell_\alpha^+}) \, J^P_\rho(\vec{p}_{P_F}, \vec{p}_{P_I})$$

$e^{-S_k^P(\vec{p})}$ replaces energy-momentum δ-function

$$S_k^P(\vec{p}) \equiv \frac{(\vec{p}_P - \vec{p})^2}{4\,\sigma_{pP}^2} + \frac{\left[E_P - E_{\nu_k}(\vec{p}) - (\vec{p}_P - \vec{p}) \cdot \vec{v}_P\right]^2}{4\,\sigma_{pP}^2 \lambda_P}$$

$$\sigma_{pP} = 0 \quad \Longrightarrow \quad \vec{p} = \vec{p}_P \,, \quad E_{\nu_k}(\vec{p}) = E_P$$

Energy: $E_P \equiv E_{P_I} - E_{P_F} - E_{\ell_\alpha^+}$

Momentum: $\vec{p}_P \equiv \vec{p}_{P_I} - \vec{p}_{P_F} - \vec{p}_{\ell_\alpha^+}$

Space Uncertainty: $\displaystyle \frac{1}{\sigma_{xP}^2} \equiv \frac{1}{\sigma_{xP_I}^2} + \frac{1}{\sigma_{xP_F}^2} + \frac{1}{\sigma_{x\ell_\alpha^+}^2}$

Momentum Uncertainty: $\sigma_{pP}^2 = \sigma_{pP_I}^2 + \sigma_{pP_F}^2 + \sigma_{p\ell_\alpha^+}^2$

$$\vec{v}_P \equiv \sigma_{xP}^2 \left(\frac{\vec{v}_{P_I}}{\sigma_{xP_I}^2} + \frac{\vec{v}_{P_F}}{\sigma_{xP_F}^2} + \frac{\vec{v}_{\ell_\alpha^+}}{\sigma_{x\ell_\alpha^+}^2} \right) \qquad (0 \leq |\vec{v}_P| \leq 1)$$

$$\Sigma_P \equiv \sigma_{xP}^2 \left(\frac{\vec{v}_{P_I}^2}{\sigma_{xP_I}^2} + \frac{\vec{v}_{P_F}^2}{\sigma_{xP_F}^2} + \frac{\vec{v}_{\ell_\alpha^+}^2}{\sigma_{x\ell_\alpha^+}^2} \right) \qquad (0 \leq \Sigma_P \leq 1)$$

$$\lambda_P \equiv \Sigma_P - \vec{v}_P^2 \qquad (0 \leq \lambda_P \leq 1)$$

$|\nu_\alpha\rangle$ is a superposition of massive neutrino wave packets

$$|\nu_k\rangle = N_k \int \mathrm{d}^3p \, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p},h) \, |\nu_k(\vec{p},h)\rangle$$

Average Momentum $\vec{p}_k$: $\left.\dfrac{\partial S_k^P(\vec{p})}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_k} = 0 \,, \quad \vec{p}_k = p_k \vec{\ell}$

Average Energy: $E_k \equiv E_{\nu_k}(\vec{p}_k) = \sqrt{\vec{p}_k^{\,2} + m_k^2}$

Group Velocity: $\vec{v}_k \equiv \left.\dfrac{\partial E_{\nu_k}(\vec{p})}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_k} = \dfrac{\vec{p}_k}{E_k}$

Extremely Relativistic Neutrinos:

$$E_k \simeq E + \xi \, \frac{m_k^2}{2E} \qquad\qquad p_k \simeq E - (1-\xi)\, \frac{m_k^2}{2E}$$

$$\vec{p}_P = E_P \, \vec{\ell} \qquad E = E_P \qquad \xi = \frac{\lambda_P - \vec{\ell}\cdot\vec{v}_P \left(1 - \vec{\ell}\cdot\vec{v}_P\right)}{\lambda_P + \left(1 - \vec{\ell}\cdot\vec{v}_P\right)^2}$$

Squared Energy-Momentum Uncertainties:

$$\langle(\delta p)^2\rangle_k \sim \langle(\delta E)^2\rangle_k \sim \sigma_{pP}^2$$

Detection Process at $(\vec{L}, T)$: $|\nu_\alpha(\vec{L}, T)\rangle = e^{-i\widehat{E}T + i\widehat{\vec{P}}\cdot\vec{L}} |\nu_\alpha\rangle$

$$|\nu_\alpha(\vec{L}, T)\rangle = N_\alpha \sum_k U_{\alpha k}^* \int \mathrm{d}^3 p \, e^{-iE_{\nu_k}(\vec{p})T + i\vec{p}\cdot\vec{L}} \times e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p}, h) \, |\nu_k(\vec{p}, h)\rangle$$

