Theory of Neutrino Oscillations

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- ⇝ Critical Discussion of Standard Theory of Neutrino Oscillations
- ⇝ Flavor Neutrino States
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Standard Theory of Neutrino Oscillations in Vacuum

[Bilenky, Pontecorvo, Phys. Rep. 41 (1978) 225]

Neutrino Production: $j_\rho^{\rm CC} = 2 \sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}} \gamma_\rho \ell_{\alpha L}$ $\qquad \nu_{\alpha L} = \sum_k U_{\alpha k} \nu_{kL}$ Fields

$$\langle 0 | \nu_{\alpha L} | \nu_\beta \rangle = \sum_{k,j} U_{\alpha k} U_{\beta j}^* \underbrace{\langle 0 | \nu_{kL} | \nu_j \rangle}_{\propto \delta_{kj}} \propto \sum_k U_{\alpha k} U_{\beta k}^* = \delta_{\alpha\beta}$$

$|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$ States

$$\mathcal{H} |\nu_k\rangle = E_k |\nu_k\rangle \quad \Rightarrow \quad |\nu_k(t)\rangle = e^{-iE_k t} |\nu_k\rangle \quad \Rightarrow \quad |\nu_\alpha(t)\rangle = \sum_k U_{\alpha k}^* e^{-iE_k t} |\nu_k\rangle$$

$$|\nu_k\rangle = \sum_{\beta=e,\mu,\tau} U_{\beta k} |\nu_\beta\rangle$$

$$|\nu_\alpha(t)\rangle = \sum_{\beta=e,\mu,\tau} \underbrace{\left(\sum_k U_{\alpha k}^* e^{-iE_k t} U_{\beta k} \right)}_{\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t)} |\nu_\beta\rangle$$

Transition Probability: $P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \left| \mathcal{A}_{\nu_\alpha \to \nu_\beta}(t) \right|^2 = \left| \sum_k U_{\alpha k}^* e^{-iE_k t} U_{\beta k} \right|^2$

$$P_{\nu_\alpha\to\nu_\beta}(t) = \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left[-i\left(E_k - E_j\right)t\right]$$

Relativistic Approximation + Assumption $p_k = p = E$ [neutrinos with the same momentum propagate in the same direction]

$$E_k = \sqrt{p^2 + m_k^2} \simeq p + \frac{m_k^2}{2p} = E + \frac{m_k^2}{2E} \implies E_k - E_j \simeq \frac{\Delta m_{kj}^2}{2E}$$

$$\Delta m_{kj}^2 \equiv m_k^2 - m_j^2$$

Approximation $t \simeq L$ $\implies$

$$P_{\nu_\alpha\to\nu_\beta}(L) \simeq \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left[-i\frac{\Delta m_{kj}^2 L}{2E}\right]$$

$$P_{\nu_\alpha\to\nu_\beta}(L) = \sum_{k} |U_{\alpha k}|^2 |U_{\beta k}|^2$$

$\Leftarrow$ constant term

$$+ 2\mathrm{Re}\sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i\frac{\Delta m_{kj}^2 L}{2E}\right)$$

$\Leftarrow$ oscillating term $\Updownarrow$ COHERENCE

Main Assumptions of Standard Theory

(A1) Neutrinos are extremely relativistic particles OK!

(A2) Neutrinos produced in CC weak interaction processes together with charged leptons α^+ are described by the flavor state $|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$

Correct approximation for ultrarelativistic ν's [Giunti, Kim, Lee, PRD 45 (1992) 2414]

(A3) Massive neutrino states $|\nu_k\rangle$ have the same momentum $p_k = p$ ("Equal Momentum Assumption") and different energies: $E_k \simeq E + \frac{m_k^2}{2E}$

Unrealistic assumption, forbidden by energy-momentum conservation and Lorentz invariance, but gives correct result (as well as the "Equal Energy Assumption")

[Winter, LNC 30 (1981) 101], [Giunti, Kim, FPL 14 (2001) 213], [Giunti, MPLA 16 (2001) 2363], [Giunti, hep-ph/0302026]

(A4) Propagation Time $T \simeq L$ Source-Detector Distance OK!

$\Updownarrow$

WAVE PACKETS

Detectable Neutrinos are Extremely Relativistic

Only neutrinos with energy larger than some fraction of MeV are detectable!

Charged-Current Processes: Threshold

$$\nu + A \to B + C$$
$$\Downarrow$$
$$s = 2Em_A + m_A^2 \geq (m_B + m_C)^2$$
$$\Downarrow$$
$$E_{\text{th}} = \frac{(m_B + m_C)^2}{2m_A} - \frac{m_A}{2}$$

☼ $\nu_e + {}^{37}\text{Cl} \to {}^{37}\text{Ar} + e^-$ $\quad E_{\text{th}} = 0.81\,\text{MeV}$

☼ $\nu_e + {}^{71}\text{Ga} \to {}^{71}\text{Ge} + e^-$ $\quad E_{\text{th}} = 0.233\,\text{MeV}$

♁ $\bar{\nu}_e + p \to n + e^+$ $\quad E_{\text{th}} = 1.8\,\text{MeV}$

♁ $\nu_\mu + n \to p + \mu^-$ $\quad E_{\text{th}} = 110\,\text{MeV}$

♁ $\nu_\mu + e^- \to \nu_e + \mu^-$ $\quad E_{\text{th}} \simeq \frac{m_\mu^2}{2m_e} = 10.9\,\text{GeV}$

Elastic Scattering Processes: Cross Section $\propto$ Energy

☼ $\nu + e^- \to \nu + e^-$ $\qquad \sigma(E) \sim \sigma_0\, E/m_e$ $\qquad \sigma_0 \sim 10^{-44}\,\text{cm}^2$

Background $\Rightarrow E_{\text{th}} \simeq 5\,\text{MeV}$ (SK, SNO)

Laboratory and Astrophysical Limits $\Longrightarrow$ $m_\nu \lesssim 1\,\text{eV}$

POSSIBLE: very small mixing of ν_e, ν_μ, ν_τ with heavy ν_k's

IN THIS CASE

- ⋆ heavy neutrino masses must be taken into account in calculation of production and detection rates
- ⋆ oscillations due to large squared-mass differences are not observable $\Longrightarrow$ constant flavor-changing transition probability due to mixing $\Longrightarrow$ no coherence problem
- ⋆ almost degenerate very heavy massive neutrinos (such that the corresponding squared-mass differences generate observable oscillations) seem very unlikely

IN THE FOLLOWING: we study oscillations due to light extremely relativistic massive neutrinos

Flavor States

Mixing of Fields: $\nu_{\alpha L} = \sum_k U_{\alpha k} \nu_{kL}$ Mixing of States: $|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$

$$\langle 0|\nu_{\alpha L}|\nu_\beta\rangle = \sum_{k,j} U_{\alpha k} U_{\beta j}^* \underbrace{\langle 0|\nu_{kL}|\nu_j\rangle}_{\propto \delta_{kj}} \propto \sum_k U_{\alpha k} U_{\beta k}^* = \delta_{\alpha\beta}$$

Neutrino Masses have been neglected!

$$\nu_{kL}(x) \propto \int \mathrm{d}^3p \sum_h \left[a_k(\vec{p},h) u_{kL}(\vec{p},h) e^{-ip\cdot x} + b_k^\dagger(\vec{p},h) v_{kL}(\vec{p},h) e^{ip\cdot x} \right]$$

$$|\nu_j(\vec{p},h)\rangle = a_j^\dagger(\vec{p},h)|0\rangle \qquad \left\{ a_k(\vec{p},h)\,,\, a_j^\dagger(\vec{p}\,',h') \right\} = \delta^3(\vec{p}-\vec{p}\,')\,\delta_{hh'}\,\delta_{kj}$$

$$\langle 0|\nu_{kL}(0)|\nu_j(\vec{p},h)\rangle \propto u_{kL}(\vec{p},h)\,\delta_{kj} \quad \Rightarrow \quad \langle 0|\nu_{\alpha L}(0)|\nu_\beta(\vec{p},h)\rangle \propto \sum_k U_{\alpha k} U_{\beta k}^*\, u_{kL}(\vec{p},h) \not\propto \delta_{\alpha\beta}$$

MIXING $\Rightarrow$ Flavor States are only approximations for Extremely Relativistic Neutrinos!

States $|\nu_\alpha\rangle$ are not quanta of Field ν_α [Giunti, Kim, Lee, PRD 45 (1992) 2414]

Fock Space of Flavor States?

Mixed Dirac Neutrinos: $\nu_\alpha(x) = \sum_k U_{\alpha k}\, \nu_k(x)$

Quantized Massive Neutrino Fields: $\{\nu_k(t,\vec{x})\,,\, \nu_j^\dagger(t,\vec{y})\} = \delta(\vec{x}-\vec{y})\, \delta_{kj}$

Free Dirac Equation: $(i\partial\!\!\!/ - m_k)\, \nu_k(x) = 0$

$$\nu_k(x) = \int \frac{\mathrm{d}\vec{p}}{(2\pi)^{3/2}} \sum_{h=\pm1} \left[a_{\nu_k}(\vec{p},h)\, u_{\nu_k}(\vec{p},h)\, e^{-iE_{\nu_k}t + i\vec{p}\vec{x}} + b_{\nu_k}^\dagger(\vec{p},h)\, v_{\nu_k}(\vec{p},h)\, e^{iE_{\nu_k}t - i\vec{p}\vec{x}} \right]$$

$$E_{\nu_k} = \sqrt{\vec{p}^{\,2} + m_{\nu_k}^2} \qquad (p\!\!\!/ - m_{\nu_k})\, u_{\nu_k}(\vec{p},h) = 0 \qquad (p\!\!\!/ + m_{\nu_k})\, v_{\nu_k}(\vec{p},h) = 0$$

$$a_{\nu_k}(\vec{p},h) = \int \frac{\mathrm{d}\vec{x}}{(2\pi)^{3/2}}\, e^{iE_{\nu_k}t - i\vec{p}\vec{x}}\, u_{\nu_k}^\dagger(\vec{p},h)\, \nu_k(x)$$

$$b_{\nu_k}(\vec{p},h) = \int \frac{\mathrm{d}\vec{x}}{(2\pi)^{3/2}}\, \nu_k^\dagger(x)\, v_{\nu_k}(\vec{p},h)\, e^{iE_{\nu_k}t - i\vec{p}\vec{x}}$$

$$\{a_{\nu_k}(\vec{p},h)\,,\, a_{\nu_j}^\dagger(\vec{p}\,',h')\} = \{b_{\nu_k}(\vec{p},h)\,,\, b_{\nu_j}^\dagger(\vec{p}\,',h')\} = \delta(\vec{p}-\vec{p}\,')\, \delta_{hh'}\, \delta_{kj}$$

arbitrary definitions with arbitrary unphysical mass parameters $\widetilde{m}_{\nu_\alpha}$

[Fujii, Habe, Yabuki, Phys. Rev. D59 (1999) 113003, hep-ph/9807266]

$$E_{\nu_\alpha} = \sqrt{\vec{p}^2 + \widetilde{m}_{\nu_\alpha}^2} \qquad (\not{p} - \widetilde{m}_{\nu_\alpha})\, u_{\nu_\alpha}(\vec{p}, h) = 0 \qquad (\not{p} + \widetilde{m}_{\nu_\alpha})\, v_{\nu_\alpha}(\vec{p}, h) = 0$$

$$\nu_\alpha(x) = \int \frac{\mathrm{d}\vec{p}}{(2\pi)^{3/2}} \sum_{h=\pm1} \left[a_{\nu_\alpha}(\vec{p}, h)\, u_{\nu_\alpha}(\vec{p}, h)\, e^{-iE_{\nu_\alpha} t + i\vec{p}\vec{x}} + b^\dagger_{\nu_\alpha}(\vec{p}, h)\, v_{\nu_\alpha}(\vec{p}, h)\, e^{iE_{\nu_\alpha} t - i\vec{p}\vec{x}} \right]$$

but mixing implies

$$\nu_\alpha(x) = \int \frac{\mathrm{d}\vec{p}}{(2\pi)^{3/2}} \sum_{h=\pm1} \sum_k U_{\alpha k} \left[a_{\nu_k}(\vec{p}, h)\, u_{\nu_k}(\vec{p}, h)\, e^{-iE_{\nu_k} t + i\vec{p}\vec{x}} + b^\dagger_{\nu_k}(\vec{p}, h)\, v_{\nu_k}(\vec{p}, h)\, e^{iE_{\nu_k} t - i\vec{p}\vec{x}} \right]$$

assumption: $a_{\nu_\alpha}(\vec{p}, h)$ are linear combinations of $a_{\nu_k}(\vec{p}, h)$ only [Giunti, Kim, Lee, PRD 45 (1992) 2414]

$$a_{\nu_\alpha}(\vec{p}, h) = \sum_k U_{\alpha k}\, a_{\nu_k}(\vec{p}, h) \left(u^\dagger_{\nu_\alpha}(\vec{p}, h)\, u_{\nu_k}(\vec{p}, h) \right) e^{i\left(E_{\nu_\alpha} - E_{\nu_k}\right)t}$$

