

Neutrino Physics

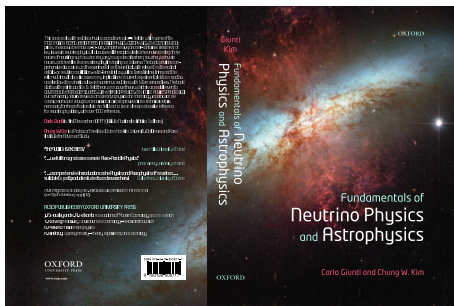
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Neutrino Unbound: <http://www.nu.to.infn.it>

Doctoral Program in Physics, EPFL, Lausanne, May 2011



C. Giunti and C.W. Kim
Fundamentals of Neutrino Physics
and Astrophysics
Oxford University Press
15 March 2007 – 728 pages

Part I: Theory of Neutrino Masses and Mixing

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
- Number of Flavor and Massive Neutrinos?
- Sterile Neutrinos

Part II: Neutrino Oscillations

- Neutrino Oscillations in Vacuum
- CPT, CP and T Symmetries
- Two-Neutrino Oscillations
- Neutrino Oscillations in Matter

Part III: Phenomenology

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- Atmospheric and LBL Oscillation Experiments
- Phenomenology of Three-Neutrino Mixing
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Part II

Neutrino Oscillations

- Neutrino Oscillations in Vacuum
- CPT, CP and T Symmetries
- Two-Neutrino Oscillations
- Neutrino Oscillations in Matter

Neutrino Oscillations in Vacuum

- Neutrino Oscillations in Vacuum
 - Ultrarelativistic Approximation
 - Easy Example of Neutrino Production
 - Neutrino Oscillations
 - Neutrinos and Antineutrinos
- CPT, CP and T Symmetries
- Two-Neutrino Oscillations
- Neutrino Oscillations in Matter

Ultrarelativistic Approximation

Only neutrinos with energy $\gtrsim 0.1 \text{ MeV}$ are detectable!

Charged-Current Processes: Threshold

$$\begin{aligned} \nu + A &\rightarrow B + C \\ &\Downarrow \\ s &= 2Em_A + m_A^2 \geq (m_B + m_C)^2 \\ &\Downarrow \\ E_{\text{th}} &= \frac{(m_B + m_C)^2}{2m_A} - \frac{m_A}{2} \end{aligned}$$

$$\nu_e + {}^{71}\text{Ga} \rightarrow {}^{71}\text{Ge} + e^- \quad E_{\text{th}} = 0.233 \text{ MeV}$$

$$\nu_e + {}^{37}\text{Cl} \rightarrow {}^{37}\text{Ar} + e^- \quad E_{\text{th}} = 0.81 \text{ MeV}$$

$$\bar{\nu}_e + p \rightarrow n + e^+ \quad E_{\text{th}} = 1.8 \text{ MeV}$$

$$\nu_\mu + n \rightarrow p + \mu^- \quad E_{\text{th}} = 110 \text{ MeV}$$

$$\nu_\mu + e^- \rightarrow \nu_e + \mu^- \quad E_{\text{th}} \simeq \frac{m_\mu^2}{2m_e} = 10.9 \text{ GeV}$$

Elastic Scattering Processes: Cross Section \propto Energy

$$\nu + e^- \rightarrow \nu + e^- \quad \sigma(E) \sim \sigma_0 E/m_e \quad \sigma_0 \sim 10^{-44} \text{ cm}^2$$

Background $\implies E_{\text{th}} \simeq 5 \text{ MeV}$ (SK, SNO), 0.25 MeV (Borexino)

Laboratory and Astrophysical Limits $\implies m_\nu \lesssim 1 \text{ eV}$

Easy Example of Neutrino Production

$$\underline{\pi^+ \rightarrow \mu^+ + \nu_\mu}$$

$$\nu_\mu = \sum_k U_{\mu k} \nu_k$$

two-body decay \implies fixed kinematics

$$E_k^2 = p_k^2 + m_k^2$$

$$\pi \text{ at rest: } \begin{cases} p_k^2 = \frac{m_\pi^2}{4} \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 - \frac{m_k^2}{2} \left(1 + \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_k^4}{4 m_\pi^2} \\ E_k^2 = \frac{m_\pi^2}{4} \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 + \frac{m_k^2}{2} \left(1 - \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_k^4}{4 m_\pi^2} \end{cases}$$

$$0^{\text{th}} \text{ order: } m_k = 0 \implies p_k = E_k = E = \frac{m_\pi}{2} \left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 30 \text{ MeV}$$

$$1^{\text{st}} \text{ order: } E_k \simeq E + \xi \frac{m_k^2}{2E}$$

$$p_k \simeq E - (1 - \xi) \frac{m_k^2}{2E}$$

$$\xi = \frac{1}{2} \left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 0.2$$

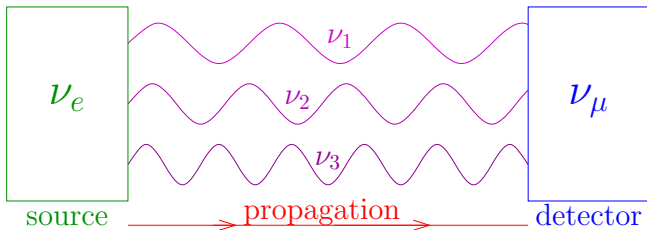
Neutrino Oscillations

- ▶ 1957: Bruno Pontecorvo proposed Neutrino Oscillations in analogy with $K^0 \leftrightarrow \bar{K}^0$ oscillations (Gell-Mann and Pais, 1955)
- ▶ Flavor Neutrinos: ν_e, ν_μ, ν_τ produced in Weak Interactions
- ▶ Massive Neutrinos: ν_1, ν_2, ν_3 propagate from Source to Detector
- ▶ A Flavor Neutrino is a superposition of Massive Neutrinos

$$\begin{aligned} |\nu_e\rangle &= U_{e1} |\nu_1\rangle + U_{e2} |\nu_2\rangle + U_{e3} |\nu_3\rangle \\ |\nu_\mu\rangle &= U_{\mu1} |\nu_1\rangle + U_{\mu2} |\nu_2\rangle + U_{\mu3} |\nu_3\rangle \\ |\nu_\tau\rangle &= U_{\tau1} |\nu_1\rangle + U_{\tau2} |\nu_2\rangle + U_{\tau3} |\nu_3\rangle \end{aligned}$$

- ▶ U is the 3×3 Neutrino Mixing Matrix

$$|\nu(t=0)\rangle = |\nu_e\rangle = U_{e1} |\nu_1\rangle + U_{e2} |\nu_2\rangle + U_{e3} |\nu_3\rangle$$



$$|\nu(t > 0)\rangle = U_{e1} e^{-iE_1 t} |\nu_1\rangle + U_{e2} e^{-iE_2 t} |\nu_2\rangle + U_{e3} e^{-iE_3 t} |\nu_3\rangle \neq |\nu_e\rangle$$

at the detector there is a **probability** > 0 to see the neutrino as a ν_μ

Neutrino Oscillations are Flavor Transitions

$$\nu_e \rightarrow \nu_\mu \quad \nu_e \rightarrow \nu_\tau \quad \nu_\mu \rightarrow \nu_e \quad \nu_\mu \rightarrow \nu_\tau$$

$$\bar{\nu}_e \rightarrow \bar{\nu}_\mu \quad \bar{\nu}_e \rightarrow \bar{\nu}_\tau \quad \bar{\nu}_\mu \rightarrow \bar{\nu}_e \quad \bar{\nu}_\mu \rightarrow \bar{\nu}_\tau$$

