Neutrino Phenomenology

Carlo Giunti

INFN, Sezione di Torino, and Dipartimento di Fisica Teorica, Università di Torino

mailto://giunti@to.infn.it

Neutrino Unbound: http://www.nu.to.infn.it

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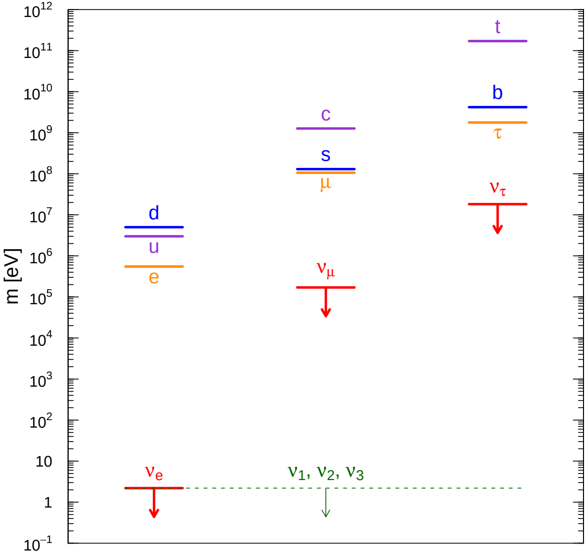
Neutrino Phenomenology

- Neutrino Masses and Mixing
- Neutrino Oscillations in Vacuum
- Neutrino Oscillations in Matter

Neutrino Masses and Mixing

Fermion Mass Spectrum



SM Extension: Dirac ν Masses

$$L_L \equiv \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} \qquad \ell_R \qquad \nu_R$$

Lepton-Higgs Yukawa Lagrangian

$$\mathscr{L}^{\text{D}} = -y^{\ell}\,\overline{L_L}\,\Phi\,\ell_R - y^{\nu}\,\overline{L_L}\,\widetilde{\Phi}\,\nu_R + \text{H.c.}$$

Symmetry Breaking

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \qquad \widetilde{\Phi} = i\sigma_2\,\Phi^* = \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix}$$

$$\begin{aligned}\mathscr{L}^{\text{D}} = & -\frac{y^{\ell}}{\sqrt{2}}\begin{pmatrix}\overline{\nu_L} & \overline{\ell_L}\end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix}\ell_R \\ & -\frac{y^{\nu}}{\sqrt{2}}\begin{pmatrix}\overline{\nu_L} & \overline{\ell_L}\end{pmatrix}\begin{pmatrix} v \\ 0 \end{pmatrix}\nu_R + \text{H.c.}\end{aligned}$$

$$\mathscr{L}^{\mathrm{D}} = -y^{\ell}\,\frac{v}{\sqrt{2}}\,\overline{\ell_L}\,\ell_R - y^{\nu}\,\frac{v}{\sqrt{2}}\,\overline{\nu_L}\,\nu_R + \mathrm{H.c.}$$

$$m_{\ell} = y^{\ell}\,\frac{v}{\sqrt{2}} \qquad\qquad m_{\nu} = y^{\nu}\,\frac{v}{\sqrt{2}}$$

$$v = \left(\sqrt{2}\,G_{\mathrm{F}}\right)^{1/2} = 246\,\mathrm{GeV}$$

Three-Generations Dirac Neutrino Masses

$L'_{eL} \equiv \begin{pmatrix} \nu'_{eL} \\ \ell'_{eL} \equiv e'_L \end{pmatrix}$	$L'_{\mu L} \equiv \begin{pmatrix} \nu'_{\mu L} \\ \ell'_{\mu L} \equiv \mu'_L \end{pmatrix}$	$L'_{\tau L} \equiv \begin{pmatrix} \nu'_{\tau L} \\ \ell'_{\tau L} \equiv \tau'_L \end{pmatrix}$
$\ell'_{eR} \equiv e'_R$	$\ell'_{\mu R} \equiv \mu'_R$	$\ell'_{\tau R} \equiv \tau'_R$
ν'_{eR}	$\nu'_{\mu R}$	$\nu'_{\tau R}$

Lepton-Higgs Yukawa Lagrangian

$$\mathscr{L}^{\text{D}} = - \sum_{\alpha,\beta=e,\mu,\tau} \left[Y'^{\ell}_{\alpha\beta} \, \overline{L'_{\alpha L}} \, \Phi \, \ell'_{\beta R} + Y'^{\nu}_{\alpha\beta} \, \overline{L'_{\alpha L}} \, \widetilde{\Phi} \, \nu'_{\beta R} \right] + \text{H.c.}$$

Symmetry Breaking

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad \widetilde{\Phi} = i\sigma_2 \, \Phi^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix}$$

$$\mathscr{L}^{\mathrm{D}} = -\sum_{\alpha,\beta=e,\mu,\tau} \left[\frac{v}{\sqrt{2}} Y'^{\ell}_{\alpha\beta} \, \overline{\ell'_{\alpha L}} \, \ell'_{\beta R} + \frac{v}{\sqrt{2}} Y'^{\nu}_{\alpha\beta} \, \overline{\nu'_{\alpha L}} \, \nu'_{\beta R} \right] + \mathrm{H.c.}$$

$$\mathscr{L}^{\mathrm{D}} = -\left[\overline{\boldsymbol{\ell}'_L} \, M'^{\ell} \, \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L} \, M'^{\nu} \, \boldsymbol{\nu}'_R \right] + \mathrm{H.c.}$$

$$\boldsymbol{\ell}'_L \equiv \begin{pmatrix} e'_L \\ \mu'_L \\ \tau'_L \end{pmatrix} \qquad \boldsymbol{\ell}'_R \equiv \begin{pmatrix} e'_R \\ \mu'_R \\ \tau'_R \end{pmatrix} \qquad \boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \qquad \boldsymbol{\nu}'_R \equiv \begin{pmatrix} \nu'_{eR} \\ \nu'_{\mu R} \\ \nu'_{\tau R} \end{pmatrix}$$

$$M'^{\ell} = \frac{v}{\sqrt{2}} \, Y'^{\ell} \qquad\qquad M'^{\nu} = \frac{v}{\sqrt{2}} \, Y'^{\nu}$$

$$M'^{\ell} \equiv \begin{pmatrix} M'^{\ell}_{ee} & M'^{\ell}_{e\mu} & M'^{\ell}_{e\tau} \\ M'^{\ell}_{\mu e} & M'^{\ell}_{\mu\mu} & M'^{\ell}_{\mu\tau} \\ M'^{\ell}_{\tau e} & M'^{\ell}_{\tau\mu} & M'^{\ell}_{\tau\tau} \end{pmatrix} \qquad M'^{\nu} \equiv \begin{pmatrix} M'^{\nu}_{ee} & M'^{\nu}_{e\mu} & M'^{\nu}_{e\tau} \\ M'^{\nu}_{\mu e} & M'^{\nu}_{\mu\mu} & M'^{\nu}_{\mu\tau} \\ M'^{\nu}_{\tau e} & M'^{\nu}_{\tau\mu} & M'^{\nu}_{\tau\tau} \end{pmatrix}$$

$$\mathscr{L}^{\mathrm{D}} = -\overline{\boldsymbol{\ell}'_L}\, M'^{\ell}\, \boldsymbol{\ell}'_R - \overline{\boldsymbol{\nu}'_L}\, M'^{\nu}\, \boldsymbol{\nu}'_R + \text{H.c.}$$