Detection Process: $\nu_\beta + D_I \to D_F + \ell_\beta^-$

Detection Amplitude:

$$\mathcal{A}_{\alpha\beta}(\vec{L}, T) = \langle D_F, \ell_\beta^- | -i \int \mathrm{d}^4 x \, \mathcal{H}_I^D(x) \, |D_I, \nu_\alpha(\vec{L}, T)\rangle$$

Effective Interaction Hamiltonian:

$$\begin{aligned} \mathcal{H}_I^D(x) &= \frac{G_F}{\sqrt{2}} \, \bar{\ell}_\beta(x) \, \gamma^\rho \, (1 - \gamma_5) \, \nu_\beta(x) \, J_\rho^D(x) \\ &= \frac{G_F}{\sqrt{2}} \sum_j U_{\beta j} \, \bar{\ell}_\beta(x) \, \gamma^\rho \, (1 - \gamma_5) \, \nu_j(x) \, J_\rho^D(x) \end{aligned}$$

Result:

$$\mathcal{A}_{\alpha\beta}(\vec{L}, T) \propto \sum_k U_{\alpha k}^* U_{\beta k} \sum_h \int \mathrm{d}^3 p \, \mathcal{A}_k^P(\vec{p}, h) \mathcal{A}_k^D(\vec{p}, h) \, e^{-S_k(\vec{p})} \times \exp\left[-iE_{\nu_k}(\vec{p})T + i\vec{p}\cdot\vec{L}\right]$$

$$S_k(\vec{p}) = S_k^P(\vec{p}) + S_k^D(\vec{p})$$

$$S_k(\vec{p}) = S_k^P(\vec{p}) + S_k^D(\vec{p})$$

$$= \frac{(\vec{p}_P - \vec{p})^2}{4\,\sigma_{pP}^2} + \frac{[(E_P - E_{\nu_k}(\vec{p})) - (\vec{p}_P - \vec{p}) \cdot \vec{v}_P]^2}{4\,\sigma_{pP}^2 \lambda_P}$$

$$+ \frac{(\vec{p}_D - \vec{p})^2}{4\,\sigma_{pD}^2} + \frac{[(E_D - E_{\nu_k}(\vec{p})) - (\vec{p}_D - \vec{p}) \cdot \vec{v}_D]^2}{4\,\sigma_{pD}^2 \lambda_D}$$

$e^{-S_k(\vec{p})}$ replaces energy-momentum δ-function

$$\sigma_{pP} = 0 \implies \vec{p} = \vec{p}_P\,, \quad E_{\nu_k}(\vec{p}) = E_P$$

$$\sigma_{pD} = 0 \implies \vec{p} = \vec{p}_D\,, \quad E_{\nu_k}(\vec{p}) = E_D$$

$\Downarrow$

only one massive neutrino contribution

$\Downarrow$

! no oscillations !

Integration over $\mathrm{d}^3 p$ with saddle-point approximation around minimum of $S_k(\vec{p})$:

$$\left.\frac{\partial S_k}{\partial \vec{p}}\right|_{\vec{p}=\vec{q}_k} = 0 \qquad \text{Energy:} \quad \varepsilon_k \equiv E_{\nu_k}(\vec{q}_k) = \sqrt{\vec{q}_k^{\,2} + m_k^2}$$

! In general $\vec{q}_k, \varepsilon_k \neq \vec{p}_k, E_k$!

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \sum_k U_{\alpha k}^* U_{\beta k} \sum_h \frac{\mathcal{A}_k^P(\vec{q}_k,h)\mathcal{A}_k^D(\vec{q}_k,h)}{\sqrt{\mathrm{Det}\Omega_k}} \, e^{-S_k(\vec{q}_k)}$$

$$\times \exp\left[-i\varepsilon_k T + i q_k L - \frac{(L - u_k T)^2}{4\eta_k^2}\right]$$