Neutrino Oscillations in Vacuum

[Eliezer, Swift, NPB 105 (1976) 45] [Fritzsch, Minkowski, PLB 62 (1976) 72] [Bilenky, Pontecorvo, SJNP 24 (1976) 316]

$$\mathcal{L}_{CC} \sim W_\rho (\bar{\nu}_e \gamma^\rho e_L + \bar{\nu}_\mu \gamma^\rho \mu_L + \bar{\nu}_\tau \gamma^\rho \tau_L)$$

Fields $\nu_\alpha = \sum_k U_{\alpha k} \nu_k \quad \Rightarrow \quad |\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$ States

initial flavor: $\alpha = e \text{ or } \mu \text{ or } \tau$

$$|\nu_k(t, x)\rangle = e^{-iE_k t + ip_k x} |\nu_k\rangle \quad \Rightarrow \quad |\nu_\alpha(t, x)\rangle = \sum_k U_{\alpha k}^* e^{-iE_k t + ip_k x} |\nu_k\rangle$$

$$|\nu_k\rangle = \sum_{\beta=e,\mu,\tau} U_{\beta k} |\nu_\beta\rangle \quad \Rightarrow \quad |\nu_\alpha(t, x)\rangle = \sum_{\beta=e,\mu,\tau} \underbrace{\left(\sum_k U_{\alpha k}^* e^{-iE_k t + ip_k x} U_{\beta k} \right)}_{\mathcal{A}_{\nu_\alpha \rightarrow \nu_\beta}(t, x)} |\nu_\beta\rangle$$

$$\mathcal{A}_{\nu_\alpha \rightarrow \nu_\beta}(0, 0) = \sum_k U_{\alpha k}^* U_{\beta k} = \delta_{\alpha\beta}$$

$$\mathcal{A}_{\nu_\alpha \rightarrow \nu_\beta}(t > 0, x > 0) \neq \delta_{\alpha\beta}$$

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t, x) = \left| \mathcal{A}_{\nu_\alpha \rightarrow \nu_\beta}(t, x) \right|^2 = \left| \sum_k U_{\alpha k}^* e^{-iE_k t + i p_k x} U_{\beta k} \right|^2$$

ultra-relativistic neutrinos $\implies t \simeq x = L$ source-detector distance

$$E_k t - p_k x \simeq (E_k - p_k) L = \frac{E_k^2 - p_k^2}{E_k + p_k} L = \frac{m_k^2}{E_k + p_k} L \simeq \frac{m_k^2}{2E} L$$

$$\begin{aligned} P_{\nu_\alpha \rightarrow \nu_\beta}(L, E) &= \left| \sum_k U_{\alpha k}^* e^{-im_k^2 L/2E} U_{\beta k} \right|^2 \\ &= \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right) \end{aligned}$$

$$\Delta m_{kj}^2 \equiv m_k^2 - m_j^2$$

Neutrinos and Antineutrinos

Right-handed antineutrinos are described by CP-conjugated fields:

$$\nu^{\text{CP}} = \gamma^0 C \bar{\nu}^T = -C \nu^*$$

C \implies Particle \iff Antiparticle

P \implies Left-Handed \iff Right-Handed

Fields: $\nu_{\alpha L} = \sum_k U_{\alpha k} \nu_{kL} \xrightarrow{\text{CP}} \nu_{\alpha L}^{\text{CP}} = \sum_k U_{\alpha k}^* \nu_{kL}^{\text{CP}}$

States: $|\nu_{\alpha}\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle \xrightarrow{\text{CP}} |\bar{\nu}_{\alpha}\rangle = \sum_k U_{\alpha k} |\bar{\nu}_k\rangle$

NEUTRINOS $U \iff U^*$ ANTINEUTRINOS

$$P_{\nu_{\alpha} \rightarrow \nu_{\beta}}(L, E) = \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right)$$

$$P_{\bar{\nu}_{\alpha} \rightarrow \bar{\nu}_{\beta}}(L, E) = \sum_{k,j} U_{\alpha k} U_{\beta k}^* U_{\alpha j}^* U_{\beta j} \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right)$$

CPT, CP and T Symmetries

- Neutrino Oscillations in Vacuum
- CPT, CP and T Symmetries
 - CPT Symmetry
 - CP Symmetry
 - T Symmetry
- Two-Neutrino Oscillations
- Neutrino Oscillations in Matter

CPT Symmetry

$$P_{\nu_\alpha \rightarrow \nu_\beta} \xrightarrow{\text{CPT}} P_{\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha}$$

$$\text{CPT Asymmetries: } A_{\alpha\beta}^{\text{CPT}} = P_{\nu_\alpha \rightarrow \nu_\beta} - P_{\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha}$$

$$\text{Local Quantum Field Theory} \implies A_{\alpha\beta}^{\text{CPT}} = 0 \quad \text{CPT Symmetry}$$

$$P_{\nu_\alpha \rightarrow \nu_\beta}(L, E) = \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right)$$

is invariant under CPT: $U \leftrightarrow U^* \quad \alpha \leftrightarrow \beta$

$$P_{\nu_\alpha \rightarrow \nu_\beta} = P_{\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha}$$

$$P_{\nu_\alpha \rightarrow \nu_\alpha} = P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\alpha}$$

(solar ν_e , reactor $\bar{\nu}_e$, accelerator ν_μ)

CP Symmetry

$$P_{\nu_\alpha \rightarrow \nu_\beta} \xrightarrow{\text{CP}} P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta}$$

CP Asymmetries: $A_{\alpha\beta}^{\text{CP}} = P_{\nu_\alpha \rightarrow \nu_\beta} - P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta}$

CPT $\Rightarrow A_{\alpha\beta}^{\text{CP}} = -A_{\beta\alpha}^{\text{CP}}$

$$A_{\alpha\beta}^{\text{CP}}(L, E) = 4 \sum_{k>j} \text{Im} \left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \right] \sin \left(\frac{\Delta m_{kj}^2 L}{2E} \right)$$