Diagonalization of M'^{ℓ} and M'^{ν} with unitary V^{ℓ}_L, V^{ℓ}_R, V^{ν}_L, V^{ν}_R

$$\boldsymbol{\ell}'_L = V^{\ell}_L\, \boldsymbol{\ell}_L \qquad \boldsymbol{\ell}'_R = V^{\ell}_R\, \boldsymbol{\ell}_R \qquad \boldsymbol{\nu}'_L = V^{\nu}_L\, \mathbf{n}_L \qquad \boldsymbol{\nu}'_R = V^{\nu}_R\, \mathbf{n}_R$$

Kinetic terms are invariant under unitary transformations of the fields

$$\mathscr{L}^{\mathrm{D}} = -\overline{\boldsymbol{\ell}_L} V^{\ell\dagger}_L M'^{\ell} V^{\ell}_R \boldsymbol{\ell}_R - \overline{\boldsymbol{\nu}_L} V^{\nu\dagger}_L M'^{\nu} V^{\nu}_R \boldsymbol{\nu}_R + \text{H.c.}$$

$$V^{\ell\dagger}_L \; M'^{\ell} \; V^{\ell}_R = \; M^{\ell} \qquad M^{\ell}_{\alpha\beta} = m^{\ell}_{\alpha}\, \delta_{\alpha\beta} \qquad (\alpha, \beta = e, \mu, \tau)$$

$$V^{\nu\dagger}_L \; M'^{\nu} \; V^{\nu}_R = \; M^{\nu} \qquad M^{\nu}_{kj} = m^{\nu}_{k}\, \delta_{kj} \qquad (k, j = 1, 2, 3)$$

Real and Positive m^{ℓ}_{α}, m^{ν}_{k}

Massive Chiral Lepton Fields

$\boldsymbol{\ell}_L = V_L^{\ell\dagger}\,\boldsymbol{\ell}'_L \equiv \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}$	$\boldsymbol{\ell}_R = V_R^{\ell\dagger}\,\boldsymbol{\ell}'_R \equiv \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}$
$\mathbf{n}_L = V_L^{\nu\dagger}\,\boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$	$\mathbf{n}_R = V_R^{\nu\dagger}\,\boldsymbol{\nu}'_R \equiv \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix}$

$$
\begin{aligned}
\mathscr{L}^{\text{D}} &= -\,\overline{\boldsymbol{\ell}_L}\,M^{\ell}\,\boldsymbol{\ell}_R - \overline{\mathbf{n}_L}\,M^{\nu}\,n_R + \text{H.c.} \\
&= -\sum_{\alpha=e,\mu,\tau} m_{\alpha}^{\ell}\,\overline{\ell_{\alpha L}}\,\ell_{\alpha R} - \sum_{k=1}^{3} m_k^{\nu}\,\overline{\nu_{kL}}\,\nu_{kR} + \text{H.c.}
\end{aligned}
$$

Mixing

Charged-Current Weak Interaction Lagrangian

$$\mathscr{L}_{\mathsf{I}}^{(\mathsf{CC})} = -\frac{g}{2\sqrt{2}}\, j_W^\rho\, W_\rho + \text{H.c.}$$

Weak Charged Current: $j_W^\rho = j_{W,\mathsf{L}}^\rho + j_{W,\mathsf{Q}}^\rho$

Leptonic Weak Charged Current

$$j_{W,\mathsf{L}}^\rho = 2 \sum_{\alpha=e,\mu,\tau} \overline{\nu'_{\alpha L}}\, \gamma^\rho\, \ell'_{\alpha L} = 2\, \overline{\boldsymbol{\nu}'_L}\, \gamma^\rho\, \boldsymbol{\ell}'_L$$

$$\underline{\boldsymbol{\ell}'_L = V_L^\ell\, \boldsymbol{\ell}_L} \qquad \underline{\boldsymbol{\nu}'_L = V_L^\nu\, \mathbf{n}_L}$$

$$j_{W,\mathsf{L}}^\rho = 2\, \overline{\mathbf{n}_L}\, V_L^{\nu\dagger}\, \gamma^\rho\, V_L^\ell\, \boldsymbol{\ell}_L = 2\, \overline{\mathbf{n}_L}\, V_L^{\nu\dagger}\, V_L^\ell\, \gamma^\rho\, \boldsymbol{\ell}_L = 2\, \overline{\mathbf{n}_L}\, U^\dagger\, \gamma^\rho\, \boldsymbol{\ell}_L$$

Mixing Matrix

$$U^\dagger = V_L^{\nu\dagger}\, V_L^\ell \qquad \boxed{U = V_L^{\ell\dagger}\, V_L^\nu}$$

- Definition: Left-Handed Flavor Neutrino Fields

$$\boldsymbol{\nu}_L = U\,\mathbf{n}_L = V_L^{\ell\dagger}\,\boldsymbol{\nu}'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$$

- They allow us to write the Leptonic Weak Charged Current as in the SM:

$$j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\boldsymbol{\nu}_L}\,\gamma^{\rho}\,\boldsymbol{\ell}_L = 2\sum_{\alpha=e,\mu,\tau}\overline{\nu_{\alpha L}}\,\gamma^{\rho}\,\ell_{\alpha L}$$

- Each left-handed flavor neutrino field is associated with the corresponding charged lepton field which describes a massive charged lepton:

$$j^{\rho}_{W,\mathrm{L}} = 2\left(\overline{\nu_{eL}}\,\gamma^{\rho}\,e_L + \overline{\nu_{\mu L}}\,\gamma^{\rho}\,\mu_L + \overline{\nu_{\tau L}}\,\gamma^{\rho}\,\tau_L\right)$$

- In practice left-handed flavor neutrino fields are useful for calculations in the SM approximation of massless neutrinos (interactions).

- If neutrino masses must be taken into account, it is necessary to use

$$j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\mathbf{n}_L}\,U^{\dagger}\,\gamma^{\rho}\,\boldsymbol{\ell}_L = 2\sum_{k=1}^{3}\sum_{\alpha=e,\mu,\tau}U^{*}_{\alpha k}\,\overline{\nu_{kL}}\,\gamma^{\rho}\,\ell_{\alpha L}$$

Flavor Lepton Numbers

Flavor Neutrino Fields are useful for defining
Flavor Lepton Numbers
as in the SM

	L_e	L_μ	L_τ
$(\nu_e\,,\,e^-)$	$+1$	0	0
$(\nu_\mu\,,\,\mu^-)$	0	$+1$	0
$(\nu_\tau\,,\,\tau^-)$	0	0	$+1$

	L_e	L_μ	L_τ
$(\nu_e^c\,,\,e^+)$	-1	0	0
$(\nu_\mu^c\,,\,\mu^+)$	0	-1	0
$(\nu_\tau^c\,,\,\tau^+)$	0	0	-1

$$\boxed{L = L_e + L_\mu + L_\tau}$$

Standard Model: Lepton numbers are conserved

$$\mathscr{L}^{\mathrm{D}} = - \left(\overline{\nu_{eL}} \quad \overline{\nu_{\mu L}} \quad \overline{\nu_{\tau L}} \right) \begin{pmatrix} m^{\mathrm{D}}_{ee} & m^{\mathrm{D}}_{e\mu} & m^{\mathrm{D}}_{e\tau} \\ m^{\mathrm{D}}_{\mu e} & m^{\mathrm{D}}_{\mu\mu} & m^{\mathrm{D}}_{\mu\tau} \\ m^{\mathrm{D}}_{\tau e} & m^{\mathrm{D}}_{\tau\mu} & m^{\mathrm{D}}_{\tau\tau} \end{pmatrix} \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix} + \mathrm{H.c.}$$