Extremely relativistic neutrinos:

$$\varepsilon_k \simeq E + \rho\,\frac{m_k^2}{2E} \qquad q_k \simeq E - (1-\rho)\,\frac{m_k^2}{2E} \qquad u_k \simeq 1 - \frac{m_k^2}{2E^2}$$

$$\vec{p}_P = E_P\,\vec{\ell} \qquad \vec{p}_D = E_D\,\vec{\ell} \qquad \vec{q}_k = q_k\,\vec{\ell} \qquad E_P = E_D = E$$

$$\sigma_x^2 = \sigma_{xP}^2 + \sigma_{xD}^2 \qquad \frac{1}{\sigma_p^2} = \frac{1}{\sigma_{pP}^2} + \frac{1}{\sigma_{pD}^2} \qquad \eta_k \sim \sigma_x$$

$$\rho = \frac{\frac{1}{\sigma_p^2} - \frac{\vec{\ell}\cdot\vec{v}_P\left(1-\vec{\ell}\cdot\vec{v}_P\right)}{\sigma_{pP}^2\,\lambda_P} - \frac{\vec{\ell}\cdot\vec{v}_D\left(1-\vec{\ell}\cdot\vec{v}_D\right)}{\sigma_{pD}^2\,\lambda_D}}{\frac{1}{\sigma_p^2} + \frac{\left(1-\vec{\ell}\cdot\vec{v}_P\right)^2}{\sigma_{pP}^2\,\lambda_P} + \frac{\left(\vec{\ell}\cdot\vec{v}_D - 1\right)^2}{\sigma_{pD}^2\,\lambda_D}}$$

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \sum_k U_{\alpha k}^* U_{\beta k} \exp\left[-i\varepsilon_k T + i q_k L - \frac{(L-u_k T)^2}{4\eta^2}\right]$$

Formally $\mathcal{A}_{\alpha\beta}(\vec{L},T)$ can be obtained in the standard way:

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \langle \nu_\beta | \nu_\alpha(\vec{L},T) \rangle$$ with

$$|\nu_\beta\rangle = N_\beta \sum_k U_{\beta k}^* \int \mathrm{d}^3 p \, e^{-S_k^D(\vec{p})} \sum_h \mathcal{A}_k^D(\vec{p},h) \, |\nu_k(\vec{p},h)\rangle$$

But calculation of coefficients of massive neutrino components of $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle$ needs Quantum Field Theory!

Space-Time Transition Probability:

$$P_{\alpha\beta}(\vec{L},T) \propto |\mathcal{A}_{\alpha\beta}(\vec{L},T)|^2$$

Transition Probability in Space:

$$P_{\alpha\beta}(\vec{L}) \propto \int \mathrm{d}T \, |\mathcal{A}_{\alpha\beta}(\vec{L},T)|^2$$

Transition Probability in Space:

$$P_{\alpha\beta}(\vec{L}) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \times \exp\left[-2\pi i \frac{L}{L_{ab}^{\mathrm{osc}}} - \left(\frac{L}{L_{ab}^{\mathrm{coh}}} \right)^2 - 2\pi^2 \rho^2 \omega \left(\frac{\sigma_x}{L_{ab}^{\mathrm{osc}}} \right)^2 \right]$$

Oscillation Lengths: $L_{ab}^{\mathrm{osc}} = \frac{4\pi E}{|\Delta m_{ab}^2|}$

Coherence Lengths: $L_{ab}^{\mathrm{coh}} = \frac{4\sqrt{2\omega} E^2}{|\Delta m_{ab}^2|} \sigma_x$

Normally $\rho \neq 0$ and $\omega \sim 1$

Necessary Localization: $\sigma_x \ll L_{ab}^{\mathrm{osc}}$

Special unrealistic possibility:

$$\rho = 0 \,, \quad \omega \to \infty \,, \quad \rho^2 \omega = 0 \quad \Longrightarrow \quad L_{ab}^{\mathrm{coh}} \to \infty$$

Example: $\vec{v}_{D_I} = 0 \,, \ \sigma_{pD_F} = \sigma_{p\ell_\beta^-} = 0 \Rightarrow \rho = 0 \,, \ \omega = \infty$

if $\sigma_{xD_F} = \sigma_{x\ell_\beta^-} = \infty$ how ν is detected?

CONCLUSIONS

- Standard expression for oscillation length is robust.

- Relativistic approximation is crucial.

- In general Equal Momentum or Energy Assumptions do not correspond to reality (incompatible with Lorentz invariance).
  Oscillation probability must be Lorentz invariant because different observers measure the same flavor transition probability ⇒ oscillations in space and time.

- Wave Packet treatment is necessary for approximation $T \simeq L$ ⇒ oscillations in space.

- Quantum Field Theory allows to calculate the effects of Production and Detection processes.
  Free propagating neutrino can be described by appropriate state.
  Effect of Detection process: energies and momenta of massive neutrino components relevant for oscillations are different from those of propagating neutrino state.