(one vacuum)

$$\{a_{\nu_\alpha}(\vec{p}, h)\,,\, a^\dagger_{\nu_\beta}(\vec{p}\,', h')\} = \delta(\vec{p} - \vec{p}\,')\, \delta_{hh'}\, e^{i\left(E_{\nu_\alpha} - E_{\nu_\beta}\right)t}$$
$$\times u^\dagger_{\nu_\alpha}(\vec{p}, h) \left(\sum_k U_{\alpha k}\, U^*_{\beta k}\, u_{\nu_k}(\vec{p}, h)\, u^\dagger_{\nu_k}(\vec{p}, h) \right) u_{\nu_\beta}(\vec{p}, h) \not\propto \delta_{\alpha\beta}$$

no Fock space of flavor neutrinos [Giunti, Kim, Lee, PRD 45 (1992) 2414] wrong: inconsistent derivation!

$$a_{\nu_\alpha}(\vec{p},h) = \int \frac{\mathrm{d}\vec{x}}{(2\pi)^{3/2}} \, e^{iE_{\nu_\alpha}t - i\vec{p}\vec{x}} \, u^\dagger_{\nu_\alpha}(\vec{p},h) \, \nu_\alpha(x)$$

$$a_{\nu_\alpha}(\vec{p},h) \, = \, e^{iE_{\nu_\alpha}t} \sum_k U_{\alpha k} \left[a_{\nu_k}(\vec{p},h) \Big(u^\dagger_{\nu_\alpha}(\vec{p},h) \, u_{\nu_k}(\vec{p},h) \Big) e^{-iE_{\nu_k}t} \right.$$
$$\left. + \, b^\dagger_{\nu_k}(-\vec{p},h) \Big(u^\dagger_{\nu_\alpha}(\vec{p},h) \, v_{\nu_k}(-\vec{p},h) \Big) e^{iE_{\nu_k}t} \right]$$

$$b_{\nu_\alpha}(\vec{p},h) = \int \frac{\mathrm{d}\vec{x}}{(2\pi)^{3/2}} \, \nu^\dagger_\alpha(x) \, v_{\nu_\alpha}(\vec{p},h) \, e^{iE_{\nu_\alpha}t - i\vec{p}\vec{x}}$$

$$b_{\nu_\alpha}(\vec{p},h) \, = \, e^{iE_{\nu_\alpha}t} \sum_k U^*_{\alpha k} \left[a^\dagger_{\nu_k}(-\vec{p},h) \Big(u^\dagger_{\nu_k}(-\vec{p},h) \, v_{\nu_\alpha}(\vec{p},h) \Big) e^{iE_{\nu_k}t} \right.$$
$$\left. + \, b_{\nu_k}(\vec{p},h) \Big(v^\dagger_{\nu_k}(\vec{p},h) \, v_{\nu_\alpha}(\vec{p},h) \Big) e^{-iE_{\nu_k}t} \right]$$

$$\{ a_{\nu_\alpha}(\vec{p},h) \, , \, a^\dagger_{\nu_\beta}(\vec{p}^{\,\prime},h') \} = \{ b_{\nu_\alpha}(\vec{p},h) \, , \, b^\dagger_{\nu_\beta}(\vec{p}^{\,\prime},h') \} = \delta(\vec{p} - \vec{p}^{\,\prime}) \, \delta_{hh'} \, \delta_{\alpha\beta}$$

Fock space of flavor neutrinos exists!

[Blasone, Vitiello, Ann. Phys. 244 (1995) 283, hep-ph/9501263] [Fujii, Habe, Yabuki, Phys. Rev. D59 (1999) 113003, hep-ph/9807266]

$$a_{\nu_\alpha}(\vec{p},h) \, |0\rangle \neq 0 \qquad\qquad b_{\nu_\alpha}(\vec{p},h) \, |0\rangle \neq 0$$

infinity of Fock spaces of flavor neutrinos

depending on the values of the arbitrary parameters $\widetilde{m}_{\nu_\alpha}$

flavor vacuums: $a_{\nu_\alpha}(\vec{p},h)\,|0_{\{\widetilde{m}\}}\rangle = 0 \qquad b_{\nu_\alpha}(\vec{p},h)\,|0_{\{\widetilde{m}\}}\rangle = 0$

can flavor Fock states describe real neutrinos? No!

☹ arbitrary and non unique construction

☹ measurable quantities depend on the arbitrary unphysical mass parameter $\widetilde{m}_{\nu_\mu}$

Example: $\pi^+ \to \mu^+ + \nu_\mu \qquad |\nu_\mu(\vec{p},h)\rangle = a_\mu^\dagger(\vec{p},h)\,|0_{\{\widetilde{m}\}}\rangle \qquad E_{\nu_\mu} = \sqrt{\vec{p} + \widetilde{m}^2_{\nu_\mu}}$

$$\mathcal{A} = \langle \mu^+(\vec{p}_\mu, h_\mu), \nu_\mu(\vec{p},h)| - i \int \mathrm{d}^4x\, \mathcal{H}_\mathsf{I}(x) |\pi^+(\vec{p}_\pi), 0_{\{\widetilde{m}\}}\rangle$$

$$\mathcal{H}_\mathsf{I}(x) = \frac{G_\mathsf{F}}{\sqrt{2}}\,\overline{\nu_\mu}(x)\,\gamma^\rho\,(1-\gamma_5)\;\mu(x)\,J_\rho(x) \qquad \langle 0|J_\rho(x)|\pi^+(\vec{p}_\pi)\rangle = i\,\vec{p}_{\pi\rho}\,f_\pi\,\cos\vartheta_\mathsf{C}\,e^{-ip_\pi x}$$

$$\mathcal{A} = 2\pi\,\frac{G_\mathsf{F}}{\sqrt{2}}\,\vec{p}_{\pi\rho}\,f_\pi\,\cos\vartheta_\mathsf{C}\,\delta^4(p_\pi - p_\mu - p)\,\overline{u_{\nu_\mu}}(\vec{p},h)\,\gamma^\rho\,(1-\gamma_5)\;v_\mu(\vec{p}_\mu,h_\mu)$$

flavor neutrino Fock spaces are clever mathematical constructs without physical relevance

Simplest Example of Neutrino Production: $\pi^+ \to \mu^+ + \nu_\mu \quad \pi^- \to \mu^- + \bar{\nu}_\mu$

two-body decay $\Longrightarrow$ fixed kinematics

$$E_k^2 = p_k^2 + m_k^2$$

π at rest:

$$\begin{cases} p_k^2 = \dfrac{m_\pi^2}{4}\left(1-\dfrac{m_\mu^2}{m_\pi^2}\right)^2 - \dfrac{m_k^2}{2}\left(1+\dfrac{m_\mu^2}{m_\pi^2}\right) + \dfrac{m_k^4}{4\,m_\pi^2} \\ E_k^2 = \dfrac{m_\pi^2}{4}\left(1-\dfrac{m_\mu^2}{m_\pi^2}\right)^2 + \dfrac{m_k^2}{2}\left(1-\dfrac{m_\mu^2}{m_\pi^2}\right) + \dfrac{m_k^4}{4\,m_\pi^2} \end{cases}$$

0^{th} order: $m_k = 0 \Rightarrow p_k = E_k = E = \dfrac{m_\pi}{2}\left(1-\dfrac{m_\mu^2}{m_\pi^2}\right) \simeq 30\,\text{MeV}$

1^{st} order:

$$E_k \simeq E + \xi\,\frac{m_k^2}{2E}$$

$$p_k \simeq E - (1-\xi)\,\frac{m_k^2}{2E}$$

$$\xi = \frac{1}{2}\left(1-\frac{m_\mu^2}{m_\pi^2}\right) \simeq 0.2$$

↖ ↗ general!

Equal Momentum of Massive Neutrinos?

(standard assumption: neutrinos with same momentum propagate in same direction)

$$p_k \simeq E - (1 - \xi)\,\frac{m_k^2}{2E}$$

the special case $\xi = 1 \Rightarrow p_k = p_j = E$ in general does not correspond to reality

BUT

for Extremely Relativistic Neutrinos the phase of $P_{\nu_\alpha \to \nu_\beta}$ is independent from ξ

$\Downarrow$

Equal Momentum Assumption gives correct Oscillation Phase

Different Momentum Contributions $\Longleftrightarrow$ Lorentz–Invariant Oscillations

$$|\nu_k(x,t)\rangle = e^{-iE_k t + i p_k x} |\nu_k\rangle \quad \Longrightarrow \quad |\nu_\alpha(x,t)\rangle = \sum_k U_{\alpha k}^* \, e^{-iE_k t + i p_k x} \, |\nu_k\rangle$$

$$|\nu_k\rangle = \sum_{\beta=e,\mu,\tau} U_{\beta k} \, |\nu_\beta\rangle$$

$$|\nu_\alpha(x,t)\rangle = \sum_{\beta=e,\mu,\tau} \underbrace{\left(\sum_k U_{\alpha k}^* e^{-iE_k t + i p_k x} U_{\beta k} \right)}_{\mathcal{A}_{\nu_\alpha \to \nu_\beta}(x,t)} |\nu_\beta\rangle$$

Transition Probability: $P_{\nu_\alpha \to \nu_\beta}(x,t) = |\langle \nu_\beta | \nu_\alpha(x,t) \rangle|^2 = \left| \mathcal{A}_{\nu_\alpha \to \nu_\beta}(x,t) \right|^2$

LORENTZ INVARIANT

OSCILLATION PROBABILITY

$$P_{\nu_\alpha \to \nu_\beta}(x,t) = \left| \sum_k U_{\alpha k}^* \, e^{-iE_k t + i p_k x} \, U_{\beta k} \right|^2$$

[Dolgov, Morozov, Okun, Shchepkin, NPB 502 (1997) 3], [Dolgov, hep-ph/0004032], [Dolgov, Phys. Rept. 370 (2002) 333],

[Giunti, Kim, FPL 14 (2001) 213], [Bilenky, Giunti, IJMPA 16 (2001) 3931], [Beuthe, Phys. Rept. 375 (2003) 105]

important: Flavor is Lorentz Invariant $\Leftrightarrow$ different observers measure same $P_{\nu_\alpha \to \nu_\beta}$

ultrarelativistic neutrinos $\Longrightarrow$ $t \simeq x = L$ source-detector distance

$$E_k t - p_k x \simeq (E_k - p_k)\, L = \frac{E_k^2 - p_k^2}{E_k + p_k}\, L = \frac{m_k^2}{E_k + p_k}\, L \simeq \frac{m_k^2}{2E}\, L$$

$$P_{\nu_\alpha \to \nu_\beta}(L) = \left| \sum_k U_{\alpha k}^*\, e^{-i m_k^2 L / 2E}\, U_{\beta k} \right|^2$$

$$= \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E} \right)$$

STANDARD OSCILLATION PROBABILITY!

$$\frac{\Delta m_{kj}^2 L}{2E} = 2\pi \quad \Longrightarrow \quad L_{kj}^{\mathrm{osc}} = \frac{4\pi E}{\Delta m_{kj}^2}$$

Oscillation Length

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-2\pi i \frac{L}{L_{kj}^{\mathrm{osc}}} \right)$$

Equal Energy of Massive Neutrinos?