L_e, L_μ, L_τ are not conserved

L is conserved: $L(\nu_{\alpha R}) = L(\nu_{\beta L}) \Rightarrow |\Delta L| = 0$

Mixing Matrix

- $U = V_L^{\ell\dagger}\, V_L^{\nu} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}$

- Unitary $N \times N$ matrix depends on N^2 independent real parameters

$$N = 3 \qquad \Longrightarrow \qquad \begin{array}{ll} \dfrac{N\,(N-1)}{2} = 3 & \quad \text{Mixing Angles} \\ \dfrac{N\,(N+1)}{2} = 6 & \quad \text{Phases} \end{array}$$

- Not all phases are physical observables

- Only physical effect of mixing matrix occurs through its presence in the Leptonic Weak Charged Current

- Weak Charged Current: $j^{\rho}_{W,\mathrm{L}} = 2 \sum_{k=1}^{3} \sum_{\alpha=e,\mu,\tau} \overline{\nu_{kL}} \, U^{*}_{\alpha k} \, \gamma^{\rho} \, \ell_{\alpha L}$
- Apart from the Weak Charged Current, the Lagrangian is invariant under the global phase transformations (6 arbitrary phases)
  $$\nu_k \to e^{i\varphi_k} \, \nu_k \quad (k = 1, 2, 3) \, , \qquad \ell_\alpha \to e^{i\varphi_\alpha} \, \ell_\alpha \quad (\alpha = e, \mu, \tau)$$
- Performing this transformation, the Charged Current becomes

$$j^{\rho}_{W,\mathrm{L}} = 2 \sum_{k=1}^{3} \sum_{\alpha=e,\mu,\tau} \overline{\nu_{kL}} \, e^{-i\varphi_k} \, U^{*}_{\alpha k} \, e^{i\varphi_\alpha} \, \gamma^{\rho} \, \ell_{\alpha L}$$

$$j^{\rho}_{W,\mathrm{L}} = 2 \, \underbrace{e^{-i(\varphi_1 - \varphi_e)}}_{1} \sum_{k=1}^{3} \sum_{\alpha=e,\mu,\tau} \overline{\nu_{kL}} \, \underbrace{e^{-i(\varphi_k - \varphi_1)}}_{2} \, U^{*}_{\alpha k} \, \underbrace{e^{i(\varphi_\alpha - \varphi_e)}}_{2} \, \gamma^{\rho} \, \ell_{\alpha L}$$

- There are 5 arbitrary phases of the fields that can be chosen to eliminate 5 of the 6 phases of the mixing matrix
- 5 and not 6 phases of the mixing matrix can be eliminated because a common rephasing of all the fields leaves the Charged Current invariant $\Longleftrightarrow$ conservation of Total Lepton Number.

- The mixing matrix contains 1 Physical Phase.
- It is convenient to express the 3×3 unitary mixing matrix only in terms of the four physical parameters:

3 Mixing Angles and 1 Phase

Standard Parameterization of Mixing Matrix

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13} \end{pmatrix}$$

$c_{ab} \equiv \cos \vartheta_{ab}$ $\quad$ $s_{ab} \equiv \sin \vartheta_{ab}$ $\quad$ $0 \leq \vartheta_{ab} \leq \frac{\pi}{2}$ $\quad$ $0 \leq \delta_{13} < 2\pi$

3 Mixing Angles ϑ_{12}, ϑ_{23}, ϑ_{13} and 1 Phase δ_{13}

Standard Parameterization

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of Different Phase Convention

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{i\delta_{23}} \\ 0 & -s_{23} e^{-i\delta_{13}} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of Different Parameterization

$$U = \begin{pmatrix} c'_{12} & s'_{12} e^{-i\delta'_{12}} & 0 \\ -s'_{12} e^{i\delta'_{12}} & c'_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c'_{23} & s'_{23} \\ 0 & -s'_{23} & c'_{23} \end{pmatrix} \begin{pmatrix} c'_{13} & 0 & s'_{13} \\ 0 & 1 & 0 \\ -s'_{13} & 0 & c'_{13} \end{pmatrix}$$

Jarlskog Rephasing Invariant

- Simplest rephasing invariants: $|U_{\alpha k}|^2 = U_{\alpha k} U_{\alpha k}^*$, $U_{\alpha k} U_{\alpha j}^* U_{\beta k}^* U_{\beta j}$

$$\Im\mathfrak{m}\left[U_{\alpha k} U_{\alpha j}^* U_{\beta k}^* U_{\beta j}\right] = \pm J$$

$$J = \Im\mathfrak{m}\left[U_{e2} U_{e3}^* U_{\mu 2}^* U_{\mu 3}\right] = \Im\mathfrak{m}\begin{pmatrix} \cdot & \circ & \times \\ \cdot & \times & \circ \\ \cdot & \cdot & \cdot \end{pmatrix}$$

- In standard parameterization:

$$\begin{aligned} J &= c_{12} s_{12} c_{23} s_{23} c_{13}^2 s_{13} \sin\delta_{13} \\ &= \frac{1}{8} \sin 2\vartheta_{12} \sin 2\vartheta_{23} \cos\vartheta_{13} \sin 2\vartheta_{13} \sin\delta_{13} \end{aligned}$$

- Jarlskog invariant is useful for quantifying CP violation in a parameterization-independent way
- All measurable CP-violation effects depend on J.

- exercise: Show that U is real if $\vartheta_{12} = 0$
- exercise: Show that U is real if $\vartheta_{13} = \pi/2$
- exercise: Show that U is real if $m_{\nu_2} = m_{\nu_3}$
- exercise: Show that $|J|_{\mathrm{max}} = 1/6\sqrt{3}$ (maximal CP violation). In this case which is the form of the mixing matrix U?

Majorana Mass

- Majorana Constraint: $\nu = \nu^{c}$

$$\nu^{c} = \mathcal{C}\,\overline{\nu}^{T} \qquad\qquad \mathcal{C}\,\gamma_{\mu}^{T}\,\mathcal{C}^{-1} = -\gamma_{\mu}$$

- $\nu_L + \nu_R = \nu_L^c + \nu_R^c \quad \Longrightarrow \quad \nu_L = \nu_R^c$ and $\nu_R = \nu_L^c$
- Same equation, because from the second $\nu_R^c = (\nu_L^c)^c = \nu_L$
- ν_L and ν_R are not independent!
- We can take as independent ν_L
- Substitute $\nu_R = \nu_L^c$ in $\dfrac{1}{2}\,\mathscr{L}^{\text{D}} = -m\,(\overline{\nu_R}\,\nu_L + \overline{\nu_L}\,\nu_R)$
- We obtain the Majorana Mass Lagrangian

$$\mathscr{L}^{\text{M}} = -\frac{m}{2}\left(\overline{\nu_L^c}\,\nu_L + \overline{\nu_L}\,\nu_L^c\right)$$

Lepton Number

$$\cancel{L = +1} \quad \longleftarrow \quad \boxed{\nu = \nu^c} \quad \longrightarrow \quad \cancel{L = -1}$$

$$\nu_L \quad \Longrightarrow \quad L = +1 \qquad\qquad \nu_L^c \quad \Longrightarrow \quad L = -1$$

$$\mathscr{L}^{\mathrm{M}} = -\frac{m}{2} \left(\overline{\nu_L^c} \, \nu_L + \overline{\nu_L} \, \nu_L^c \right)$$