Jarlskog rephasing invariant: $\text{Im} \left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \right] = \pm J$

$$J = c_{12} s_{12} c_{23} s_{23} c_{13}^2 s_{13} \sin \delta_{13}$$

violation of CP in neutrino oscillations is proportional to

$$|U_{e3}| = \sin \vartheta_{13} \quad \text{and} \quad \sin \delta_{13}$$

T Symmetry

$$P_{\nu_\alpha \rightarrow \nu_\beta} \xrightarrow{T} P_{\nu_\beta \rightarrow \nu_\alpha}$$

$$T \text{ Asymmetries: } A_{\alpha\beta}^T = P_{\nu_\alpha \rightarrow \nu_\beta} - P_{\nu_\beta \rightarrow \nu_\alpha}$$

$$\begin{aligned} \text{CPT} \implies 0 &= A_{\alpha\beta}^{\text{CPT}} = P_{\nu_\alpha \rightarrow \nu_\beta} - P_{\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha} \\ &= P_{\nu_\alpha \rightarrow \nu_\beta} - P_{\nu_\beta \rightarrow \nu_\alpha} + P_{\nu_\beta \rightarrow \nu_\alpha} - P_{\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha} \\ &= A_{\alpha\beta}^T + A_{\beta\alpha}^{\text{CP}} = A_{\alpha\beta}^T - A_{\alpha\beta}^{\text{CP}} \implies \boxed{A_{\alpha\beta}^T = A_{\alpha\beta}^{\text{CP}}} \end{aligned}$$

$$\boxed{A_{\alpha\beta}^T(L, E) = 4 \sum_{k>j} \text{Im} \left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \right] \sin \left(\frac{\Delta m_{kj}^2 L}{2E} \right)}$$

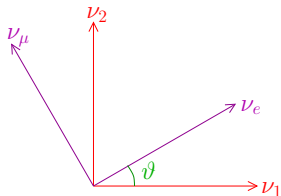
$$\text{Jarlskog rephasing invariant: } \text{Im} \left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \right] = \pm J$$

Two-Neutrino Oscillations

- Neutrino Oscillations in Vacuum
- CPT, CP and T Symmetries
- Two-Neutrino Oscillations
 - Two-Neutrino Mixing and Oscillations
 - Types of Experiments
 - Average over Energy Resolution of the Detector
- Neutrino Oscillations in Matter

Two-Neutrino Mixing and Oscillations

$$|\nu_\alpha\rangle = \sum_{k=1}^2 U_{\alpha k} |\nu_k\rangle \quad (\alpha = e, \mu)$$



$$U = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix}$$

$$\begin{aligned} |\nu_e\rangle &= \cos \vartheta |\nu_1\rangle + \sin \vartheta |\nu_2\rangle \\ |\nu_\mu\rangle &= -\sin \vartheta |\nu_1\rangle + \cos \vartheta |\nu_2\rangle \end{aligned}$$

$$\Delta m^2 \equiv \Delta m_{21}^2 \equiv m_2^2 - m_1^2$$

Transition Probability: $P_{\nu_e \rightarrow \nu_\mu} = P_{\nu_\mu \rightarrow \nu_e} = \sin^2 2\vartheta \sin^2 \left(\frac{\Delta m^2 L}{4E} \right)$

Survival Probabilities: $P_{\nu_e \rightarrow \nu_e} = P_{\nu_\mu \rightarrow \nu_\mu} = 1 - P_{\nu_e \rightarrow \nu_\mu}$

two-neutrino mixing transition probability

$$\alpha \neq \beta$$

$$\alpha, \beta = e, \mu, \tau$$

$$\begin{aligned} P_{\nu_\alpha \rightarrow \nu_\beta}(L, E) &= \sin^2 2\vartheta \sin^2 \left(\frac{\Delta m^2 L}{4E} \right) \\ &= \sin^2 2\vartheta \sin^2 \left(1.27 \frac{\Delta m^2 [\text{eV}^2] L [\text{m}]}{E [\text{MeV}]} \right) \\ &= \sin^2 2\vartheta \sin^2 \left(1.27 \frac{\Delta m^2 [\text{eV}^2] L [\text{km}]}{E [\text{GeV}]} \right) \end{aligned}$$

oscillation length

$$L^{\text{osc}} = \frac{4\pi E}{\Delta m^2} = 2.47 \frac{E [\text{MeV}]}{\Delta m^2 [\text{eV}^2]} \text{ m} = 2.47 \frac{E [\text{GeV}]}{\Delta m^2 [\text{eV}^2]} \text{ km}$$

Types of Experiments

Two-Neutrino
Mixing

$$P_{\nu_\alpha \rightarrow \nu_\beta}(L, E) = \sin^2 2\vartheta \sin^2 \left(\frac{\Delta m^2 L}{4E} \right)$$

observable if
 $\frac{\Delta m^2 L}{4E} \gtrsim 1$

SBL

$L/E \lesssim 10 \text{ eV}^{-2} \Rightarrow \Delta m^2 \gtrsim 0.1 \text{ eV}^2$

Reactor: $L \sim 10 \text{ m}$, $E \sim 1 \text{ MeV}$

Accelerator: $L \sim 1 \text{ km}$, $E \gtrsim 0.1 \text{ GeV}$

ATM & LBL

$L/E \lesssim 10^4 \text{ eV}^{-2}$

Reactor: $L \sim 1 \text{ km}$, $E \sim 1 \text{ MeV}$ CHOOZ, PALO VERDE

Accelerator: $L \sim 10^3 \text{ km}$, $E \gtrsim 1 \text{ GeV}$ K2K, MINOS, CNGS

Atmospheric: $L \sim 10^2 - 10^4 \text{ km}$, $E \sim 0.1 - 10^2 \text{ GeV}$

$\Delta m^2 \gtrsim 10^{-4} \text{ eV}^2$ Kamiokande, IMB, Super-Kamiokande, Soudan, MACRO, MINOS

SUN

$L \sim 10^8 \text{ km}$, $E \sim 0.1 - 10 \text{ MeV}$

$\frac{L}{E} \sim 10^{11} \text{ eV}^{-2} \Rightarrow \Delta m^2 \gtrsim 10^{-11} \text{ eV}^2$

Homestake, Kamiokande, GALLEX, SAGE,

Super-Kamiokande, GNO, SNO, Borexino

Matter Effect (MSW) $\Rightarrow 10^{-4} \lesssim \sin^2 2\vartheta \lesssim 1$, $10^{-8} \text{ eV}^2 \lesssim \Delta m^2 \lesssim 10^{-4} \text{ eV}^2$

VLBL

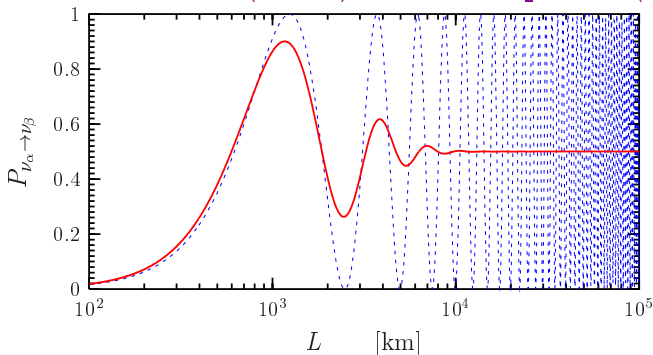
$L/E \lesssim 10^5 \text{ eV}^{-2} \Rightarrow \Delta m^2 \gtrsim 10^{-5} \text{ eV}^2$