[Grossman, Lipkin, PRD 55 (1997) 2760], [Lipkin, PLB 477 (2000) 195]

$$E_k \simeq E + \xi \frac{m_k^2}{2E}$$

the special case $\xi = 0 \Rightarrow E_k = E_j = E$ in general does not correspond to reality

BUT

for Extremely Relativistic Neutrinos the phase of $P_{\nu_\alpha \to \nu_\beta}$ is independent from ξ

$\Downarrow$

Equal Energy Assumption gives correct Oscillation Phase

advantage of Equal Energy Assumption: in the Lorentz Invariant treatment the approximation $T \simeq L$ is not needed

$$\Phi_{kj} = (p_k - p_j)\, L - (E_k - E_j)\, T = (p_k - p_j)\, L \simeq - \frac{\Delta m_{kj}^2 L}{2E}$$

BUT Equal Energy Assumption is incompatible with Lorentz Invariance!

Equal Energy Assumption and Equal Momentum Assumption are incompatible with Lorentz Invariance

[Giunti, MPLA 16 (2001) 2363]

assume for illustration that in frame S $\quad E_k = E \quad \Rightarrow \quad p_k = \sqrt{E^2 - m_k^2} \simeq E - \frac{m_k^2}{2E}$

(E ↑ energy of massless ν)

in another frame S' with velocity v along the neutrino path

$$E'_k = \gamma\,(E_k + v\,p_k) = \underbrace{\gamma\,(1+v)\,E}_{E'} - \boxed{\gamma v\,\frac{m_k^2}{2E}}$$

energies are different!

$$\boxed{\Delta E'_{kj} = -\frac{v}{1-v}\,\frac{m_k^2}{2E'}} \qquad \boxed{\Delta p'_{kj} = -\frac{1}{1-v}\,\frac{m_k^2}{2E'}}$$

$\Delta E'_{kj} \sim \Delta p'_{kj}$ for relativistic velocities COMMON IN PRACTICE!

Example: $\pi^+ \to \mu^+ + \nu_\mu$

Assume for sake of illustration S with $E_k = E$ is the rest frame of π

Many experiments measure oscillations of ν's produced in π decay in flight

e.g. short and long baseline, atmospheric ν experiments with $E_\pi \sim 100\,\mathrm{MeV} - 100\,\mathrm{GeV}$

Example: $E_\pi \simeq 200\,\mathrm{MeV} \quad \Rightarrow \quad v \simeq 0.71 \quad \Rightarrow \quad \dfrac{v}{1-v} \simeq 2.4 \quad \dfrac{1}{1-v} \simeq 3.4$

$$\Delta E'_{kj} = -\frac{v}{1-v}\,\frac{m_k^2}{2E'} \quad \sim \quad \Delta p'_{kj} = -\frac{1}{1-v}\,\frac{m_k^2}{2E'}$$

same order of magnitude!

for higher energies $\Delta E'_{kj} \simeq \Delta p'_{kj}$

Lorentz Invariance $\Longrightarrow$ In general Equal Energy Assumption and Equal Momentum Assumption do not correspond to reality

CONCLUSION: Forget Equal Energy Assumption and Equal Momentum Assumption. In any case they are not needed in Lorentz-invariant derivation of flavor transition probability of ultrarelativistic neutrinos.

Problem

$$P_{\nu_\alpha \to \nu_\beta}(x,t) = \left| \sum_k U_{\alpha k}^* \, U_{\beta k} \, \exp\left[-i \left(E_k t - p_k x \right) \right] \right|^2$$

explicitely Lorentz-invariant

$\uparrow$ product of two four-vectors

ultrarelativistic $\Downarrow$ neutrinos

$$P_{\nu_\alpha \to \nu_\beta}(L) = \left| \sum_k U_{\alpha k}^* \, U_{\beta k} \, \exp\left(-i \, \frac{m_k L}{2E} \right) \right|^2$$

apparently not Lorentz-invariant

Boost $\mathcal{O} \to \mathcal{O}'$ moving with respect to $\mathcal{O}$ with velocity v in the x direction

$$\Delta x' = \gamma \left(\Delta x - v \, \Delta t \right) \qquad \Delta t' = \gamma \left(-v \, \Delta x + \Delta t \right) \qquad \gamma \equiv \left(1 - v^2 \right)^{-1/2}$$

contraction of distances

$$\Delta t' = 0 \Rightarrow \Delta t = v \, \Delta x \Rightarrow \Delta x' = \gamma \left(1 - v^2 \right) \Delta x = \frac{\Delta x}{\gamma} \Rightarrow L' = \frac{L}{\gamma}$$

$$\left. \begin{aligned} p' &= \gamma \left(p - v \, E \right) \\ E' &= \gamma \left(-v \, p + E \right) \end{aligned} \right\} \xrightarrow[\text{limit}]{\text{massless}} E = p \implies E' = p' = \gamma \left(1 - v \right) E$$

$L'/E' \neq L/E \implies$ something is wrong!

phase $\frac{m_k L}{2E}$ derived from $E_k t - p_k x$ without assumptions about frame

$\Downarrow$

must be frame-invariant

crucial point: approximation $T = L$

$$\phi_k(T,L) = E_k\, T - p_k\, L \quad \xrightarrow[T=L]{} \quad \phi_k(T=L) = (E_k - p_k)\, L = \frac{E_k^2 - p_k^2}{E_k + p_k}\, L = \frac{m_k^2}{E_k + p_k}\, L \simeq \frac{m_k^2}{2\,E}\, L$$

L is not the equal-time distance between source and detector

L is the distance traveled by the neutrino in the time T

L is the spatial distance between the space-time events of neutrino production and detection

$$\mathcal{O} : \Delta x = \Delta t \quad \Longrightarrow \quad \mathcal{O}' : \Delta x' = \Delta t' = \gamma\,(1 - v)\,\Delta x$$

$$\Delta x' = \gamma\,(\Delta x - v\,\Delta t)$$
$$\Delta t' = \gamma\,(-v\,\Delta x + \Delta t)$$

$$L' = \gamma\,(1 - v)\,L \qquad E' = \gamma\,(1 - v)\,E \qquad \Longrightarrow \qquad L'/E' = L/E$$

OK!

Simple Example

source and detector at rest in the system $\mathcal{O}$

$\Downarrow$

in the system $\mathcal{O}'$ the detector is moving with velocity $-v$ along the x axis

$\Downarrow$

since the detector moves after the propagating neutrino has left the source,
the spatial distance traveled by the neutrino is shorter than
the instantaneous source-detector distance

$\Downarrow$

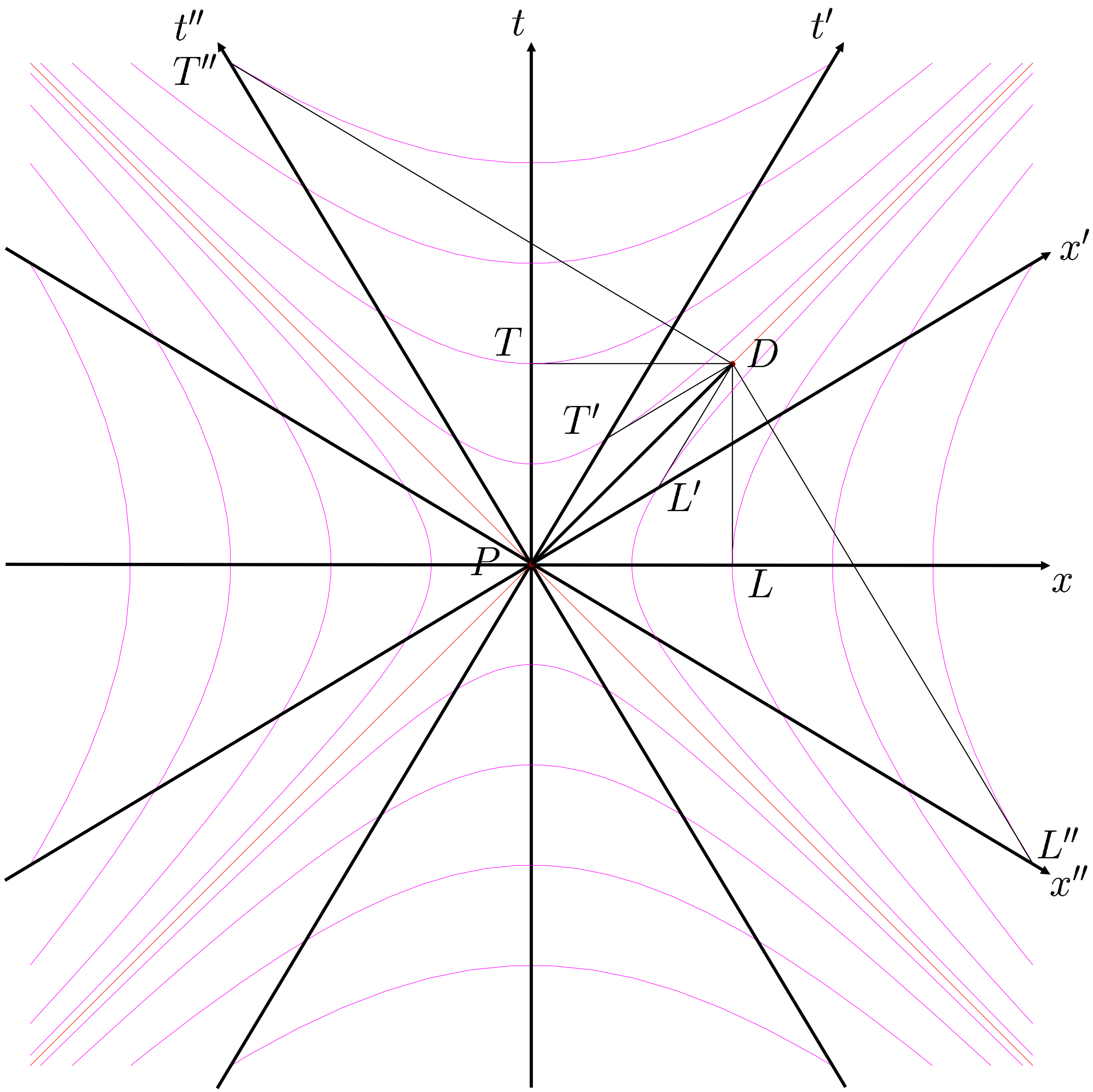
the distance traveled by the neutrino is different from
the Lorentz-contracted source-detector distance

---

what happens if source and detector are in relative motion in the system $\mathcal{O}$?

since the oscillation probability is measured by the detector,
the velocity of the source with respect to the detector does not matter

the system $\mathcal{O}$ in which L coincides with the instantaneous distance between source and detector at the time of neutrino emission is always the rest system of the detector



$\mathcal{O}: \ (x,t) \qquad v = 0$

$\mathcal{O}': \ (x',t') \qquad v = 3/5$

$\mathcal{O}'': \ (x'',t'') \qquad v = -3/5$

$$\widehat{x'x} = \arctan 3/5$$

$$\widehat{t't} = -\arctan 3/5$$

$$\widehat{x''x} = -\arctan 3/5$$

$$\widehat{t''t} = \arctan 3/5$$

$$L' = L/2 \qquad L'' = 2L$$

world-line of detector is constrained to be the vertical line passing through D

world-line of source can be any time-like line passing through P

hyperbolas $t^2 - x^2 = t'^2 - x'^2 = t''^2 - x''^2 = \text{constant}$ fix the scale on axes