Total Lepton Number is not conserved: $\boxed{\Delta L = \pm 2}$

Best process to find violation of Total Lepton Number:

Neutrinoless Double-β Decay

$$\mathcal{N}(A, Z) \to \mathcal{N}(A, Z + 2) + 2e^- + \cancel{2\bar{\nu}_e} \qquad (\beta\beta^-_{0\nu})$$

$$\mathcal{N}(A, Z) \to \mathcal{N}(A, Z - 2) + 2e^+ + \cancel{2\nu_e} \qquad (\beta\beta^+_{0\nu})$$

No Majorana Neutrino Mass in the SM

- Majorana Mass Term $\propto \left[\nu_L^T \, \mathcal{C}^\dagger \, \nu_L - \overline{\nu_L} \, \mathcal{C} \, \overline{\nu_L}^T\right]$ involves only the neutrino left-handed chiral field ν_L, which is present in the SM (one for each lepton generation)
- Eigenvalues of the weak isospin I, of its third component I_3, of the hypercharge Y and of the charge Q of the lepton and Higgs multiplets:

	I	I_3	Y	$Q = I_3 + \frac{Y}{2}$
lepton doublet $L_L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}$	$1/2$	$1/2$ $-1/2$	-1	0 -1
lepton singlet ℓ_R	0	0	-2	-1
Higgs doublet $\Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}$	$1/2$	$1/2$ $-1/2$	$+1$	1 0

- $\nu_L^T \, \mathcal{C}^\dagger \, \nu_L$ has $I_3 = 1$ and $Y = -2 \Longrightarrow$ needed Higgs triplet with $Y = 2$

Mixing of Three Majorana Neutrinos

- $\boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix}$

$$\mathscr{L}^{\mathrm{M}} = \frac{1}{2} \boldsymbol{\nu}'^{T}_{L} \, \mathcal{C}^{\dagger} \, M^{L} \, \boldsymbol{\nu}'_{L} + \mathrm{H.c.}$$

$$= \frac{1}{2} \sum_{\alpha,\beta=e,\mu,\tau} \nu'^{T}_{\alpha L} \, \mathcal{C}^{\dagger} \, M^{L}_{\alpha\beta} \, \nu'_{\beta L} + \mathrm{H.c.}$$

- In general, the matrix M^L is a complex symmetric matrix

$$\sum_{\alpha,\beta} \nu'^{T}_{\alpha L} \, \mathcal{C}^{\dagger} \, M^{L}_{\alpha\beta} \, \nu'_{\beta L} = \sum_{\alpha,\beta} \left(\nu'^{T}_{\alpha L} \, \mathcal{C}^{\dagger} \, M^{L}_{\alpha\beta} \, \nu'_{\beta L} \right)^{T}$$

$$= - \sum_{\alpha,\beta} \nu'^{T}_{\beta L} \, M^{L}_{\alpha\beta} \, (\mathcal{C}^{\dagger})^{T} \, \nu'_{\alpha L} = \sum_{\alpha,\beta} \nu'^{T}_{\beta L} \, \mathcal{C}^{\dagger} \, M^{L}_{\alpha\beta} \, \nu'_{\alpha L}$$

$$= \sum_{\alpha,\beta} \nu'^{T}_{\alpha L} \, \mathcal{C}^{\dagger} \, M^{L}_{\beta\alpha} \, \nu'_{\beta L}$$

$$M^{L}_{\alpha\beta} = M^{L}_{\beta\alpha} \quad \Longleftrightarrow \quad M^{L} = M^{L\,T}$$

Diagonalization of Majorana Mass Matrix

- $\mathscr{L}^{\text{M}} = \frac{1}{2} \boldsymbol{\nu}_L'^T \, \mathcal{C}^\dagger \, M^L \, \boldsymbol{\nu}_L' + \text{H.c.}$
- $\boldsymbol{\nu}_L' = V_L^\nu \, \mathbf{n}_L \qquad \Longrightarrow \qquad \mathscr{L}^{\text{M}} = \frac{1}{2} \boldsymbol{\nu}_L'^T \, (V_L^\nu)^T \, \mathcal{C}^\dagger \, M^L \, V_L^\nu \, \boldsymbol{\nu}_L' + \text{H.c.}$
- $(V_L^\nu)^T \, M^L \, V_L^\nu = M \,, \qquad M_{kj} = m_k \, \delta_{kj} \qquad (k, j = 1, 2, 3)$
- Neutrino fields with definite mass: $\mathbf{n}_L = V_L^{\nu\dagger} \, \boldsymbol{\nu}_L' = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$

$$\mathscr{L}^{\text{M}} = \frac{1}{2} \sum_{k=1}^{3} m_k \left(\nu_{kL}^T \, \mathcal{C}^\dagger \, \nu_{kL} - \overline{\nu_{kL}} \, \mathcal{C} \, \nu_{kL}^T \right)$$

Mixing Matrix

- Leptonic Weak Charged Current:

$$j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\mathbf{n}_L}\,U^{\dagger}\,\gamma^{\rho}\,\boldsymbol{\ell}_L \qquad \text{with} \qquad U = V_L^{\ell\dagger}\,V_L^{\nu}$$

- Definition of the left-handed flavor neutrino fields:

$$\boldsymbol{\nu}_L = U\,\mathbf{n}_L = V_L^{\ell\dagger}\,\boldsymbol{\nu}'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$$

- Leptonic Weak Charged Current has the SM form

$$j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\boldsymbol{\nu}_L}\,\gamma^{\rho}\,\boldsymbol{\ell}_L = 2 \sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}}\,\gamma^{\rho}\,\ell_{\alpha L}$$

- Important difference with respect to Dirac case:
  Two additional CP-violating phases: Majorana phases

- Majorana Mass Term $\mathscr{L}^{\mathrm{M}} = \frac{1}{2} \sum_{k=1}^{3} m_k \, \nu_{kL}^{T} \, \mathcal{C}^{\dagger} \, \nu_{kL} + \mathrm{H.c.}$ is not invariant under global U(1) gauge transformations $\nu_{kL} \to e^{i\varphi_k} \, \nu_{kL} \quad (k = 1, 2, 3)$

$$j_{W,\mathrm{L}}^{\rho\dagger} = 2 \sum_{\alpha=e,\mu,\tau} \overline{\ell_{\alpha L}} \, \gamma^{\rho} \, U_{\alpha k} \, \nu_{kL}$$

- Two Majorana phases factorized on the right of mixing matrix cannot be eliminated:

$$\boxed{U = U^{\mathrm{D}} \, D^{\mathrm{M}}} \qquad D^{\mathrm{M}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_2} & 0 \\ 0 & 0 & e^{i\lambda_3} \end{pmatrix}$$

- U^{D} is analogous to a Dirac mixing matrix, with one Dirac phase
- Standard parameterization:

$$U = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_2} & 0 \\ 0 & 0 & e^{i\lambda_3} \end{pmatrix}$$

One-Generation Dirac-Majorana Mass Term

If ν_R exists, the most general mass term is the

Dirac-Majorana Mass Term

$$\mathscr{L}^{\text{D+M}} = \mathscr{L}^{\text{D}} + \cancel{\mathscr{L}^{L}} + \mathscr{L}^{R}$$

$$\mathscr{L}^{\text{D}} = -m^{\text{D}}\,\overline{\nu_R}\,\nu_L + \text{H.c.}$$ Dirac Mass Term

$$\mathscr{L}^{L} = \frac{1}{2}\,m_L^{\text{M}}\,\nu_L^T\,\mathcal{C}^\dagger\,\nu_L + \text{H.c.}$$

ν_L Majorana Mass Term forbidden by SM Symmetries

$$\mathscr{L}^{R} = \frac{1}{2}\,m_R^{\text{M}}\,\nu_R^T\,\mathcal{C}^\dagger\,\nu_R + \text{H.c.}$$

New ν_R Majorana Mass Term allowed by SM Symmetries!