Reactor: $L \sim 10^2 \text{ km}$, $E \sim 1 \text{ MeV}$

KamLAND

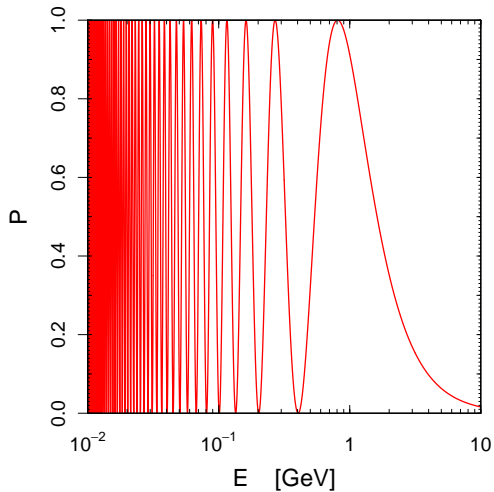
Average over Energy Resolution of the Detector

$$P_{\nu_\alpha \rightarrow \nu_\beta}(L, E) = \sin^2 2\vartheta \sin^2 \left(\frac{\Delta m^2 L}{4E} \right) = \frac{1}{2} \sin^2 2\vartheta \left[1 - \cos \left(\frac{\Delta m^2 L}{2E} \right) \right]$$



$$\Delta m^2 = 10^{-3} \text{ eV} \quad \sin^2 2\vartheta = 1 \quad \langle E \rangle = 1 \text{ GeV} \quad \Delta E = 0.2 \text{ GeV}$$

$$\langle P_{\nu_\alpha \rightarrow \nu_\beta}(L, E) \rangle = \frac{1}{2} \sin^2 2\vartheta \left[1 - \int \cos \left(\frac{\Delta m^2 L}{2E} \right) \phi(E) dE \right] \quad (\alpha \neq \beta)$$

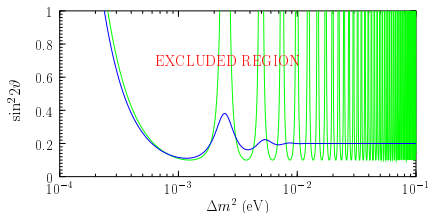


$$\Delta m^2 = 1 \text{ eV} \quad \sin^2 2\vartheta = 1 \quad L = 1 \text{ km}$$

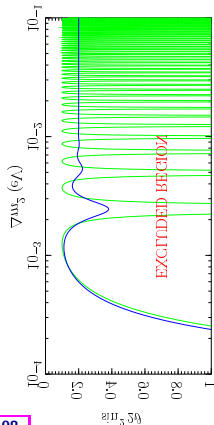
Exclusion Curves

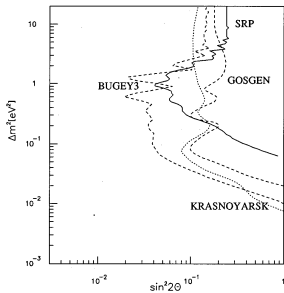
$$\langle P_{\nu_\alpha \rightarrow \nu_\beta}(L, E) \rangle = \frac{1}{2} \sin^2 2\vartheta \left[1 - \int \cos\left(\frac{\Delta m^2 L}{2E}\right) \phi(E) dE \right] \quad (\alpha \neq \beta)$$

$$\langle P_{\nu_\alpha \rightarrow \nu_\beta}(L, E) \rangle \leq P_{\nu_\alpha \rightarrow \nu_\beta}^{\max} \implies \sin^2 2\vartheta \leq \frac{2 P_{\nu_\alpha \rightarrow \nu_\beta}^{\max}}{1 - \int \cos\left(\frac{\Delta m^2 L}{2E}\right) \phi(E) dE}$$

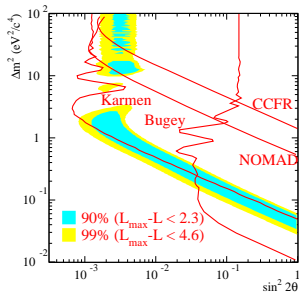


rotate
and
mirror

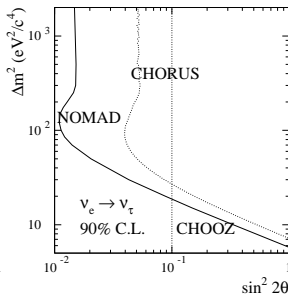
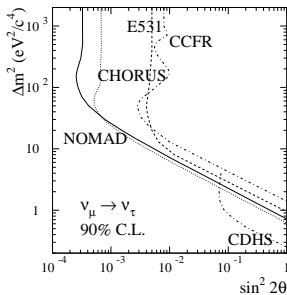




Reactor SBL Experiments: $\bar{\nu}_e \rightarrow \bar{\nu}_e$

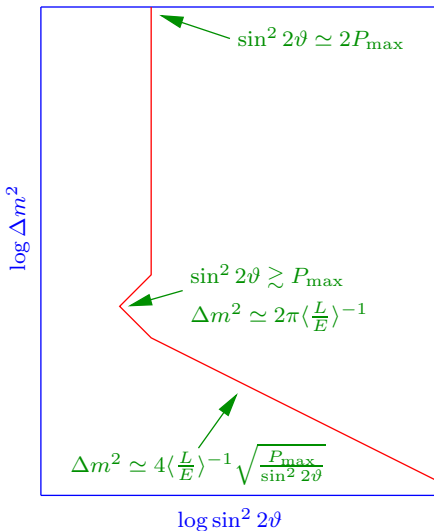


Accelerator SBL Experiments: $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$



Accelerator SBL Experiments: $\bar{\nu}_\mu \rightarrow \bar{\nu}_\tau$ and $\bar{\nu}_e \rightarrow \bar{\nu}_\tau$

Anatomy of Exclusion Plots



▶ $\Delta m^2 \gg \langle L/E \rangle^{-1}$

$$P_{\max} \simeq \frac{1}{2} \sin^2 2\vartheta \Rightarrow \sin^2 2\vartheta \simeq 2P_{\max}$$

▶ $\text{Min} \left\langle \cos \left(\frac{\Delta m^2 L}{2E} \right) \right\rangle \geq -1$

$$\sin^2 2\vartheta = \frac{2P_{\max}}{1 - \text{Min} \left\langle \cos \left(\frac{\Delta m^2 L}{2E} \right) \right\rangle} \geq P_{\max}$$

$$\Delta m^2 \simeq 2\pi \langle L/E \rangle^{-1}$$

▶ $\Delta m^2 \ll 2\pi \langle L/E \rangle^{-1}$

$$\cos \left(\frac{\Delta m^2 L}{2E} \right) \simeq 1 - \frac{1}{2} \left(\frac{\Delta m^2 L}{2E} \right)^2$$