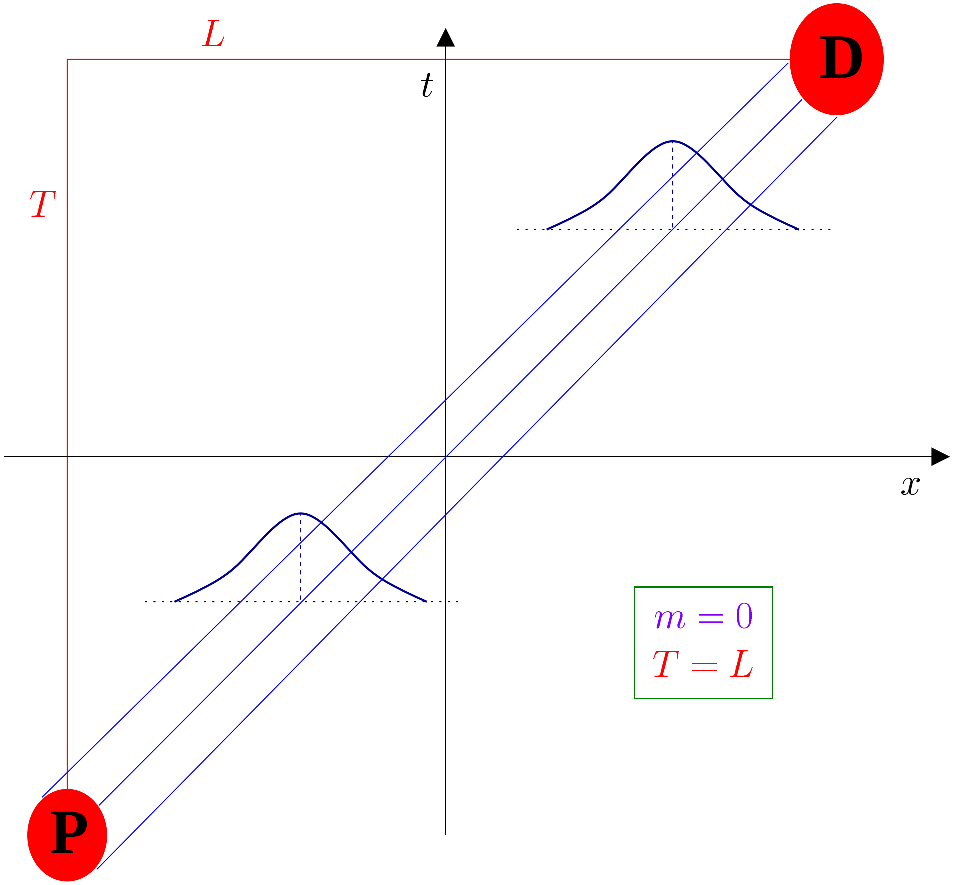
$t \simeq x = L \iff$ Wave Packets

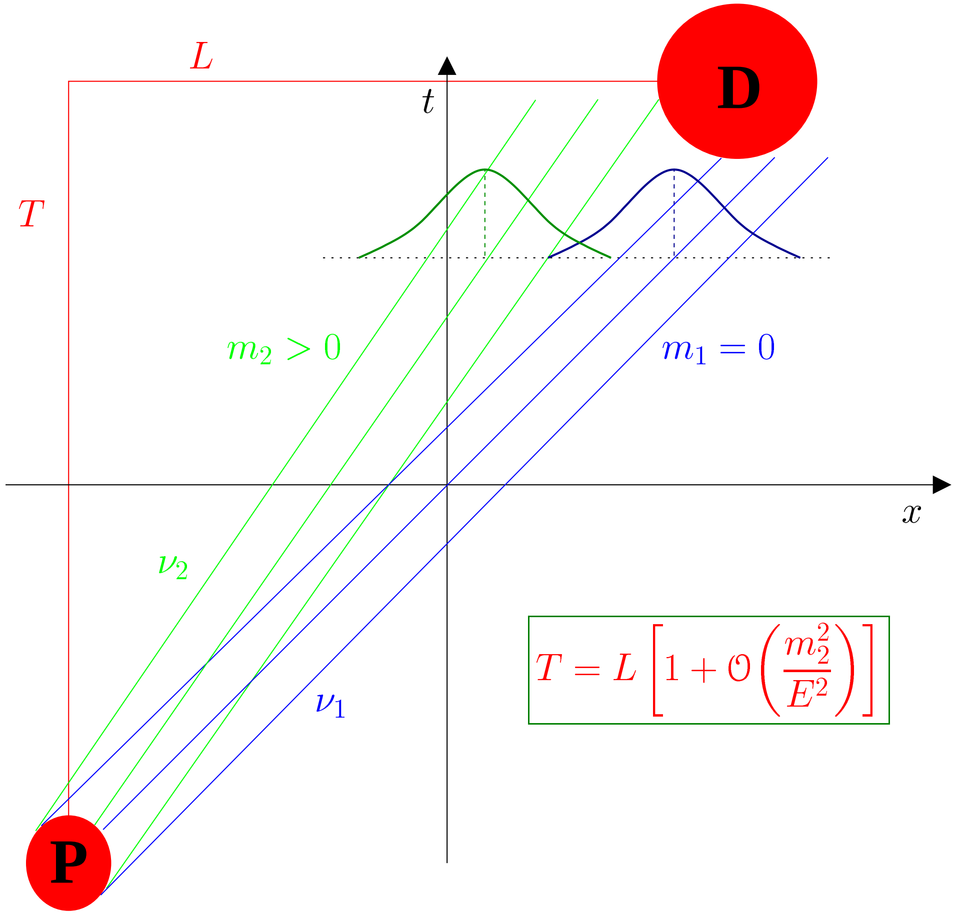
Other Motivations:

[Kayser, PRD 24 (1981) 110]

[Giunti, hep-ph/0302026]

- ⇝ Localization of Production and Detection Processes
- ⇝ Exact Energy-Momentum conservation would imply creation and detection of only one massive neutrino (neutrino mass measurement)





Corrections to $T = L$

Size of wave packets is determined by coherence size of Production Process δt_{P}

($\delta t_{\text{P}} \gtrsim \delta x_{\text{P}}$ because coherence region must be causally connected)

velocity of neutrino wave packets: $v_k = \frac{p_k}{E_k} \simeq 1 - \frac{m_k^2}{2E^2}$

Wave packets arrive at Detection Process at different times: $t_k = \frac{L}{v_k} \simeq L\left(1 + \frac{m_k^2}{2E^2}\right)$

$$m_k > m_j \quad \Longrightarrow \quad v_k < v_j \quad \Longrightarrow \quad t_k > t_j$$

average time: $\bar{t} = \frac{t_k + t_j}{2} \simeq L\left(1 + \frac{\overline{m_{kj}^2}}{2E^2}\right) \qquad \overline{m_{kj}^2} = \frac{m_k^2 + m_j^2}{2}$

wave packets overlap with detection process $\Longrightarrow$ range of $T \simeq [\bar{t} - \delta t, \bar{t} + \delta t]$

$$\delta t \simeq \sqrt{\delta t_{\text{P}}^2 + \delta t_{\text{D}}^2}$$

phase of oscillations: $\Phi_{kj} = (p_k - p_j)\, L - (E_k - E_j)\, T$

$$T = L \quad \Longrightarrow \quad \Phi_{kj} = - \frac{\Delta m^2_{kj} L}{2E} \qquad \text{correction:} \quad \Delta\Phi_{kj} \simeq - (E_k - E_j) \left(\frac{\overline{m^2_{kj}}}{2E^2} L \pm \delta t \right)$$

$$E_k \simeq E + \xi \frac{m^2_k}{2E} \quad \Longrightarrow \quad E_k - E_j \simeq \xi \frac{\Delta m^2_{kj}}{2E} \quad \Longrightarrow \quad \Delta\Phi_{kj} \simeq -\xi \frac{\Delta m^2_{kj} L}{2E} \left(\frac{\overline{m^2_{kj}}}{2E^2} \pm \frac{\delta t}{L} \right)$$

$\overline{m^2_{kj}} \ll E^2$ (ultrarelativistic neutrinos) $\qquad \delta t \ll L^{\text{osc}}_{kj} \lesssim L$

flux energy spectrum
+
detector energy resolution
+
distance uncertainty

$\Longrightarrow$

oscillations observable if $\Phi_{kj} \sim 1 \Rightarrow \dfrac{\Delta m^2_{kj} L}{2E} \sim 1$

$$\Delta\Phi_{kj} \simeq -\xi \frac{\Delta m^2_{kj} L}{2E} \left(\underbrace{\frac{\overline{m^2_{kj}}}{2E^2}}_{\text{negligible}} \pm \underbrace{\frac{\delta t}{L}}_{\text{negligible}} \right)$$

phase practically constant during wave packets overlap with detection process

Very Important: ξ is irrelevant $\Rightarrow$ oscillations of ultrarelativistic neutrinos are independent from the kinematics of the production process

Coherence Length

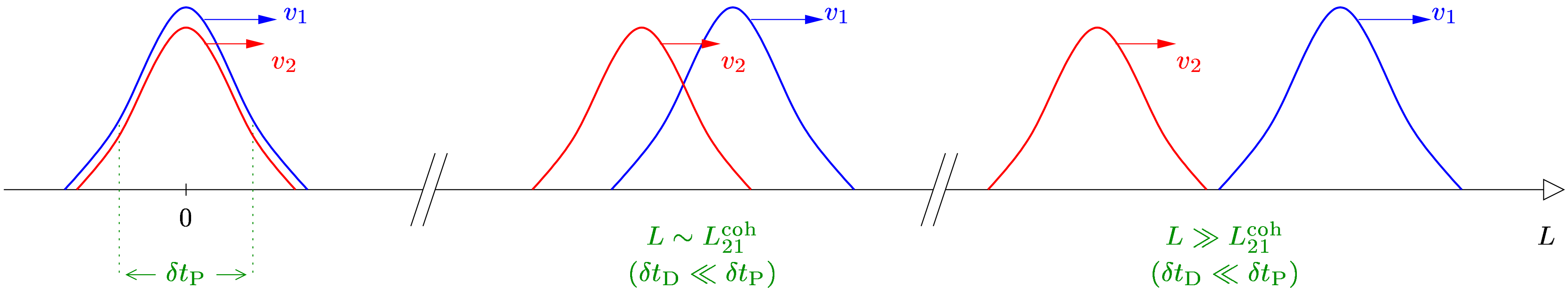
[Nussinov, PLB 63 (1976) 201], [Kiers, Nussinov, Weiss, PRD 53 (1996) 537]

Wave Packets have different velocities and separate

> different massive neutrinos can interfere
> **if and only if**
> wave packets arrive with $\delta t_{kj} < \delta t_{\mathrm{D}}$

$\Longrightarrow \quad L \lesssim L^{\mathrm{coh}}_{kj}$

$$|\delta t_{kj}| \simeq |v_k - v_j|\, T \simeq \frac{|\Delta m^2_{kj}|}{2E^2} L \qquad \Longrightarrow \qquad L^{\mathrm{coh}}_{kj} \sim \frac{2E^2}{|\Delta m^2_{kj}|} \sqrt{\delta t_{\mathrm{P}}^2 + \delta t_{\mathrm{D}}^2}$$



Quantum Mechanical Wave Packet Model

[Giunti, Kim, Lee, PRD 44 (1991) 3635], [Giunti, Kim, PRD 58 (1998) 017301]

also called "Intermediate Wave Packet Model" [Beuthe, Phys. Rept. 375 (2003) 105]

neglecting mass effects in amplitudes of production and detection processes

$$|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* \int \mathrm{d}p \, \psi_k^P(p) \, |\nu_k(p)\rangle \qquad |\nu_\beta\rangle = \sum_k U_{\beta k}^* \int \mathrm{d}p \, \psi_k^D(p) \, |\nu_k(p)\rangle$$

$$\begin{aligned} \mathcal{A}_{\alpha\beta}(x,t) &= \langle \nu_\beta | \, e^{-i\widehat{E}t + i\widehat{P}x} \, |\nu_\alpha\rangle \\ &= \sum_k U_{\alpha k}^* \, U_{\beta k} \int \mathrm{d}p \, \psi_k^P(p) \, \psi_k^{D*}(p) \, e^{-iE_k(p)t + ipx} \end{aligned}$$