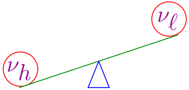
See-Saw Mechanism

$$\mathcal{L}^{\text{D+M}} = -\frac{1}{2} \begin{pmatrix} \overline{\nu_L^c} & \overline{\nu_R} \end{pmatrix} \begin{pmatrix} 0 & m^{\text{D}} \\ m^{\text{D}} & m_R^{\text{M}} \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \text{H.c.}$$

m_R^{M} can be arbitrarily large (not protected by SM symmetries)

$m_R^{\text{M}} \sim$ scale of new physics beyond Standard Model $\Rightarrow m_R^{\text{M}} \gg m^{\text{D}}$

diagonalization of $\begin{pmatrix} 0 & m^{\text{D}} \\ m^{\text{D}} & m_R^{\text{M}} \end{pmatrix} \quad \Rightarrow \quad m_\ell \simeq \frac{(m^{\text{D}})^2}{m_R^{\text{M}}}, \qquad m_h \simeq m_R^{\text{M}}$



see-saw mechanism

natural explanation of smallness
of light neutrino masses

massive neutrinos are Majorana!

3-GEN ⇒ effective low-energy 3-ν mixing

[Minkowski, PLB 67 (1977) 42]

[Yanagida (1979); Gell-Mann, Ramond, Slansky (1979); Mohapatra, Senjanovic, PRL 44 (1980) 912]

Neutrino Oscillations in Vacuum

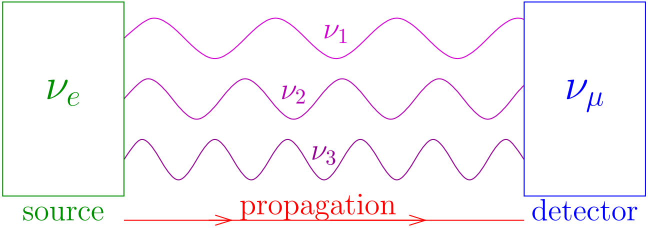
Flavor Neutrino Oscillations

- Flavor Neutrinos: ν_e, ν_μ, ν_τ produced in Weak Interactions
- Massive Neutrinos: ν_1, ν_2, ν_3 propagate from Source to Detector
- A Flavor Neutrino is a superposition of Massive Neutrinos

$$\begin{aligned} |\nu_e\rangle &= U_{e1}\,|\nu_1\rangle + U_{e2}\,|\nu_2\rangle + U_{e3}\,|\nu_3\rangle \\ |\nu_\mu\rangle &= U_{\mu1}\,|\nu_1\rangle + U_{\mu2}\,|\nu_2\rangle + U_{\mu3}\,|\nu_3\rangle \\ |\nu_\tau\rangle &= U_{\tau1}\,|\nu_1\rangle + U_{\tau2}\,|\nu_2\rangle + U_{\tau3}\,|\nu_3\rangle \end{aligned}$$

- U is the 3×3 unitary Neutrino Mixing Matrix

$$|\nu(t=0)\rangle = |\nu_e\rangle = U_{e1}\,|\nu_1\rangle + U_{e2}\,|\nu_2\rangle + U_{e3}\,|\nu_3\rangle$$



$$|\nu(t>0)\rangle = U_{e1}\,e^{-iE_1 t}\,|\nu_1\rangle + U_{e2}\,e^{-iE_2 t}\,|\nu_2\rangle + U_{e3}\,e^{-iE_3 t}\,|\nu_3\rangle \neq |\nu_e\rangle$$

$$E_k^2 = p^2 + m_k^2$$

at the detector there is a probability > 0 to see the neutrino as a ν_μ

Neutrino Oscillations are Flavor Transitions

$$\begin{array}{cccc} \nu_e \to \nu_\mu & \nu_e \to \nu_\tau & \nu_\mu \to \nu_e & \nu_\mu \to \nu_\tau \\ \bar{\nu}_e \to \bar{\nu}_\mu & \bar{\nu}_e \to \bar{\nu}_\tau & \bar{\nu}_\mu \to \bar{\nu}_e & \bar{\nu}_\mu \to \bar{\nu}_\tau \end{array}$$

transition probabilities depend on U and $\Delta m_{kj}^2 \equiv m_k^2 - m_j^2$

Early History of Neutrino Oscillations

- 1957: Pontecorvo proposed Neutrino Oscillations in analogy with $K^0 \leftrightarrows \bar{K}^0$ oscillations (Gell-Mann and Pais, 1955) $\quad \Longrightarrow \quad \nu \leftrightarrows \bar{\nu}$
- In 1957 only one neutrino $\nu = \nu_e$ was known!
- 1958: Goldhaber, Grodzins and Sunyar measure neutrino helicity: ν_L
- Then, in weak interactions ν_L and $\bar{\nu}_R$
- Helicity conservation $\quad \Longrightarrow \quad \nu_L \leftrightarrows \bar{\nu}_L$
- $\bar{\nu}_L$ is a sterile neutrino (Pontecorvo, 1967)
- 1962: Lederman, Schwartz and Steinberger discover ν_μ
- 1962: Maki, Nakagava, Sakata proposed a model with neutrino mixing:

$$\begin{aligned} \nu_e &= \cos\vartheta\,\nu_1 + \sin\vartheta\,\nu_2 \\ \nu_\mu &= -\sin\vartheta\,\nu_1 + \cos\vartheta\,\nu_2 \end{aligned}$$

  "weak neutrinos are not stable due to the occurrence of a virtual transmutation $\nu_e \leftrightarrows \nu_\mu$"
- 1967: Pontecorvo: $\nu_e \leftrightarrows \nu_\mu$ oscillations and applications (solar neutrinos)

Ultrarelativistic Approximation

Only neutrinos with energy $\gtrsim 0.1\,\text{MeV}$ are detectable!

Charged-Current Processes: Threshold

$$\boxed{\begin{gathered} \nu + A \to B + C \\ \Downarrow \\ s = 2Em_A + m_A^2 \geq (m_B + m_C)^2 \\ \Downarrow \\ E_{\text{th}} = \frac{(m_B + m_C)^2}{2m_A} - \frac{m_A}{2} \end{gathered}}$$

$$\begin{array}{ll} \nu_e + {}^{71}\text{Ga} \to {}^{71}\text{Ge} + e^- & E_{\text{th}} = 0.233\,\text{MeV} \\ \nu_e + {}^{37}\text{Cl} \to {}^{37}\text{Ar} + e^- & E_{\text{th}} = 0.81\,\text{MeV} \\ \bar{\nu}_e + p \to n + e^+ & E_{\text{th}} = 1.8\,\text{MeV} \\ \nu_\mu + n \to p + \mu^- & E_{\text{th}} = 110\,\text{MeV} \\ \nu_\mu + e^- \to \nu_e + \mu^- & E_{\text{th}} \simeq \frac{m_\mu^2}{2m_e} = 10.9\,\text{GeV} \end{array}$$