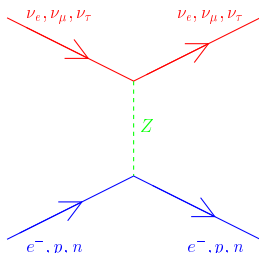
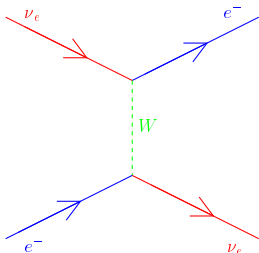
$$\Delta m^2 \simeq 4 \left\langle \frac{L}{E} \right\rangle^{-1} \sqrt{\frac{P_{\max}}{\sin^2 2\vartheta}}$$

$$\frac{1}{2} \sin^2 2\vartheta \left[1 - \left\langle \cos \left(\frac{\Delta m^2 L}{2E} \right) \right\rangle \right] = P_{\max}$$

Neutrino Oscillations in Matter

- Neutrino Oscillations in Vacuum
- CPT, CP and T Symmetries
- Two-Neutrino Oscillations
- Neutrino Oscillations in Matter
 - Effective Potentials in Matter
 - Evolution of Neutrino Flavors in Matter
 - MSW Effect (Resonant Transitions in Matter)
 - Solar Neutrinos
 - In Neutrino Oscillations Dirac = Majorana

Effective Potentials in Matter



$$V_{CC} = \sqrt{2} G_F N_e$$

$$V_{NC}^{(e^-)} = -V_{NC}^{(p)} \Rightarrow$$

$$V_{NC} = V_{NC}^{(n)} = -\frac{\sqrt{2}}{2} G_F N_n$$

$$V_e = V_{CC} + V_{NC}$$

$$V_{\mu} = V_{\tau} = V_{NC}$$

only $V_{CC} = V_e - V_{\mu} = V_e - V_{\tau}$ is important for flavor transitions

antineutrinos: $\bar{V}_{CC} = -V_{CC}$ $\bar{V}_{NC} = -V_{NC}$

Matter Effects

a flavor neutrino ν_α with momentum p is described by

$$|\nu_\alpha(p)\rangle = \sum_k U_{\alpha k}^* |\nu_k(p)\rangle$$

$$\mathcal{H}_0 |\nu_k(p)\rangle = E_k |\nu_k(p)\rangle \quad E_k = \sqrt{p^2 + m_k^2}$$

in matter $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \quad \mathcal{H}_I |\nu_\alpha(p)\rangle = V_\alpha |\nu_\alpha(p)\rangle$

$V_\alpha =$ effective potential due to coherent interactions with the medium

forward elastic CC and NC scattering

Evolution of Neutrino Flavors in Matter

Schrödinger picture: $i \frac{d}{dt} |\nu(p, t)\rangle = \mathcal{H}|\nu(p, t)\rangle, \quad |\nu(p, 0)\rangle = |\nu_\alpha(p)\rangle$

flavor transition amplitudes: $\varphi_\beta(p, t) = \langle \nu_\beta(p) | \nu(p, t) \rangle, \quad \varphi_\beta(p, 0) = \delta_{\alpha\beta}$

$$i \frac{d}{dt} \varphi_\beta(p, t) = \langle \nu_\beta(p) | \mathcal{H} | \nu(p, t) \rangle = \langle \nu_\beta(p) | \mathcal{H}_0 | \nu(p, t) \rangle + \langle \nu_\beta(p) | \mathcal{H}_I | \nu(p, t) \rangle$$

$$\begin{aligned} \langle \nu_\beta(p) | \mathcal{H}_0 | \nu(p, t) \rangle &= \sum_\rho \langle \nu_\beta(p) | \mathcal{H}_0 | \nu_\rho(p) \rangle \underbrace{\langle \nu_\rho(p) | \nu(p, t) \rangle}_{\varphi_\rho(p, t)} \\ &= \sum_\rho \sum_{kj} U_{\beta k} \underbrace{\langle \nu_k(p) | \mathcal{H}_0 | \nu_j(p) \rangle}_{\delta_{kj} E_k} U_{\rho j}^* \varphi_\rho(p, t) \end{aligned}$$

$$\langle \nu_\beta(p) | \mathcal{H}_I | \nu(p, t) \rangle = \sum_\rho \underbrace{\langle \nu_\beta(p) | \mathcal{H}_I | \nu_\rho(p) \rangle}_{\delta_{\beta\rho} V_\beta} \varphi_\rho(p, t) = V_\beta \varphi_\beta(p, t)$$

$$i \frac{d}{dt} \varphi_\beta = \sum_\rho \left(\sum_k U_{\beta k} E_k U_{\rho k}^* + \delta_{\beta\rho} V_\beta \right) \varphi_\rho$$

ultrarelativistic neutrinos: $E_k = p + \frac{m_k^2}{2E}$ $E = p$ $t = x$

$$V_e = V_{CC} + V_{NC} \qquad V_\mu = V_\tau = V_{NC}$$

$$i \frac{d}{dx} \varphi_\beta(p, x) = (p + V_{NC}) \varphi_\beta(p, x) + \sum_\rho \left(\sum_k U_{\beta k} \frac{m_k^2}{2E} U_{\rho k}^* + \delta_{\beta e} \delta_{\rho e} V_{CC} \right) \varphi_\rho(p, x)$$

$$\psi_\beta(p, x) = \varphi_\beta(p, x) e^{ipx + i \int_0^x V_{NC}(x') dx'}$$

$$i \frac{d}{dx} \psi_\beta = e^{ipx + i \int_0^x V_{NC}(x') dx'} \left(-p - V_{NC} + i \frac{d}{dx} \right) \varphi_\beta$$

$$i \frac{d}{dx} \psi_\beta = \sum_\rho \left(\sum_k U_{\beta k} \frac{m_k^2}{2E} U_{\rho k}^* + \delta_{\beta e} \delta_{\rho e} V_{CC} \right) \psi_\rho$$

$$P_{\nu_\alpha \rightarrow \nu_\beta} = |\varphi_\beta|^2 = |\psi_\beta|^2$$

evolution of flavor transition amplitudes in matrix form

$$i \frac{d}{dx} \Psi_\alpha = \frac{1}{2E} (U M^2 U^\dagger + A) \Psi_\alpha$$

$$\Psi_\alpha = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \quad M^2 = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} \quad A = \begin{pmatrix} A_{CC} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_{CC} = 2EV_{CC} = 2\sqrt{2}EG_F N_e$$

effective
mass-squared
matrix
in vacuum

$$M_{\text{VAC}}^2 = U M^2 U^\dagger \xrightarrow{\text{matter}} U M^2 U^\dagger + 2E V = M_{\text{MAT}}^2$$

potential due to coherent
forward elastic scattering

effective
mass-squared
matrix
in matter

Two-Neutrino Mixing

$$\nu_e \rightarrow \nu_\mu \text{ transitions with } U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