Gaussian Approximation of Wave Packets

$$\psi_k^P(p) = \left(2\pi\sigma_{pP}^2\right)^{-1/4} \exp\left[-\frac{(p-p_k)^2}{4\sigma_{pP}^2}\right] \qquad \psi_k^D(p) = \left(2\pi\sigma_{pD}^2\right)^{-1/4} \exp\left[-\frac{(p-p_k)^2}{4\sigma_{pD}^2}\right]$$

the value of p_k is determined by the production process (causality)

$$\mathcal{A}_{\alpha\beta}(x,t) \propto \sum_k U_{\alpha k}^* \, U_{\beta k} \int \mathrm{d}p \, \exp\left[-iE_k(p)t + ipx - \frac{(p-p_k)^2}{4\sigma_p^2} \right]$$

global energy-momentum uncertainty: $\displaystyle \frac{1}{\sigma_p^2} = \frac{1}{\sigma_{pP}^2} + \frac{1}{\sigma_{pD}^2}$

sharply peaked wave packets

$$\sigma_p \ll E_k^2(p_k)/m_k \quad \Longrightarrow \quad E_k(p) = \sqrt{p^2 + m_k^2} \simeq E_k + v_k \, (p - p_k)$$

$$E_k = E_k(p_k) = \sqrt{p_k^2 + m_k^2} \qquad v_k = \left. \frac{\partial E_k(p)}{\partial p} \right|_{p=p_k} = \frac{p_k}{E_k}$$ group velocity

$$\mathcal{A}_{\alpha\beta}(x,t) \propto \sum_k U_{\alpha k}^* \, U_{\beta k} \, \exp\left[-iE_k t + ip_k x - \underbrace{\frac{(x - v_k t)^2}{4\sigma_x^2}}\right]$$

suppression factor for $|x - v_k t| \gtrsim \sigma_x$
due to size of wave packets

$$\boxed{\sigma_x \, \sigma_p = \frac{1}{2}}$$

global space-time uncertainty: $\sigma_x^2 = \sigma_{xP}^2 + \sigma_{xD}^2$

$$
\begin{aligned}
-E_k t + p_k x &= -(E_k - p_k)\,x + E_k\,(x - t) = -\frac{E_k^2 - p_k^2}{E_k + p_k}\,x + E_k\,(x-t) \\
&= -\frac{m_k^2}{E_k + p_k}\,x + E_k\,(x-t) \simeq -\frac{m_k^2}{2E}\,x + E_k\,(x-t)
\end{aligned}
$$

$$
\mathcal{A}_{\alpha\beta}(x,t) \propto \sum_k U_{\alpha k}^* \, U_{\beta k} \, \exp\left[\underbrace{-\,i\,\frac{m_k^2}{2E}\,x}_{\text{standard phase for } t = x} + \underbrace{i E_k\,(x-t)}_{\text{additional phase for } t \neq x} - \frac{(x - v_k t)^2}{4\sigma_x^2} \right]
$$

Space-Time Flavor Transition Probability

$$P_{\alpha\beta}(x,t) \propto \sum_{k,j} U_{\alpha k}^{*} U_{\beta k} U_{\alpha j} U_{\beta j}^{*} \exp\left[\underbrace{- i \frac{\Delta m_{kj}^{2} x}{2E}}_{\text{standard phase for } t = x} + \underbrace{i (E_k - E_j)(x - t)}_{\text{additional phase for } t \neq x} \right]$$

$$\times \exp\left[\underbrace{- \frac{(x - \overline{v}_{kj} t)^2}{4\sigma_x^2}}_{\substack{\text{suppression factor for} \\ |x - \overline{v}_{kj} t| \gtrsim \sigma_x \\ \text{due to size of wave packets}}} \underbrace{- \frac{(v_k - v_j)^2 t^2}{8\sigma_x^2}}_{\substack{\text{suppression factor due to} \\ \text{separation of wave packets}}} \right]$$

$$v_k = \frac{p_k}{E_k} \simeq 1 - \frac{m_k^2}{2E^2} \qquad \overline{v}_{kj} = \frac{v_k + v_j}{2} \simeq 1 - \frac{m_k^2 + m_j^2}{4E^2}$$

Oscillations in Space: $P_{\alpha\beta}(L) \propto \int \mathrm{d}t \, P_{\alpha\beta}(L,t)$

Gaussian integration over $\mathrm{d}t$

$$P_{\alpha\beta}(L) \propto \sum_{k,j} U_{\alpha k}^{*} \, U_{\beta k} \, U_{\alpha j} \, U_{\beta j}^{*} \, \exp\left[-i \, \frac{\Delta m_{kj}^{2} L}{2E}\right]$$

$$\times \underbrace{\sqrt{\frac{2}{v_k^2 + v_j^2}}}_{\simeq 1} \exp\left[- \underbrace{\frac{(v_k - v_j)^2}{v_k^2 + v_j^2}}_{\simeq (\Delta m_{kj}^2)^2 / 8E^4} \frac{L^2}{4\sigma_x^2} - \underbrace{\frac{(E_k - E_j)^2}{v_k^2 + v_j^2}}_{\simeq \xi^2 (\Delta m_{kj}^2)^2 / 8E^2} \sigma_x^2 \right]$$

$$\times \exp \underbrace{\left[i \, (E_k - E_j) \left(1 - \frac{2\overline{v}_{kj}^2}{v_k^2 + v_j^2} \right) L \right]}_{\text{negligible}}$$

Ultrarelativistic Neutrinos: $p_k \simeq E - (1-\xi) \, \frac{m_k^2}{2E}$ $\qquad E_k \simeq E + \xi \, \frac{m_k^2}{2E}$

$$P_{\alpha\beta}(L) = \sum_{k,j} U_{\alpha k}^{*} \, U_{\beta k} \, U_{\alpha j} \, U_{\beta j}^{*} \, \exp\left[-i \, \frac{\Delta m_{kj}^2 L}{2E} \right]$$
$$\times \exp\left[- \left(\frac{\Delta m_{kj}^2 L}{4\sqrt{2} E^2 \sigma_x} \right)^2 - 2\xi^2 \left(\frac{\Delta m_{kj}^2 \sigma_x}{4E} \right)^2 \right]$$

Oscillation Lengths: $L_{kj}^{\text{osc}} = \dfrac{4\pi E}{\Delta m_{kj}^2}$ Coherence Lengths: $L_{kj}^{\text{coh}} = \dfrac{4\sqrt{2} E^2}{|\Delta m_{kj}^2|} \sigma_x$

$$P_{\alpha\beta}(L) = \sum_{k,j} U_{\alpha k}^{*} \, U_{\beta k} \, U_{\alpha j} \, U_{\beta j}^{*} \, \exp\left[-2\pi i \, \frac{L}{L_{kj}^{\text{osc}}} \right]$$
$$\times \exp\left[- \left(\frac{L}{L_{kj}^{\text{coh}}} \right)^2 - 2\pi^2 \xi^2 \left(\frac{\sigma_x}{L_{kj}^{\text{osc}}} \right)^2 \right]$$

new localization term: $\exp\left[-2\pi^2\xi^2\left(\frac{\sigma_x}{L^{\text{osc}}_{kj}}\right)^2\right]$

interference is suppressed for $\sigma_x \gtrsim L^{\text{osc}}_{kj}$ $(\xi \sim 1)$

equivalent to neutrino mass measurement

uncertainty of neutrino mass measurement: $\boxed{m_k^2 = E_k^2 - p_k^2}$

$$\delta m_k^2 \simeq \sqrt{(2\,E_k\,\delta E_k)^2 + (2\,p_k\,\delta p_k)^2} \sim 4\,E\,\sigma_p$$

$$\sigma_p = \frac{1}{2\,\sigma_x}\,, \qquad E = \frac{|\Delta m^2_{kj}|L^{\text{osc}}_{kj}}{4\pi} \quad \Longrightarrow \quad \delta m_k^2 \sim \frac{|\Delta m^2_{kj}|L^{\text{osc}}_{kj}}{\sigma_x}$$

$$\boxed{\sigma_x \gtrsim L^{\text{osc}}_{kj} \quad \Longrightarrow \quad \delta m_k^2 \lesssim |\Delta m^2_{kj}|}$$

$$\Downarrow$$

! only one massive neutrino !

Achievements of the Quantum Mechanical Wave Packet Model

Confirmed Standard Oscillation Length:

$$L^{\rm osc}_{kj} = \frac{4\pi E}{\Delta m^2_{kj}}$$

Derived Coherence Length:

$$L^{\rm coh}_{kj} = \frac{4\sqrt{2}E^2}{|\Delta m^2_{kj}|}\,\sigma_x$$

problem: flavor states in production and detection processes have to be assumed

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha k} \int {\rm d}p\, \psi^P_k(p)\, |\nu_k(p)\rangle \qquad |\nu_\beta\rangle = \sum_k U^*_{\beta k} \int {\rm d}p\, \psi^D_k(p)\, |\nu_k(p)\rangle$$

calculation of neutrino production and detection?

⇑

Quantum Field Theoretical Wave Packet Model

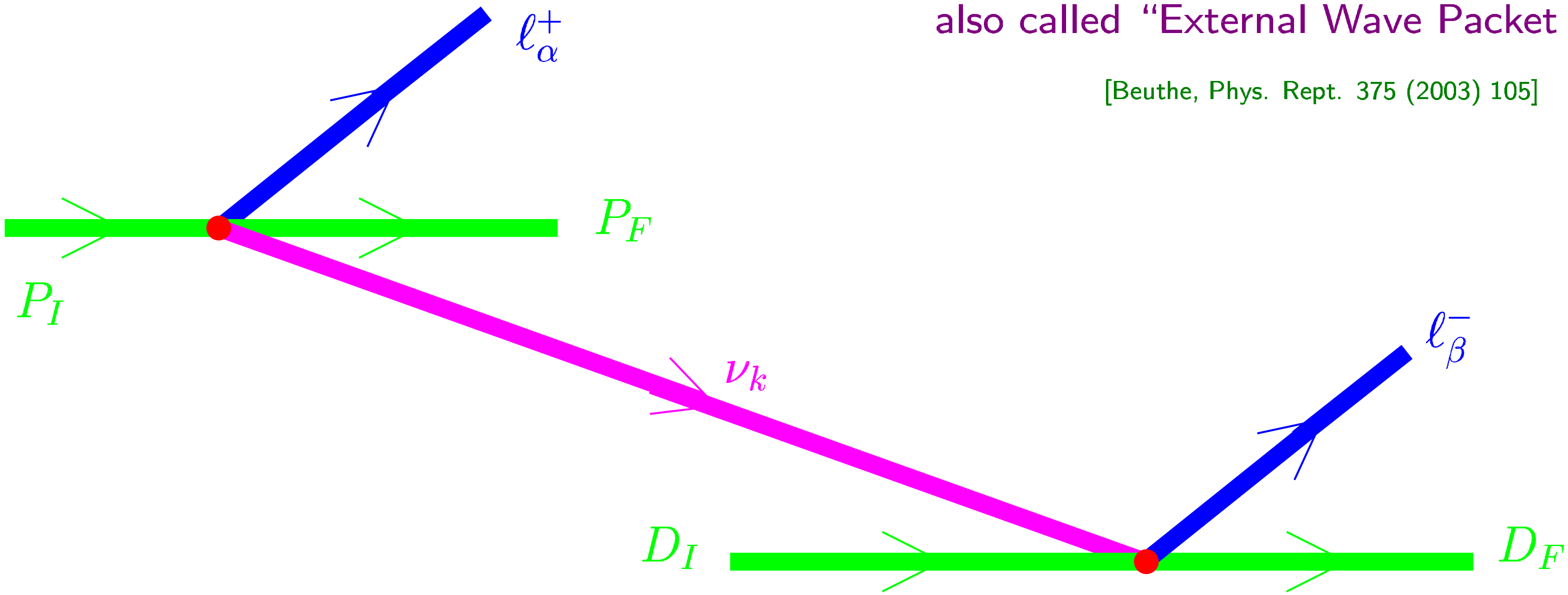
Quantum Field Model of Neutrino Oscillations with external particles in Production and Detection processes described by wave packets and intermediate virtual neutrino

[Giunti, Kim, Lee, Lee, PRD 48 (1993) 4310]
[Giunti, Kim, Lee, PLB 421 (1998) 237]
[Cardall, PRD 61 (2000) 073006]
[Beuthe, PRD 66 (2002) 013003]

$$P_I \to P_F + \ell_\alpha^+ + \nu_\alpha \xrightarrow{\nu_\alpha \to \nu_\beta} \nu_\beta + D_I \to D_F + \ell_\beta^-$$

also called "External Wave Packet Model"