Elastic Scattering Processes: Cross Section $\propto$ Energy

$$\nu + e^- \to \nu + e^- \qquad \sigma(E) \sim \sigma_0\, E/m_e \qquad \sigma_0 \sim 10^{-44}\,\text{cm}^2$$

Background $\Longrightarrow$ $E_{\text{th}} \simeq 5\,\text{MeV}$ (SK, SNO), $0.25\,\text{MeV}$ (Borexino)

Laboratory and Astrophysical Limits $\Longrightarrow$ $\boxed{m_\nu \lesssim 1\,\text{eV}}$

Easy Example of Neutrino Production

$$\underline{\pi^+ \to \mu^+ + \nu_\mu}$$

$$\boxed{\nu_\mu = \sum_k U_{\mu k} \nu_k}$$

two-body decay $\Longrightarrow$ fixed kinematics

$$\boxed{E_k^2 = p_k^2 + m_k^2}$$

$$\pi \text{ at rest: } \begin{cases} p_k^2 = \dfrac{m_\pi^2}{4}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right)^2 - \dfrac{m_k^2}{2}\left(1 + \dfrac{m_\mu^2}{m_\pi^2}\right) + \dfrac{m_k^4}{4\,m_\pi^2} \\[2ex] E_k^2 = \dfrac{m_\pi^2}{4}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right)^2 + \dfrac{m_k^2}{2}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right) + \dfrac{m_k^4}{4\,m_\pi^2} \end{cases}$$

0th order: $m_k = 0 \Rightarrow p_k = E_k = E = \dfrac{m_\pi}{2}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right) \simeq 30\,\text{MeV}$

1st order: $\boxed{E_k \simeq E + \xi \dfrac{m_k^2}{2E}}$ $\boxed{p_k \simeq E - (1-\xi)\dfrac{m_k^2}{2E}}$

$$\xi = \frac{1}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 0.2$$

Flavor Transitions

[Eliezer, Swift, NPB 105 (1976) 45] [Fritzsch, Minkowski, PLB 62 (1976) 72] [Bilenky, Pontecorvo, SJNP 24 (1976) 316]

$$\mathcal{L}_{\text{CC}} \sim W_\rho \left(\overline{\nu_{eL}} \gamma^\rho e_L + \overline{\nu_{\mu L}} \gamma^\rho \mu_L + \overline{\nu_{\tau L}} \gamma^\rho \tau_L \right)$$

Fields $\nu_\alpha = \sum_k U_{\alpha k} \nu_k \quad \Longrightarrow \quad |\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$ States

initial flavor: $\alpha \quad = \quad e$ or μ or τ

$$|\nu_k(t,x)\rangle = e^{-iE_k t + ip_k x} |\nu_k\rangle \quad \Rightarrow \quad |\nu_\alpha(t,x)\rangle = \sum_k U_{\alpha k}^* \, e^{-iE_k t + ip_k x} \, |\nu_k\rangle$$

$$|\nu_k\rangle = \sum_{\beta=e,\mu,\tau} U_{\beta k} |\nu_\beta\rangle \quad \Rightarrow \quad |\nu_\alpha(t,x)\rangle = \sum_{\beta=e,\mu,\tau} \underbrace{\left(\sum_k U_{\alpha k}^* e^{-iE_k t + ip_k x} U_{\beta k} \right)}_{\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t,x)} |\nu_\beta\rangle$$

$$\mathcal{A}_{\nu_\alpha \to \nu_\beta}(0,0) = \sum_k U_{\alpha k}^* U_{\beta k} = \delta_{\alpha\beta} \qquad \qquad \mathcal{A}_{\nu_\alpha \to \nu_\beta}(t > 0, x > 0) \neq \delta_{\alpha\beta}$$

$$P_{\nu_\alpha \to \nu_\beta}(t,x) = \left|\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t,x)\right|^2 = \left|\sum_k U_{\alpha k}^* e^{-iE_k t + i p_k x} U_{\beta k}\right|^2$$

ultra-relativistic neutrinos $\Longrightarrow$ $t \simeq x = L$ source-detector distance

$$E_k t - p_k x \simeq (E_k - p_k) L = \frac{E_k^2 - p_k^2}{E_k + p_k} L = \frac{m_k^2}{E_k + p_k} L \simeq \frac{m_k^2}{2E} L$$

$$\begin{aligned} P_{\nu_\alpha \to \nu_\beta}(L,E) &= \left|\sum_k U_{\alpha k}^* \, e^{-i m_k^2 L/2E} \, U_{\beta k}\right|^2 \qquad \Delta m_{kj}^2 \equiv m_k^2 - m_j^2 \\ &= \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right) \end{aligned}$$

- exercise: Derive $P_{\nu_\alpha \to \nu_\beta}$ assuming $p_k = p$ and $|\nu_k(t)\rangle = e^{-iE_k t} |\nu_k\rangle$.
  Why the result is the same?

Antineutrinos

Right-handed antineutrinos are described by CP-conjugated fields:

$$\nu^{\mathrm{CP}} = \gamma^0 \mathcal{C}\, \overline{\nu}^T = -\mathcal{C}\, \nu^*$$

$$\mathrm{C} \quad \Longrightarrow \quad \text{Particle} \leftrightarrows \text{Antiparticle}$$

$$\mathrm{P} \quad \Longrightarrow \quad \text{Left-Handed} \leftrightarrows \text{Right-Handed}$$

$$\text{Fields:} \;\; \nu_{\alpha L} = \sum_k U_{\alpha k} \nu_{kL} \quad \xrightarrow{\mathrm{CP}} \nu_{\alpha L}^{\mathrm{CP}} = \sum_k U_{\alpha k}^* \nu_{kL}^{\mathrm{CP}}$$

$$\text{States:} \; |\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle \xrightarrow{\mathrm{CP}} |\bar{\nu}_\alpha\rangle = \sum_k U_{\alpha k} |\bar{\nu}_k\rangle$$

NEUTRINOS $\quad U \leftrightarrows U^* \quad$ ANTINEUTRINOS

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right)$$

$$P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}(L, E) = \sum_{k,j} U_{\alpha k} U_{\beta k}^* U_{\alpha j}^* U_{\beta j} \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right)$$

- exercise: Derive $P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}$ from $\mathcal{L}_{\mathrm{CC}}$.

CPT Symmetry

$$P_{\nu_\alpha \to \nu_\beta} \xrightarrow{\text{CPT}} P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha}$$

CPT Asymmetries: $A^{\text{CPT}}_{\alpha\beta} = P_{\nu_\alpha \to \nu_\beta} - P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha}$

Local Quantum Field Theory $\Longrightarrow$ $A^{\text{CPT}}_{\alpha\beta} = 0$ CPT Symmetry

$$P_{\nu_\alpha \to \nu_\beta}(L,E) = \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \exp\left(-i \frac{\Delta m^2_{kj} L}{2E}\right)$$

is invariant under CPT: $U \leftrightarrows U^*$ $\alpha \leftrightarrows \beta$

$$\boxed{P_{\nu_\alpha \to \nu_\beta} = P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha}}$$

$\boxed{P_{\nu_\alpha \to \nu_\alpha} = P_{\bar{\nu}_\alpha \to \bar{\nu}_\alpha}}$ (solar ν_e, reactor $\bar{\nu}_e$, accelerator ν_μ)

CP Symmetry

$$P_{\nu_\alpha \to \nu_\beta} \xrightarrow{\text{CP}} P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}$$