$$\begin{aligned} U M^2 U^\dagger &= \begin{pmatrix} \cos^2\vartheta m_1^2 + \sin^2\vartheta m_2^2 & \cos\vartheta \sin\vartheta (m_2^2 - m_1^2) \\ \cos\vartheta \sin\vartheta (m_2^2 - m_1^2) & \sin^2\vartheta m_1^2 + \cos^2\vartheta m_2^2 \end{pmatrix} \\ &= \frac{1}{2} \Sigma m^2 + \frac{1}{2} \begin{pmatrix} -\Delta m^2 \cos 2\vartheta & \Delta m^2 \sin 2\vartheta \\ \Delta m^2 \sin 2\vartheta & \Delta m^2 \cos 2\vartheta \end{pmatrix} \\ &\quad \uparrow \\ &\text{irrelevant common phase} \end{aligned}$$

$$\Sigma m^2 \equiv m_1^2 + m_2^2$$

$$\Delta m^2 \equiv m_2^2 - m_1^2$$

$$i \frac{d}{dx} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\Delta m^2 \cos 2\vartheta + 2A_{CC} & \Delta m^2 \sin 2\vartheta \\ \Delta m^2 \sin 2\vartheta & \Delta m^2 \cos 2\vartheta \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\text{initial } \nu_e \implies \begin{pmatrix} \psi_e(0) \\ \psi_\mu(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$P_{\nu_e \rightarrow \nu_\mu}(x) = |\psi_\mu(x)|^2$$
$$P_{\nu_e \rightarrow \nu_e}(x) = |\psi_e(x)|^2 = 1 - P_{\nu_e \rightarrow \nu_\mu}(x)$$

Constant Matter Density

$$i \frac{d}{dx} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\Delta m^2 \cos 2\vartheta + 2A_{CC} & \Delta m^2 \sin 2\vartheta \\ \Delta m^2 \sin 2\vartheta & \Delta m^2 \cos 2\vartheta \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\frac{dA_{CC}}{dx} = 0$$

Diagonalization of Effective Hamiltonian

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos \vartheta_M & \sin \vartheta_M \\ -\sin \vartheta_M & \cos \vartheta_M \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$i \frac{d}{dx} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \left[\frac{A_{CC}}{4E} + \frac{1}{4E} \begin{pmatrix} -\Delta m_M^2 & 0 \\ 0 & \Delta m_M^2 \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

↑
irrelevant common phase

Effective Mixing Angle in Matter

$$\tan 2\vartheta_M = \frac{\tan 2\vartheta}{1 - \frac{A_{CC}}{\Delta m^2 \cos 2\vartheta}}$$

Effective Squared-Mass Difference

$$\Delta m_M^2 = \sqrt{(\Delta m^2 \cos 2\vartheta - A_{CC})^2 + (\Delta m^2 \sin 2\vartheta)^2}$$

Resonance ($\vartheta_M = \pi/4$)

$$A_{CC}^R = \Delta m^2 \cos 2\vartheta \quad \Rightarrow \quad N_e^R = \frac{\Delta m^2 \cos 2\vartheta}{2\sqrt{2}EG_F}$$

$$i \frac{d}{dx} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\Delta m_M^2 & 0 \\ 0 & \Delta m_M^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\vartheta_M & \sin\vartheta_M \\ -\sin\vartheta_M & \cos\vartheta_M \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos\vartheta_M & -\sin\vartheta_M \\ \sin\vartheta_M & \cos\vartheta_M \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\nu_e \rightarrow \nu_\mu \Rightarrow \begin{pmatrix} \psi_e(0) \\ \psi_\mu(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \begin{pmatrix} \cos\vartheta_M \\ \sin\vartheta_M \end{pmatrix}$$

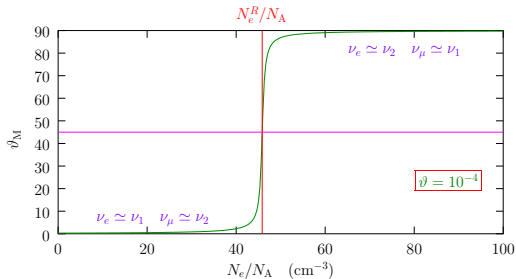
$$\psi_1(x) = \cos\vartheta_M \exp\left(i \frac{\Delta m_M^2 x}{4E}\right)$$

$$\psi_2(x) = \sin\vartheta_M \exp\left(-i \frac{\Delta m_M^2 x}{4E}\right)$$

$$P_{\nu_e \rightarrow \nu_\mu}(x) = |\psi_\mu(x)|^2 = |-\sin\vartheta_M \psi_1(x) + \cos\vartheta_M \psi_2(x)|^2$$

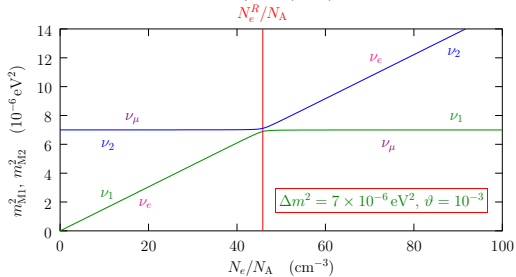
$$P_{\nu_e \rightarrow \nu_\mu}(x) = \sin^2 2\vartheta_M \sin^2\left(\frac{\Delta m_M^2 x}{4E}\right)$$

MSW Effect (Resonant Transitions in Matter)



$$\begin{aligned}\nu_e &= \cos\vartheta_M \nu_1 + \sin\vartheta_M \nu_2 \\ \nu_\mu &= -\sin\vartheta_M \nu_1 + \cos\vartheta_M \nu_2\end{aligned}$$

$$\tan 2\vartheta_M = \frac{\tan 2\vartheta}{1 - \frac{A_{\text{CC}}}{\Delta m^2 \cos 2\vartheta}}$$



$$\Delta m_M^2 = \left[(\Delta m^2 \cos 2\vartheta - A_{\text{CC}})^2 + (\Delta m^2 \sin 2\vartheta)^2 \right]^{1/2}$$

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\vartheta_M & \sin\vartheta_M \\ -\sin\vartheta_M & \cos\vartheta_M \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$i \frac{d}{dx} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \left[\underbrace{\frac{A_{CC}}{4E}}_{\text{irrelevant common phase}} + \frac{1}{4E} \begin{pmatrix} -\Delta m_M^2 & 0 \\ 0 & \Delta m_M^2 \end{pmatrix} + \begin{pmatrix} 0 & -i \frac{d\vartheta_M}{dx} \\ i \frac{d\vartheta_M}{dx} & 0 \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

irrelevant common phase

maximum near resonance

$$\begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} = \begin{pmatrix} \cos\vartheta_M^0 & -\sin\vartheta_M^0 \\ \sin\vartheta_M^0 & \cos\vartheta_M^0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\vartheta_M^0 \\ \sin\vartheta_M^0 \end{pmatrix}$$

$$\psi_1(x) \simeq \left[\cos\vartheta_M^0 \exp\left(i \int_0^{x_R} \frac{\Delta m_M^2(x')}{4E} dx'\right) \mathcal{A}_{11}^R + \sin\vartheta_M^0 \exp\left(-i \int_0^{x_R} \frac{\Delta m_M^2(x')}{4E} dx'\right) \mathcal{A}_{21}^R \right] \\ \times \exp\left(i \int_{x_R}^x \frac{\Delta m_M^2(x')}{4E} dx'\right)$$

$$\psi_2(x) \simeq \left[\cos\vartheta_M^0 \exp\left(i \int_0^{x_R} \frac{\Delta m_M^2(x')}{4E} dx'\right) \mathcal{A}_{12}^R + \sin\vartheta_M^0 \exp\left(-i \int_0^{x_R} \frac{\Delta m_M^2(x')}{4E} dx'\right) \mathcal{A}_{22}^R \right] \\ \times \exp\left(-i \int_{x_R}^x \frac{\Delta m_M^2(x')}{4E} dx'\right)$$