[Beuthe, Phys. Rept. 375 (2003) 105]



$$P_{\nu_\alpha \to \nu_\beta} \propto \left| \sum_k \left\{ P_I \to P_F + \ell_\alpha^+ + \nu_k \xrightarrow{\text{propagator}} \nu_k + D_I \to D_F + \ell_\beta^- \right\} \right|^2$$

Confirmed Standard Oscillation Length

Derived Coherence Length

“philosophical” problem of External Wave Packet Model: neutrino has no properties!

in oscillation experiments neutrinos propagate as free particles over macroscopically large distance, sometimes astronomical distances (solar, atmospheric neutrinos)

it must be possible to describe neutrinos in oscillation experiments with appropriate state, as in the quantum-mechanical approach

Neutrino Wave Packets in Quantum Field Theory

[Giunti, JHEP 11 (2002) 017]

also called "Interacting Wave Packet Model" [Beuthe, Phys. Rept. 375 (2003) 105]

In Quantum Field Theory $\quad |f\rangle \propto (\mathcal{S} - \mathbf{1})|i\rangle \simeq -i \int \mathrm{d}^4x \, \mathcal{H}_I(x) \, |i\rangle$

Entangled Final State in Production Process:

$$P_I \to P_F + \ell_\alpha^+ + \nu_\alpha$$

$$|\widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle \propto -i \int \mathrm{d}^4x \, \mathcal{H}_I(x) \, |P_I\rangle$$

Disentangled by Interaction with Surrounding Medium (Measurement):

$$|\nu_\alpha\rangle \propto \langle P_F, \ell_\alpha^+ | \widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle \propto \langle P_F, \ell_\alpha^+ | - i \int \mathrm{d}^4x \, \mathcal{H}_I(x) \, |P_I\rangle$$

Effective Interaction Hamiltonian:

$$\mathcal{H}_I^P(x) = \frac{G_F}{\sqrt{2}} \, \bar{\nu}_\alpha(x) \, \gamma^\rho \, (1 - \gamma_5) \, \ell_\alpha(x) \, J_\rho^P(x) = \frac{G_F}{\sqrt{2}} \sum_k U_{\alpha k}^* \, \bar{\nu}_k(x) \, \gamma^\rho \, (1 - \gamma_5) \, \ell_\alpha(x) \, J_\rho^P(x)$$

Localization: $$|\chi\rangle = \int \mathrm{d}^3p\, \psi_\chi(\vec{p};\vec{p}_\chi,\sigma_{p\chi})\, |\chi(\vec{p},h_\chi)\rangle \qquad (\chi = P_I, P_F, \ell_\alpha^+)$$

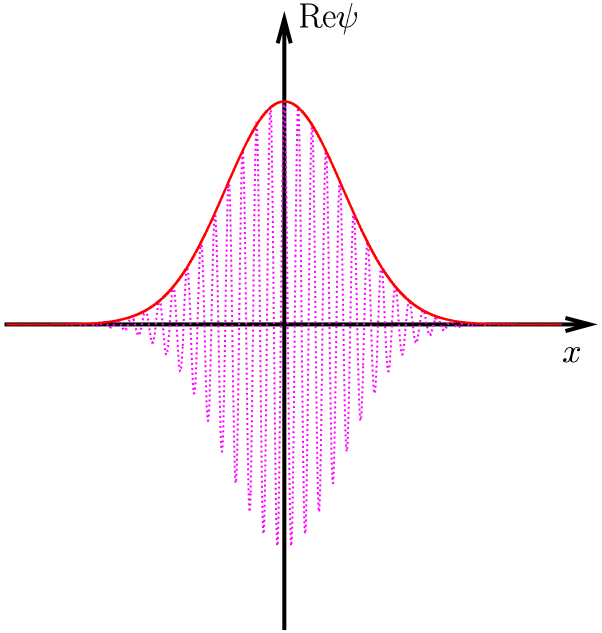
Gaussian Momentum Distribution: $$\psi_\chi(\vec{p};\vec{p}_\chi,\sigma_{p\chi}) = \left(2\pi\sigma_{p\chi}^2\right)^{-3/4} \exp\left[-\frac{(\vec{p}-\vec{p}_\chi)^2}{4\sigma_{p\chi}^2}\right]$$

Wave Function: $$\begin{aligned}\psi_\chi(\vec{x},t;\vec{p}_\chi,\sigma_{p\chi}) &= \langle 0|\chi(x)|\chi\rangle \\ &\approx \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\, \psi_\chi(\vec{p};\vec{p}_\chi,\sigma_{p\chi})\, e^{-iE_\chi(\vec{p})t+i\vec{p}\vec{x}}\end{aligned}$$

Dispersion Relation: $$E_\chi(\vec{p}) = \sqrt{\vec{p}^{\,2}+m_\chi^2} \simeq E_\chi + \vec{v}_\chi\left(\vec{p}-\vec{p}_\chi\right)$$

Average Energy: $E_\chi = \sqrt{\vec{p}_\chi^{\,2}+m_\chi^2}$ Group Velocity: $\vec{v}_\chi \equiv \left.\frac{\partial E_\chi}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_\chi} = \frac{\vec{p}_\chi}{E_\chi}$

Wave Function: $\psi_\chi(\vec{x},t;\vec{p}_\chi,\sigma_{p\chi}) \approx \left(2\pi\sigma_{x\chi}^2\right)^{-3/4} \exp\left[-iE_\chi t + i\vec{p}_\chi\cdot\vec{x} - \frac{(\vec{x}-\vec{v}_\chi t)^2}{4\sigma_{x\chi}^2}\right]$



Space and Momentum Uncertainties: $\sigma_{x\chi}\,\sigma_{p\chi} = \frac{1}{2}$

Squared Energy Uncertainty: $\langle(\delta E)^2\rangle_\chi = \langle\chi|(\widehat{E}-E_\chi)^2|\chi\rangle = \vec{v}_\chi^2\,\sigma_{p\chi}^2$ [Beuthe, priv. comm. 2002]

Neutrino State: $$|\nu_\alpha\rangle \propto \sum_k U_{\alpha k}^* \int \mathrm{d}^3p \int \mathrm{d}^3p'_{P_F}\, \psi^*_{P_F}(\vec{p}'_{P_F}; \vec{p}_{P_F}, \sigma_{pP_F})$$
$$\times \int \mathrm{d}^3p'_{\ell_\alpha^+}\, \psi^*_{\ell_\alpha^+}(\vec{p}'_{\ell_\alpha^+}; \vec{p}_{\ell_\alpha^+}, \sigma_{\ell_\alpha^+}) \int \mathrm{d}^3p'_{P_I}\, \psi_{P_I}(\vec{p}'_{P_I}; \vec{p}_{P_I}, \sigma_{pP_I})$$
$$\times \sum_h \bar{u}_{\nu_k}(\vec{p}, h)\gamma^\rho (1-\gamma_5)\, v_{\ell_\alpha^+}(\vec{p}'_{\ell_\alpha^+}, h_{\ell_\alpha^+})\, J^P_\rho(\vec{p}'_{P_F}, h_{P_F}; \vec{p}'_{P_I}, h_{P_I})$$
$$\times \int \mathrm{d}^4x\, e^{i(p+p'_{\ell_\alpha^+}+p'_{P_F}-p'_{P_I})x}\, |\nu_k(\vec{p}, h)\rangle$$

integrations over d^4x, $\mathrm{d}^3p'_{P_I}$, $\mathrm{d}^3p'_{\ell_\alpha^+}$, $\mathrm{d}^3p'_{P_F}$ $\Rightarrow$

$$\boxed{|\nu_\alpha\rangle = N_\alpha \sum_k U_{\alpha k}^* \int \mathrm{d}^3p\, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p}, h)\, |\nu_k(\vec{p}, h)\rangle}$$

Amplitudes: $$\mathcal{A}_k^P(\vec{p}, h) = \bar{u}_{\nu_k}(\vec{p}, h)\, \gamma^\rho (1-\gamma_5)\, v_{\ell_\alpha^+}(\vec{p}_{\ell_\alpha^+}, h_{\ell_\alpha^+})\, J^P_\rho(\vec{p}'_{P_F}, h_{P_F}; \vec{p}'_{P_I}, h_{P_I})$$

$e^{-S_k^P(\vec{p})}$ replaces energy-momentum δ-function

$$S_k^P(\vec{p}) \equiv \frac{(\vec{p}_P - \vec{p})^2}{4\,\sigma_{pP}^2} + \frac{\left[E_P - E_{\nu_k}(\vec{p}) - (\vec{p}_P - \vec{p}) \cdot \vec{v}_P\right]^2}{4\,\sigma_{pP}^2 \lambda_P}$$

Energy: $E_P \equiv E_{P_I} - E_{P_F} - E_{\ell_\alpha^+}$ Momentum: $\vec{p}_P \equiv \vec{p}_{P_I} - \vec{p}_{P_F} - \vec{p}_{\ell_\alpha^+}$

Space Uncertainty: $$\frac{1}{\sigma_{xP}^2} \equiv \frac{1}{\sigma_{xP_I}^2} + \frac{1}{\sigma_{xP_F}^2} + \frac{1}{\sigma_{x\ell_\alpha^+}^2}$$

Momentum Uncertainty: $$\sigma_{pP}^2 = \sigma_{pP_I}^2 + \sigma_{pP_F}^2 + \sigma_{p\ell_\alpha^+}^2$$

$$\vec{v}_P \equiv \sigma_{xP}^2 \left(\frac{\vec{v}_{P_I}}{\sigma_{xP_I}^2} + \frac{\vec{v}_{P_F}}{\sigma_{xP_F}^2} + \frac{\vec{v}_{\ell_\alpha^+}}{\sigma_{x\ell_\alpha^+}^2} \right)$$

$(0 \le |\vec{v}_P| \le 1)$

$$\Sigma_P \equiv \sigma_{xP}^2 \left(\frac{\vec{v}_{P_I}^2}{\sigma_{xP_I}^2} + \frac{\vec{v}_{P_F}^2}{\sigma_{xP_F}^2} + \frac{\vec{v}_{\ell_\alpha^+}^2}{\sigma_{x\ell_\alpha^+}^2} \right)$$

$(0 \le \Sigma_P \le 1)$

$$\lambda_P \equiv \Sigma_P - \vec{v}_P^2 \qquad (0 \le \lambda_P \le 1)$$

$$\sigma_{pP} = 0 \quad \Longrightarrow \quad \vec{p} = \vec{p}_P\,, \quad E_{\nu_k}(\vec{p}) = E_P \quad \Longrightarrow$$ only one massive neutrinos

$\updownarrow$

plane waves

$\Updownarrow$

exact energy-momentum conservation

flavor state: $$|\nu_\alpha\rangle = N_\alpha \sum_k U_{\alpha k}^* \int \mathrm{d}^3p \, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p},h) \, |\nu_k(\vec{p},h)\rangle$$

superposition of massive neutrino states $$|\nu_k\rangle = N_k \int \mathrm{d}^3p \, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p},h) \, |\nu_k(\vec{p},h)\rangle$$

Average Momentum: $$\left. \frac{\partial S_k^P(\vec{p})}{\partial \vec{p}} \right|_{\vec{p}=\vec{p}_k} = 0 \qquad \vec{p}_k = p_k \vec{\ell}$$

Average Energy: $E_k = E_{\nu_k}(\vec{p}_k) = \sqrt{\vec{p}_k^{\,2} + m_k^2}$ Group Velocity: $\vec{v}_k = \left. \frac{\partial E_{\nu_k}(\vec{p})}{\partial \vec{p}} \right|_{\vec{p}=\vec{p}_k} = \frac{\vec{p}_k}{E_k}$

Ultrarelativistic Neutrinos: $$p_k \simeq E - (1-\xi) \frac{m_k^2}{2E} \qquad E_k \simeq E + \xi \frac{m_k^2}{2E}$$

$$\vec{p}_P = E_P \, \vec{\ell} \qquad E = E_P \qquad \xi = \frac{\lambda_P - \vec{\ell} \cdot \vec{v}_P \left(1 - \vec{\ell} \cdot \vec{v}_P\right)}{\lambda_P + \left(1 - \vec{\ell} \cdot \vec{v}_P\right)^2}$$