CP Asymmetries: $A^{\text{CP}}_{\alpha\beta} = P_{\nu_\alpha \to \nu_\beta} - P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}$

$$\boxed{A^{\text{CP}}_{\alpha\beta}(L, E) = 4 \sum_{k>j} \text{Im}\left[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}\right] \sin\left(\frac{\Delta m^2_{kj} L}{2E}\right)}$$

Jarlskog rephasing invariant: $\text{Im}\left[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}\right] = \pm J$

$$J = c_{12} s_{12} c_{23} s_{23} c^2_{13} s_{13} \sin\delta_{13}$$

CP violation is proportional to $\text{Im}[U_{e3}] = -s_{13} \sin\delta_{13}$

$$\boxed{\text{CPT} \Rightarrow A^{\text{CP}}_{\alpha\beta} = -A^{\text{CP}}_{\beta\alpha}}$$

exercise

T Symmetry

$$P_{\nu_\alpha \to \nu_\beta} \quad \xrightarrow{\mathrm{T}} \quad P_{\nu_\beta \to \nu_\alpha}$$

T Asymmetries: $A^{\mathrm{T}}_{\alpha\beta} = P_{\nu_\alpha \to \nu_\beta} - P_{\nu_\beta \to \nu_\alpha}$

$$\boxed{\mathrm{CPT} \quad \Rightarrow \quad A^{\mathrm{T}}_{\alpha\beta} = A^{\mathrm{CP}}_{\beta\alpha}}$$

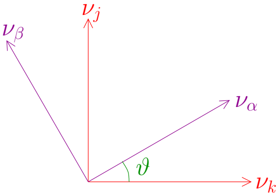
exercise

$$\boxed{A^{\mathrm{T}}_{\alpha\beta}(L,E) = 4 \sum_{k>j} \mathrm{Im}\left[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}\right] \sin\left(\frac{\Delta m^2_{kj} L}{2E}\right)}$$

Jarlskog rephasing invariant: $\mathrm{Im}\left[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}\right] = \pm J$

Two-Neutrino Mixing and Oscillations

$$\boxed{\begin{aligned} |\nu_\alpha\rangle &= \cos\vartheta\,|\nu_k\rangle + \sin\vartheta\,|\nu_j\rangle \\ |\nu_\beta\rangle &= -\sin\vartheta\,|\nu_k\rangle + \cos\vartheta\,|\nu_j\rangle \end{aligned}}$$



$$U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

$$\Delta m^2 \equiv \Delta m^2_{kj} \equiv m^2_k - m^2_j$$

Transition Probability: $$P_{\nu_\alpha\to\nu_\beta} = P_{\nu_\beta\to\nu_\alpha} = \sin^2 2\vartheta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

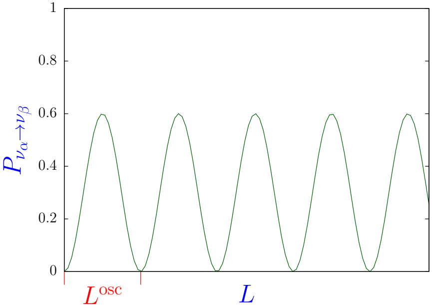
Survival Probabilities: $$P_{\nu_\alpha\to\nu_\alpha} = P_{\nu_\beta\to\nu_\beta} = 1 - P_{\nu_\alpha\to\nu_\beta}$$

oscillation phase

$$\frac{\Delta m^2 L}{4E} = 1.27\,\frac{\Delta m^2[\text{eV}^2]\,L[\text{m}]}{E[\text{MeV}]} = 1.27\,\frac{\Delta m^2[\text{eV}^2]\,L[\text{km}]}{E[\text{GeV}]}$$

oscillation length

$$L^{\text{osc}} = \frac{4\pi E}{\Delta m^2} = 2.47\,\frac{E\,[\text{MeV}]}{\Delta m^2\,[\text{eV}^2]}\,\text{m} = 2.47\,\frac{E\,[\text{GeV}]}{\Delta m^2\,[\text{eV}^2]}\,\text{km}$$



Types of Experiments

transitions due to Δm^2 observable only if $\frac{\Delta m^2 L}{E} \gtrsim 1 \Leftrightarrow \Delta m^2 \gtrsim \left(\frac{L}{E}\right)^{-1}$

SBL

$L/E \lesssim 10\,\text{eV}^{-2} \Rightarrow \Delta m^2 \gtrsim 0.1\,\text{eV}^2$

Reactor: $L \sim 10\,\text{m}\,,\ E \sim 1\,\text{MeV}$

Accelerator: $L \sim 1\,\text{km}\,,\ E \gtrsim 0.1\,\text{GeV}$

ATM & LBL

$L/E \lesssim 10^4\,\text{eV}^{-2}$

$\Downarrow$

$\Delta m^2 \gtrsim 10^{-4}\,\text{eV}^2$

Reactor: $L \sim 1\,\text{km}\,,\ E \sim 1\,\text{MeV}$ CHOOZ, PALO VERDE

Accelerator: $L \sim 10^3\,\text{km}\,,\ E \gtrsim 1\,\text{GeV}$ K2K, MINOS, CNGS

Atmospheric: $L \sim 10^2 - 10^4\,\text{km}\,,\ E \sim 0.1 - 10^2\,\text{GeV}$

Kamiokande, IMB, Super-Kamiokande, Soudan, MACRO, MINOS

SUN

$\frac{L}{E} \sim 10^{11}\,\text{eV}^{-2} \Rightarrow \Delta m^2 \gtrsim 10^{-11}\,\text{eV}^2$

$L \sim 10^8\,\text{km}\,,\quad E \sim 0.1 - 10\,\text{MeV}$

Homestake, Kamiokande, GALLEX, SAGE, Super-Kamiokande, GNO, SNO, Borexino

Matter Effect (MSW) $\Rightarrow 10^{-4} \lesssim \sin^2 2\vartheta \lesssim 1\,,\ 10^{-8}\,\text{eV}^2 \lesssim \Delta m^2 \lesssim 10^{-4}\,\text{eV}^2$

VLBL

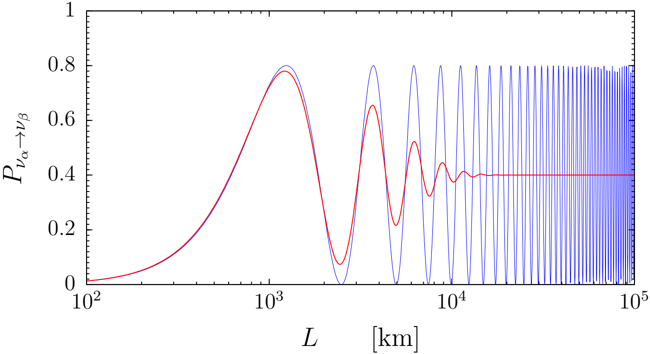
$L/E \lesssim 10^5\,\text{eV}^{-2} \Rightarrow \Delta m^2 \gtrsim 10^{-5}\,\text{eV}^2$