Averaged Survival Probability

$$\psi_e(x) = \cos\vartheta_M^x \psi_1(x) + \sin\vartheta_M^x \psi_2(x)$$

neglect interference (averaged over energy spectrum)

$$\begin{aligned}\bar{P}_{\nu_e \rightarrow \nu_e}(x) = |\langle \psi_e(x) \rangle|^2 &= \cos^2\vartheta_M^x \cos^2\vartheta_M^0 |\mathcal{A}_{11}^R|^2 + \cos^2\vartheta_M^x \sin^2\vartheta_M^0 |\mathcal{A}_{21}^R|^2 \\ &+ \sin^2\vartheta_M^x \cos^2\vartheta_M^0 |\mathcal{A}_{12}^R|^2 + \sin^2\vartheta_M^x \sin^2\vartheta_M^0 |\mathcal{A}_{22}^R|^2\end{aligned}$$

conservation of probability (unitarity)

$$|\mathcal{A}_{12}^R|^2 = |\mathcal{A}_{21}^R|^2 = P_c \quad |\mathcal{A}_{11}^R|^2 = |\mathcal{A}_{22}^R|^2 = 1 - P_c$$

$P_c \equiv$ crossing probability

$$\bar{P}_{\nu_e \rightarrow \nu_e}(x) = \frac{1}{2} + \left(\frac{1}{2} - P_c \right) \cos 2\vartheta_M^0 \cos 2\vartheta_M^x$$

[Parke, PRL 57 (1986) 1275]

Crossing Probability

$$P_c = \frac{\exp\left(-\frac{\pi}{2}\gamma F\right) - \exp\left(-\frac{\pi}{2}\gamma \frac{F}{\sin^2\vartheta}\right)}{1 - \exp\left(-\frac{\pi}{2}\gamma \frac{F}{\sin^2\vartheta}\right)}$$

[Kuo, Pantaleone, PRD 39 (1989) 1930]

adiabaticity parameter:
$$\gamma = \frac{\Delta m_M^2/2E}{2|d\vartheta_M/dx|} \Big|_R = \frac{\Delta m^2 \sin^2 2\vartheta}{2E \cos 2\vartheta} \Big|_{\frac{d \ln A_{cc}}{dx}} \Big|_R$$

$A \propto x$ $F = 1$ (Landau-Zener approximation) [Parke, PRL 57 (1986) 1275]

$A \propto 1/x$ $F = (1 - \tan^2 \vartheta)^2 / (1 + \tan^2 \vartheta)$ [Kuo, Pantaleone, PRD 39 (1989) 1930]

[Pizzochero, PRD 36 (1987) 2293]

$A \propto \exp(-x)$ $F = 1 - \tan^2 \vartheta$ [Toshev, PLB 196 (1987) 170]

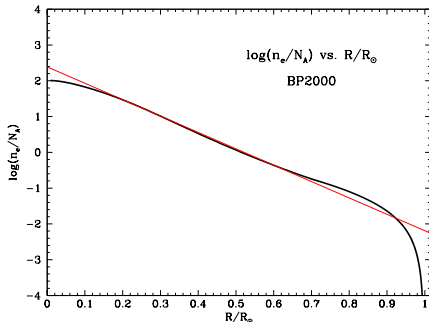
[Petcov, PLB 200 (1988) 373]

Review: [Kuo, Pantaleone, RMP 61 (1989) 937]

Solar Neutrinos

SUN: $N_e(x) \simeq N_e^c \exp\left(-\frac{x}{x_0}\right)$

$$N_e^c = 245 N_A/\text{cm}^3 \quad x_0 = \frac{R_\odot}{10.54}$$



$$\overline{P}_{\nu_e \rightarrow \nu_e}^{\text{sun}} = \frac{1}{2} + \left(\frac{1}{2} - P_c\right) \cos 2\vartheta_M^0 \cos 2\vartheta$$

$$P_c = \frac{\exp\left(-\frac{\pi}{2}\gamma F\right) - \exp\left(-\frac{\pi}{2}\gamma \frac{F}{\sin^2 \vartheta}\right)}{1 - \exp\left(-\frac{\pi}{2}\gamma \frac{F}{\sin^2 \vartheta}\right)}$$

$$\gamma = \frac{\Delta m^2 \sin^2 2\vartheta}{2E \cos 2\vartheta \left| \frac{d \ln A_{cc}}{dx} \right|_R}$$

$$F = 1 - \tan^2 \vartheta$$

$$A_{cc} = 2\sqrt{2}EG_F N_e$$

practical prescription:

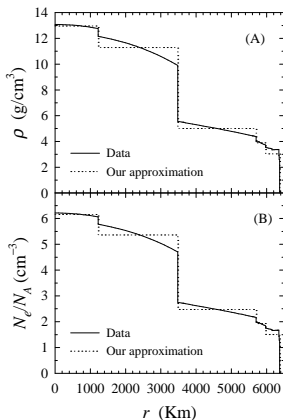
[Lisi et al., PRD 63 (2001) 093002]

$$\left\{ \begin{array}{ll} \text{numerical } \left| \frac{d \ln A_{cc}}{dx} \right|_R & \text{for } x \leq 0.904 R_\odot \\ \left| \frac{d \ln A_{cc}}{dx} \right|_R \rightarrow \frac{18.9}{R_\odot} & \text{for } x > 0.904 R_\odot \end{array} \right.$$

Electron Neutrino Regeneration in the Earth

$$P_{\nu_e \rightarrow \nu_e}^{\text{sun+earth}} = \overline{P}_{\nu_e \rightarrow \nu_e}^{\text{sun}} + \frac{\left(1 - 2\overline{P}_{\nu_e \rightarrow \nu_e}^{\text{sun}}\right) \left(P_{\nu_2 \rightarrow \nu_e}^{\text{earth}} - \sin^2\vartheta\right)}{\cos 2\vartheta}$$

[Mikheev, Smirnov, Sov. Phys. Usp. 30 (1987) 759], [Baltz, Weneser, PRD 35 (1987) 528]



$P_{\nu_2 \rightarrow \nu_e}^{\text{earth}}$ is usually calculated numerically approximating the Earth density profile with a step function.

Effective massive neutrinos propagate as plane waves in regions of constant density.

Wave functions of flavor neutrinos are joined at the boundaries of steps.