Squared Energy-Momentum Uncertainties: $\langle (\delta p)^2 \rangle_k \sim \langle (\delta E)^2 \rangle_k \sim \sigma_{pP}^2$

Detection Process at $(\vec{L},T)$: $|\nu_\alpha(\vec{L},T)\rangle = e^{-i\widehat{E}T+i\widehat{\vec{P}}\cdot\vec{L}}\,|\nu_\alpha\rangle$

$$|\nu_\alpha(\vec{L},T)\rangle = N_\alpha \sum_k U_{\alpha k}^* \int \mathrm{d}^3p\, e^{-iE_{\nu_k}(\vec{p})T+i\vec{p}\cdot\vec{L}}\, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p},h)\,|\nu_k(\vec{p},h)\rangle$$

Detection Amplitude: $\mathcal{A}_{\alpha\beta}(\vec{L},T) = \langle D_F, \ell_\beta^-| - i \int \mathrm{d}^4x\, \mathcal{H}_I(x)\,|D_I, \nu_\alpha(\vec{L},T)\rangle$

↑ Transition Amplitude

Effective Interaction Hamiltonian:
$$\mathcal{H}_I^D(x) = \frac{G_F}{\sqrt{2}}\,\bar{\ell}_\beta(x)\,\gamma^\rho\,(1-\gamma_5)\,\nu_\beta(x)\,J_\rho^D(x)$$
$$= \frac{G_F}{\sqrt{2}}\sum_j U_{\beta j}\,\bar{\ell}_\beta(x)\,\gamma^\rho\,(1-\gamma_5)\,\nu_j(x)\,J_\rho^D(x)$$

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \sum_k U_{\alpha k}^* U_{\beta k} \sum_h \int \mathrm{d}^3p \, \mathcal{A}_k^P(\vec{p},h) \mathcal{A}_k^D(\vec{p},h) \, e^{-S_k(\vec{p})} \, \exp\left[-iE_{\nu_k}(\vec{p})T + i\vec{p}\cdot\vec{L}\right]$$

$$S_k(\vec{p}) = S_k^P(\vec{p}) + S_k^D(\vec{p})$$

$e^{-S_k(\vec{p})}$ replaces energy-momentum δ-functions

$$S_k(\vec{p}) = \frac{(\vec{p}_P - \vec{p})^2}{4\,\sigma_{pP}^2} + \frac{\left[(E_P - E_{\nu_k}(\vec{p})) - (\vec{p}_P - \vec{p})\cdot\vec{v}_P\right]^2}{4\,\sigma_{pP}^2 \lambda_P} + \frac{(\vec{p}_D - \vec{p})^2}{4\,\sigma_{pD}^2} + \frac{\left[(E_D - E_{\nu_k}(\vec{p})) - (\vec{p}_D - \vec{p})\cdot\vec{v}_D\right]^2}{4\,\sigma_{pD}^2 \lambda_D}$$

$$\sigma_{pP} = 0 \quad \Longrightarrow \quad \vec{p} = \vec{p}_P \,, \quad E_{\nu_k}(\vec{p}) = E_P$$

or

$$\sigma_{pD} = 0 \quad \Longrightarrow \quad \vec{p} = \vec{p}_D \,, \quad E_{\nu_k}(\vec{p}) = E_D$$

$$\Downarrow$$

only one massive neutrino contribution $\Longrightarrow$ no oscillations formally $\mathcal{A}_{\alpha\beta}(\vec{L},T)$ can be obtained in the standard quantum-mechanical way:

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \langle\nu_\beta|\, e^{-i\widehat{E}T+i\widehat{\vec{P}}\cdot\vec{L}}\, |\nu_\alpha\rangle$$

with

$$|\nu_\alpha\rangle = N_\alpha \sum_k U_{\alpha k}^{*} \int \mathrm{d}^3p\, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p},h)\, |\nu_k(\vec{p},h)\rangle$$

$$|\nu_\beta\rangle = N_\beta \sum_k U_{\beta k}^{*} \int \mathrm{d}^3p\, e^{-S_k^D(\vec{p})} \sum_h \mathcal{A}_k^D(\vec{p},h)\, |\nu_k(\vec{p},h)\rangle$$

but calculation of coefficients of massive neutrino components of $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle$ needs Quantum Field Theory !

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \sum_k U_{\alpha k}^* U_{\beta k} \sum_h \int \mathrm{d}^3p \, \mathcal{A}_k^P(\vec{p},h) \mathcal{A}_k^D(\vec{p},h) \, e^{-S_k(\vec{p})} \, \exp\left[-iE_{\nu_k}(\vec{p})T + i\vec{p}\cdot\vec{L}\right]$$

integration over d^3p with saddle-point approximation around minimum of $S_k(\vec{p})$

$$\left.\frac{\partial S_k}{\partial \vec{p}}\right|_{\vec{p}=\vec{q}_k} = 0$$

Energy: $\varepsilon_k \equiv E_{\nu_k}(\vec{q}_k) = \sqrt{\vec{q}_k^{\,2} + m_k^2}$

in general $\vec{q}_k, \varepsilon_k \neq \vec{p}_k, E_k$

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \sum_k U_{\alpha k}^* U_{\beta k} \sum_h \frac{\mathcal{A}_k^P(\vec{q}_k,h) \mathcal{A}_k^D(\vec{q}_k,h)}{\sqrt{\mathrm{Det}\Omega_k}} \, e^{-S_k(\vec{q}_k)}$$

$$\times \exp\left[-i\varepsilon_k T + i q_k L - \frac{1}{2} \left(L^a - u_k^a T\right) (\Omega_k^{-1})^{ab} \left(L^b - u_k^b T\right)\right]$$

$$\Omega_k^{ab} \equiv \left.\frac{\partial^2 S_k(\vec{p})}{\partial p^a \partial p^b}\right|_{\vec{p}=\vec{q}_k}$$

Ultrarelativistic Neutrinos

$$\varepsilon_k \simeq E + \rho \, \frac{m_k^2}{2E} \qquad q_k \simeq E - (1-\rho) \, \frac{m_k^2}{2E} \qquad u_k \simeq 1 - \frac{m_k^2}{2E^2}$$

$$\vec{p}_P = E_P \, \vec{\ell} \qquad \vec{p}_D = E_D \, \vec{\ell} \qquad \vec{q}_k = q_k \, \vec{\ell} \qquad E_P = E_D = E$$

$$\sigma_x^2 = \sigma_{xP}^2 + \sigma_{xD}^2 \qquad \frac{1}{\sigma_p^2} = \frac{1}{\sigma_{pP}^2} + \frac{1}{\sigma_{pD}^2}$$

$$\frac{\mathcal{A}_k^P(\vec{q}_k, -)\mathcal{A}_k^D(\vec{q}_k, -)}{\sqrt{\mathrm{Det}\Omega_k}} \simeq \frac{\mathcal{A}^P(E, -)\mathcal{A}^D(E, -)}{\sqrt{\mathrm{Det}\Omega}}$$ factorized out of $\sum_k$

$\mathcal{A}_k^P(\vec{q}_k, +)\mathcal{A}_k^D(\vec{q}_k, +)$ negligible

$$\mathcal{A}_{\alpha\beta}(L,T) \propto \sum_k U_{\alpha k}^* U_{\beta k} \exp\left[-i\varepsilon_k T + i q_k L - \frac{(L - u_k T)^2}{4\eta^2} - \zeta \left(\frac{m_k^2}{4E\sigma_p} \right)^2 \right]$$

$$\eta^2 = \omega \, \sigma_x^2$$

$$\rho \sim 1 \qquad \omega \sim 1 \qquad \zeta \sim 1$$

$$\rho = \frac{\frac{1}{\sigma_p^2} - \frac{\vec{\ell}\cdot\vec{v}_P\left(1-\vec{\ell}\cdot\vec{v}_P\right)}{\sigma_{pP}^2\,\lambda_P} - \frac{\vec{\ell}\cdot\vec{v}_D\left(1-\vec{\ell}\cdot\vec{v}_D\right)}{\sigma_{pD}^2\,\lambda_D}}{\frac{1}{\sigma_p^2} + \frac{\left(1-\vec{\ell}\cdot\vec{v}_P\right)^2}{\sigma_{pP}^2\,\lambda_P} + \frac{\left(\vec{\ell}\cdot\vec{v}_D-1\right)^2}{\sigma_{pD}^2\,\lambda_D}}$$

$$\begin{aligned}\omega = & \left\{1 + \sigma_p^2\left[\frac{\left(v_P^x-1\right)^2+(v_P^y)^2+(v_P^z)^2}{\sigma_{pP}^2\lambda_P} + \frac{\left(v_D^x-1\right)^2+(v_D^y)^2+(v_D^z)^2}{\sigma_{pD}^2\lambda_D}\right]\right. \\ & \left. +\sigma_p^4\,\frac{\left[\left(v_P^x-1\right)v_D^y-\left(v_D^x-1\right)v_P^y\right]^2+\left[\left(v_P^x-1\right)v_D^z-\left(v_D^x-1\right)v_P^z\right]^2+\left(v_P^y v_D^z-v_P^z v_D^y\right)^2}{\sigma_{pP}^2\lambda_P\,\sigma_{pD}^2\lambda_D}\right\} \\ & \times\left\{1 + \sigma_p^2\left[\frac{(v_P^y)^2+(v_P^z)^2}{\sigma_{pP}^2\lambda_P} + \frac{(v_D^y)^2+(v_D^z)^2}{\sigma_{pD}^2\lambda_D}\right] + \sigma_p^4\,\frac{\left(v_P^y v_D^z-v_P^z v_D^y\right)^2}{\sigma_{pP}^2\lambda_P\,\sigma_{pD}^2\lambda_D}\right\}^{-1}\end{aligned}$$

$$S_k(\vec{q}_k) = \zeta\left(\frac{m_k^2}{4E\sigma_p}\right)^2$$

$$\begin{aligned}\zeta = & \left[\frac{1}{\sigma_p^2} + \frac{\left(1-\vec{\ell}\cdot\vec{v}_P\right)^2}{\sigma_{pP}^2\lambda_P} + \frac{\left(1-\vec{\ell}\cdot\vec{v}_D\right)^2}{\sigma_{pD}^2\lambda_D}\right]^{-2}\left[\frac{1}{\sigma_p^2}\left(\frac{1}{\sigma_{pP}^2\lambda_P} + \frac{1}{\sigma_{pD}^2\lambda_D}\right) + \left(\frac{1-\vec{\ell}\cdot\vec{v}_P}{\sigma_{pP}^2\lambda_P} + \frac{1-\vec{\ell}\cdot\vec{v}_D}{\sigma_{pD}^2\lambda_D}\right)^2\right. \\ & \left. + \frac{\left[\vec{\ell}\cdot(\vec{v}_P-\vec{v}_D)\right]^2}{\sigma_{pP}^2\lambda_P\,\sigma_{pD}^2\lambda_D}\left(2 + \frac{\sigma_p^2}{\sigma_{pP}^2}\,\frac{\left(1-\vec{\ell}\cdot\vec{v}_P\right)^2}{\lambda_P} + \frac{\sigma_p^2}{\sigma_{pD}^2}\,\frac{\left(1-\vec{\ell}\cdot\vec{v}_D\right)^2}{\lambda_D}\right)\right]\end{aligned}$$

Space-Time Transition Probability: $P_{\alpha\beta}(\vec{L},T) \propto |\mathcal{A}_{\alpha\beta}(\vec{L},T)|^2$

Transition Probability in Space: $P_{\alpha\beta}(\vec{L}) \propto \int \mathrm{d}T \, |\mathcal{A}_{\alpha\beta}(\vec{L},T)|^2$