[Giunti, Kim, Monteno, NP B 521 (1998) 3]

Phenomenology of Solar Neutrinos

LMA (Large Mixing Angle):

LOW (LOW Δm^2):

SMA (Small Mixing Angle):

QVO (Quasi-Vacuum Oscillations):

VAC (VACuum oscillations):

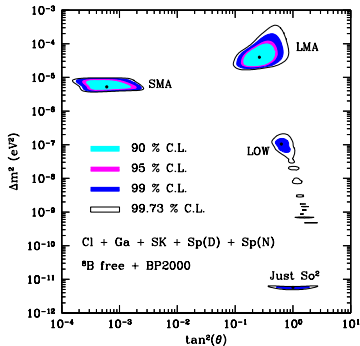
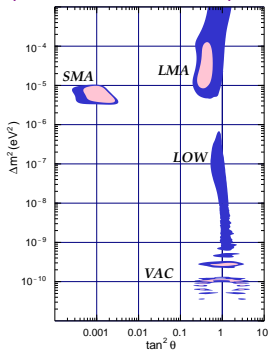
$$\Delta m^2 \sim 5 \times 10^{-5} \text{ eV}^2, \quad \tan^2 \vartheta \sim 0.8$$

$$\Delta m^2 \sim 7 \times 10^{-8} \text{ eV}^2, \quad \tan^2 \vartheta \sim 0.6$$

$$\Delta m^2 \sim 5 \times 10^{-6} \text{ eV}^2, \quad \tan^2 \vartheta \sim 10^{-3}$$

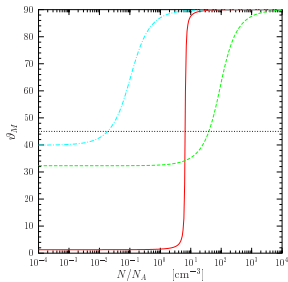
$$\Delta m^2 \sim 10^{-9} \text{ eV}^2, \quad \tan^2 \vartheta \sim 1$$

$$\Delta m^2 \lesssim 5 \times 10^{-10} \text{ eV}^2, \quad \tan^2 \vartheta \sim 1$$



[de Gouvea, Friedland, Murayama, PLB 490 (2000) 125]

[Bahcall, Krastev, Smirnov, JHEP 05 (2001) 015]



solid line:
(typical SMA)

$$\Delta m^2 = 5 \times 10^{-6} \text{ eV}^2$$

$$\tan^2 \vartheta = 5 \times 10^{-4}$$

dashed line:
(typical LMA)

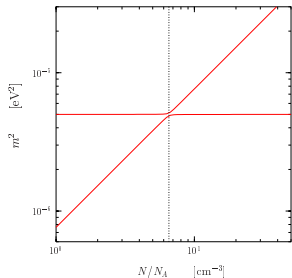
$$\Delta m^2 = 7 \times 10^{-5} \text{ eV}^2$$

$$\tan^2 \vartheta = 0.4$$

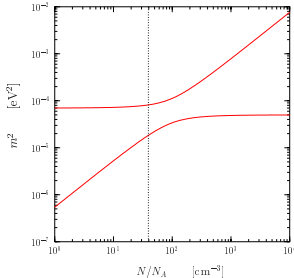
dash-dotted line:
(typical LOW)

$$\Delta m^2 = 8 \times 10^{-8} \text{ eV}^2$$

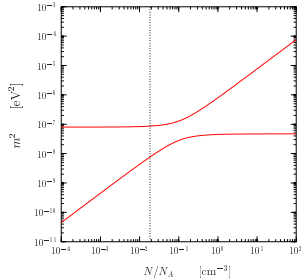
$$\tan^2 \vartheta = 0.7$$



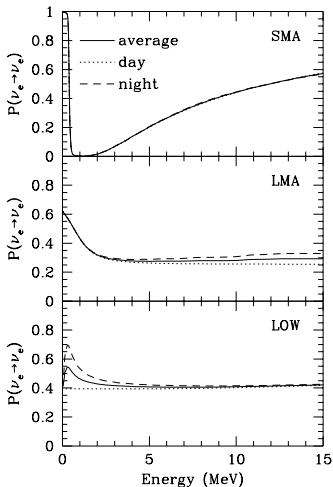
typical SMA



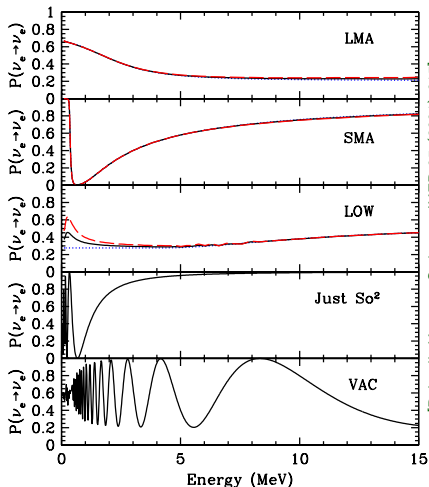
typical LMA



typical LOW



SMA: $\Delta m^2 = 5.0 \times 10^{-6} \text{ eV}^2$ $\sin^2 2\theta = 3.5 \times 10^{-3}$
 LMA: $\Delta m^2 = 1.6 \times 10^{-5} \text{ eV}^2$ $\sin^2 2\theta = 0.57$
 LOW: $\Delta m^2 = 7.9 \times 10^{-8} \text{ eV}^2$ $\sin^2 2\theta = 0.95$



LMA: $\Delta m^2 = 4.2 \times 10^{-5} \text{ eV}^2$ $\tan^2 \theta = 0.26$
 SMA: $\Delta m^2 = 5.2 \times 10^{-6} \text{ eV}^2$ $\tan^2 \theta = 5.5 \times 10^{-4}$
 LOW: $\Delta m^2 = 7.6 \times 10^{-8} \text{ eV}^2$ $\tan^2 \theta = 0.72$
 Just So^2 : $\Delta m^2 = 5.5 \times 10^{-12} \text{ eV}^2$ $\tan^2 \theta = 1.0$
 VAC: $\Delta m^2 = 1.4 \times 10^{-10} \text{ eV}^2$ $\tan^2 \theta = 0.38$

In Neutrino Oscillations Dirac = Majorana

Evolution of Amplitudes:
$$i \frac{d\psi_\alpha}{dx} = \frac{1}{2E} \sum_\beta \left(UM^2U^\dagger + 2EV \right)_{\alpha\beta} \psi_\beta$$

difference:
$$\left\{ \begin{array}{ll} \text{Dirac:} & U^{(D)} \\ \text{Majorana:} & U^{(M)} = U^{(D)} D(\lambda) \end{array} \right.$$

$$D(\lambda) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & e^{i\lambda_{21}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{i\lambda_{N1}} \end{pmatrix} \Rightarrow D^\dagger = D^{-1}$$

$$M^2 = \begin{pmatrix} m_1^2 & 0 & \dots & 0 \\ 0 & m_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m_N^2 \end{pmatrix} \Rightarrow DM^2 = M^2D \Rightarrow DM^2D^\dagger = M^2$$

$$U^{(M)} M^2 (U^{(M)})^\dagger = U^{(D)} D M^2 D^\dagger (U^{(D)})^\dagger = U^{(D)} M^2 (U^{(D)})^\dagger$$