Ultrarelativistic Neutrinos:

$$P_{\alpha\beta}(L) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*$$

$$\times \exp\left[-2\pi i \, \frac{L}{L_{kj}^{\mathrm{osc}}} - \left(\frac{L}{L_{kj}^{\mathrm{coh}}} \right)^2 - 2\pi^2 \, (\rho^2 \omega + \zeta) \left(\frac{\sigma_x}{L_{kj}^{\mathrm{osc}}} \right)^2 - \zeta \, \frac{(m_k^2 + m_j^2)^2}{32 E^2 \sigma_p^2} \right]$$

Oscillation Lengths: $L_{kj}^{\mathrm{osc}} = \dfrac{4\pi E}{\Delta m_{kj}^2}$ Coherence Lengths: $L_{kj}^{\mathrm{coh}} = \dfrac{4\sqrt{2\omega} E^2}{|\Delta m_{kj}^2|} \, \sigma_x$

ρ, ω, ζ depend on Production and Detection processes ($\rho \sim 1$, $\omega \sim 1$, $\zeta \sim 1$)

Necessary Localization: $\sigma_x \ll L_{kj}^{\mathrm{osc}}$

otherwise: neutrino mass measurement $\Longleftrightarrow$ no oscillations

$$\sigma_x \gtrsim L_{kj}^{\mathrm{osc}} \Longrightarrow \delta m_k^2 \sim E \delta E \sim E \sigma_p \sim \frac{|\Delta m_{kj}^2| L_{kj}^{\mathrm{osc}}}{\sigma_x} \lesssim |\Delta m_{kj}^2|$$

N massive neutrinos $\Longrightarrow$ $\dfrac{N(N-1)}{2}$ squared-mass differences (Δm^2_{kj})

$\dfrac{N(N-1)}{2}$ oscillation lengths $L^{\text{osc}}_{kj} = \frac{4\pi E}{\Delta m^2_{kj}}$

$\dfrac{N(N-1)}{2}$ coherence lengths $L^{\text{coh}}_{kj} \sim \frac{4\sqrt{2}E^2}{|\Delta m^2_{kj}|}\,\delta x$

$L \gg L^{\text{coh}}_{kj}$ $\Rightarrow$ no interference term

$L \gg L^{\text{coh}}_{kj} \quad \forall k,j$ $\Rightarrow$ constant flavor-changing transitions: $P_{\nu_\alpha\to\nu_\beta} = \sum_k |U_{\alpha k}|^2\,|U_{\beta k}|^2$

Example of intermediate case: $L \ll L^{\text{coh}}_{21}$ $\quad L \gg L^{\text{coh}}_{kj} \quad \forall k > 2$

$$P_{\nu_\alpha\to\nu_\beta}(L) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\,\text{Re}\,U^*_{\alpha 2}\,U_{\beta 2}\,U_{\alpha 1}\,U^*_{\beta 1}\,\exp\left(-2\pi i\,\frac{L}{L^{\text{osc}}_{21}}\right)$$

Size of Wave Packets

Line Broadening studied in atomic spectroscopy since the end of 1800

Main causes (Michelson 1895)

- ⋆ The change in wavelength due to the Doppler effect of the component of the velocity of the vibrating atom in the light of sight (Rayleigh 1889) $\Rightarrow$ Thermal Broadening, incoherent effect (solar $^{7}\mathrm{Be}$ line)
- ⋆ The exponential diminution in amplitude of the vibrations $\Rightarrow$ Natural Linewidth $\Rightarrow$ $\Delta t_{\mathrm{P}} \sim \tau_{\mathrm{dec}}$ (wave packet emission in vacuum)
- ⋆ The limitation of the number of regular vibrations by more or less abrupt changes of phase amplitude or plane of vibration caused by collisions $\Rightarrow$ Collision Broadening or Pressure Broadening $\Delta t_{\mathrm{P}} \sim \tau_{\mathrm{col}}$ (wave packet emission in a medium if $\tau_{\mathrm{col}} < \tau_{\mathrm{dec}}$)

Natural Linewidth

$$\boxed{\Delta t_{\rm P} \sim \tau_{\rm dec}} \quad \Longrightarrow \quad \boxed{\sigma_x \sim \tau_{\rm dec}}$$

$\pi^+ \to \mu^+ + \nu_\mu$ at rest in vacuum $\Longrightarrow \tau_{\rm dec} = 2.6 \times 10^{-8}\,{\rm s} \Longrightarrow \sigma_x \sim 10\,{\rm m}$

$\pi^+ \to \mu^+ + \nu_\mu$ in flight $\Longrightarrow E_\pi = 1\,{\rm GeV} \Longrightarrow \gamma \simeq 7 \Longrightarrow \sigma_x \sim 70\,{\rm m}$

$\pi^+ \to \mu^+ + \nu_\mu$ in decay tunnel of length ℓ

$$\tau_{\rm tun} = \frac{\ell}{v} \qquad \tau_{\rm tun} < \tau_{\rm dec} \quad \Longrightarrow \quad \sigma_x \sim \tau_{\rm tun}$$

$$E_\pi = 1\,{\rm GeV} \quad \Longrightarrow \quad v \simeq 0.99$$

$$\Downarrow$$

$$\ell = 10\,{\rm m} \quad \Longrightarrow \quad \sigma_x \sim 10\,{\rm m}$$

$$\Delta m^2 = 1\,{\rm eV}^2$$

$$\Downarrow$$

$$L^{\rm coh} \gtrsim \frac{E^2}{\Delta m^2}\,\sigma_x \sim 10^{18}\,{\rm m} \sim 100\,{\rm ly}$$

Collision Broadening

Mean Free Path: $\ell \implies \tau_{\text{col}} = \frac{\ell}{v}$ Problem: Estimate Mean Free Path ℓ

Nussinov, Physics Letters B 63 (1976) 201

"The effective time τ_{e} for the atom (or nucleus) to emit an uninterrupted train of waves is the time during which the random phase change of the emitter due to collisions, i.e. changes of potential energy, with neighboring atoms does not exceed $\approx \pi/2$." $\Rightarrow \tau_{\text{e}} = \tau_{\text{col}}$

Solar Neutrinos $\rho_c \approx 100\,\text{g}\,\text{cm}^{-3}$ $kT_c \approx 1\,\text{keV}$ $\Rightarrow$ $v \approx 10^{-3}$

$$N_p \approx \rho_c N_A \approx 6 \times 10^{25}\,\text{cm}^{-3} \qquad r \sim N_p^{-1/3} \sim 10^{-9}\,\text{cm} \quad \Rightarrow \quad \tau_r = \frac{r}{v} \sim 3 \times 10^{-17}\,\text{s}$$

Potential Energy: $V \sim \frac{Z_1 Z_2 e^2}{r} \approx 2\,\text{keV}$

Random Phase Shift: $\Delta\phi = \Delta E \Delta t \sim V \tau_r \sim 20 \gg \frac{\pi}{2}$

$$\sigma_x \sim \Delta t_{\text{P}} \sim \tau_{\text{col}} \sim \tau_r \sim 3 \times 10^{-17}\,\text{s} \sim 10^{-6}\,\text{cm}$$

collision $\Longleftrightarrow$ deflection $\Longleftrightarrow$ $\underset{\text{kinetic energy}}{K} \sim \frac{Z_1 Z_2 e^2}{r}$ [Anada, Nishimura, PRD 37 (1988) 552, 41 (1990) 2379]

$$K \sim T \quad \Longrightarrow \quad b \sim \frac{4\pi Z_1 Z_2 \alpha}{T}$$ impact parameter

$$\pi b^2 \ell N = 1 \text{ collision} \quad \Longrightarrow \quad \ell \sim \frac{1}{\pi b^2 N}$$

$$\ell \gtrsim \frac{1}{N^{1/3}} \quad \Longrightarrow \quad \boxed{\ell \sim \mathrm{Max}\left[\frac{1}{\pi b^2 N}, \frac{1}{N^{1/3}}\right]}$$ mean free path

effective time between collisions: $\tau_{\mathrm{col}} = \frac{\ell}{v} \quad \Longrightarrow \quad \sigma_x \sim \tau_{\mathrm{col}} = \frac{\ell}{v}$

Example: $\pi^+ \to \mu^+ + \nu_\mu$ in graphite at $T \simeq 300\,^\circ\mathrm{K}$

$$\sigma_x \sim \mathrm{Min}\left[\frac{\ell_\pi}{v_\pi}, \frac{\ell_\mu}{v_\mu}\right]$$

graphite: $Z = 6 \quad A \simeq 12 \quad \rho \simeq 2\,\mathrm{g\,cm^{-3}} \quad N \simeq 10^{23}\,\mathrm{cm^{-3}} \quad N^{-1/3} \sim 2 \times 10^{-8}\,\mathrm{cm}$

$$\ell \sim \mathrm{Max}\left[3 \times 10^{-2} \left(\frac{T}{\mathsf{MeV}}\right)^2 \mathsf{cm}\,,\ 2 \times 10^{-8}\,\mathrm{cm}\right]$$

$$T \sim 300K \Rightarrow K_\pi \sim 4 \times 10^{-2}\,\mathrm{eV} \Rightarrow \ell \sim \mathrm{Max}\left[5 \times 10^{-17}\,,\ 2 \times 10^{-8}\right]\mathrm{cm} = 2 \times 10^{-8}\,\mathrm{cm}$$

$$v_\pi \simeq 2 \times 10^{-5} \quad \Longrightarrow \quad \frac{\ell_\pi}{v_\pi} \sim 10^{-3}\,\mathrm{cm}$$

$$K_\mu = \frac{(m_\pi - m_\mu)^2}{2m_\pi} \simeq 4\,\mathrm{MeV} \quad \Rightarrow \quad \ell \sim \mathrm{Max}\left[0.5\,,\ 2 \times 10^{-8}\right]\mathrm{cm} = 0.5$$

$$v_\mu \simeq 0.3 \quad \Longrightarrow \quad \frac{\ell_\mu}{v_\mu} \sim 1\,\mathrm{cm}$$

$$\sigma_x \sim \frac{\ell_\pi}{v_\pi} \sim 10^{-3}\,\mathrm{cm}$$

Estimates of Coherence Length

$$L^{\text{osc}} = \frac{4\pi E}{\Delta m^2} = 2.5 \frac{(E/\text{MeV})}{(\Delta m^2/\text{eV}^2)} \text{m} \qquad L^{\text{coh}} \sim \frac{4\sqrt{2}E^2}{|\Delta m^2|}\sigma_x = 10^{12} \frac{(E^2/\text{MeV}^2)}{(|\Delta m^2|/\text{eV}^2)} \left(\frac{\sigma_x}{\text{m}}\right) \text{m}$$

Process	$\|\Delta m^2\|$	L^{osc}	σ_x	L^{coh}
$\pi \to \mu + \nu$ at rest in vacuum: $E \simeq 30\,\text{MeV}$ natural linewidth	$2.5 \times 10^{-3}\,\text{eV}^2$	$30\,\text{km}$	$\tau_\pi \sim 10\,\text{m}$	$\sim 10^{16}\,\text{km}$
$\pi \to \mu + \nu$ at rest in matter: $E \simeq 30\,\text{MeV}$ collision broadening	$2.5 \times 10^{-3}\,\text{eV}^2$	$30\,\text{km}$	$\tau_{\text{col}} \sim 10^{-5}\,\text{m}$	$\sim 10^{10}\,\text{km}$
$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ at rest in matter: $E \leq 50\,\text{MeV}$ collision broadening	$1\,\text{eV}^2$	$\leq 125\,\text{m}$	$\tau_{\text{col}} \sim 10^{-10}\,\text{m}$	$\lesssim 10^2\,\text{km}$
${}^7\text{Be} + e^- \to {}^7\text{Li} + \nu_e$ in solar core: $E \simeq 0.86\,\text{MeV}$ collision broadening	$7 \times 10^{-5}\,\text{eV}^2$	$31\,\text{km}$	$\tau_{\text{col}} \sim 10^{-9}\,\text{m}$	$\sim 10^4\,\text{km}$

Conclusions

- Standard expression for Oscillation Length of Ultrarelativistic Neutrinos is robust.
- Wave Packet Treatment is necessary for $T \simeq L \Leftrightarrow$ Oscillations in Space.
- Quantum Field Theoretical Wave Packet Models confirm Standard Oscillation Length and allows to calculate Coherence Length.

Neutrino Unbound

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