Reactor: $L \sim 10^2\,\text{km}\,,\ E \sim 1\,\text{MeV}$

KamLAND

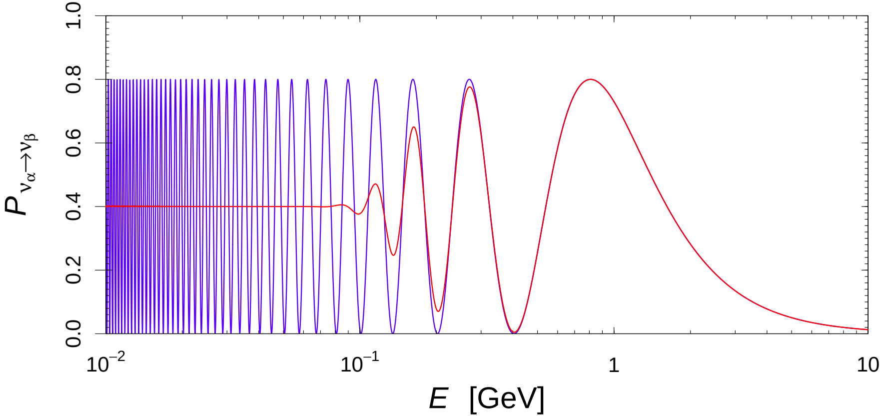
Average over Energy Resolution of the Detector

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sin^2 2\vartheta \sin^2\left(\frac{\Delta m^2 L}{4E}\right) = \frac{1}{2}\sin^2 2\vartheta \left[1 - \cos\left(\frac{\Delta m^2 L}{2E}\right)\right]$$



$\Delta m^2 = 10^{-3}\,\text{eV}$ $\sin^2 2\vartheta = 0.8$ $\langle E \rangle = 1\,\text{GeV}$ $\sigma_E = 0.1\,\text{GeV}$

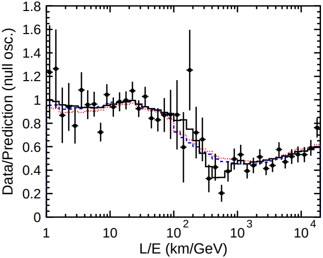
$$\langle P_{\nu_\alpha \to \nu_\beta}(L, E) \rangle = \frac{1}{2}\sin^2 2\vartheta \left[1 - \int \cos\left(\frac{\Delta m^2 L}{2E}\right) \phi(E)\,\mathrm{d}E\right] \qquad (\alpha \neq \beta)$$



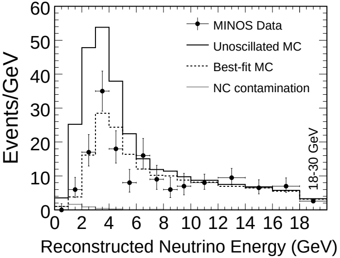
$\Delta m^2 = 10^{-3}\,\mathrm{eV} \qquad \sin^2 2\vartheta = 0.8 \qquad L = 10^3\,\mathrm{km} \qquad \sigma_E = 0.01\,\mathrm{GeV}$

$$\langle P_{\nu_\alpha \to \nu_\beta}(L,E) \rangle = \frac{1}{2} \sin^2 2\vartheta \left[1 - \int \cos\left(\frac{\Delta m^2 L}{2E} \right) \phi(E) \, \mathrm{d}E \right] \qquad (\alpha \neq \beta)$$

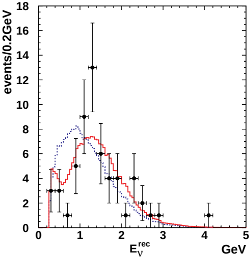
Observations of Neutrino Oscillations



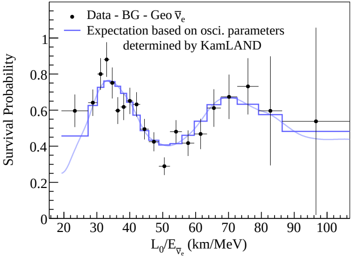
[Super-Kamiokande, PRL 93 (2004) 101801, hep-ex/0404034]



[MINOS, PRD 77 (2008) 072002, arXiv:0711.0769]



[K2K, PRD 74 (2006) 072003, hep-ex/0606032v3]

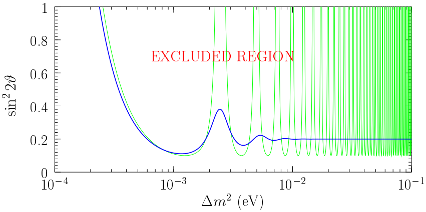


[KamLAND, PRL 100 (2008) 221803, arXiv:0801.4589]

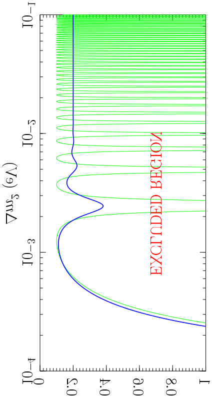
Exclusion Curves

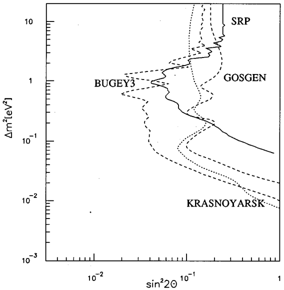
$$\langle P_{\nu_\alpha \to \nu_\beta}(L, E) \rangle = \frac{1}{2} \sin^2 2\vartheta \left[1 - \int \cos\left(\frac{\Delta m^2 L}{2E} \right) \phi(E) \, \mathrm{d}E \right] \qquad (\alpha \neq \beta)$$

$$\langle P_{\nu_\alpha \to \nu_\beta}(L, E) \rangle \leq P^{\max}_{\nu_\alpha \to \nu_\beta} \quad \Longrightarrow \quad \sin^2 2\vartheta \leq \frac{2 \, P^{\max}_{\nu_\alpha \to \nu_\beta}}{1 - \int \cos\left(\frac{\Delta m^2 L}{2E} \right) \phi(E) \, \mathrm{d}E}$$

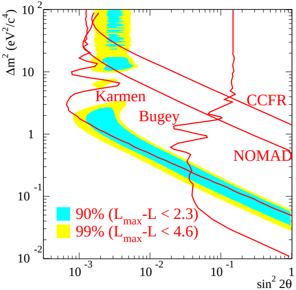


rotate
and
mirror

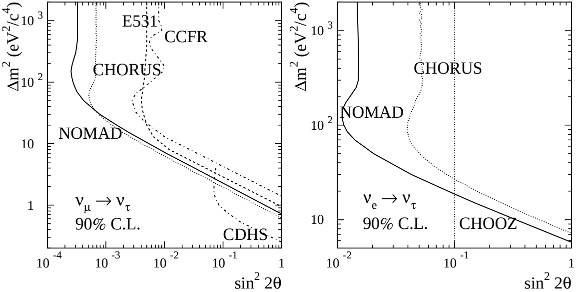




Reactor SBL Experiments: $\bar{\nu}_e \to \bar{\nu}_e$



Accelerator SBL Experiments: $\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e$



Accelerator SBL Experiments: $\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_\tau$ and $\overset{(-)}{\nu}_e \to \overset{(-)}{\nu}_\tau$

Observations of Neutrino Oscillations

Solar

$\nu_e \to \nu_\mu, \nu_\tau$

(SNO, BOREXino

Super-Kamiokande

GALLEX/GNO, SAGE

Homestake, Kamiokande)

VLBL Reactor

$\bar{\nu}_e$ disappearance

(KamLAND)

$$\to \begin{cases} \Delta m^2_{\text{S}} \simeq 7.6 \times 10^{-5}\,\text{eV}^2 \\ \sin^2\vartheta_{\text{S}} \simeq 0.30 \end{cases}$$

Atmospheric

$\nu_\mu \to \nu_\tau$

(Super-Kamiokande

Kamiokande, IMB

MACRO, Soudan-2)

LBL Accelerator

ν_μ disappearance

(K2K, MINOS, T2K)

LBL Accelerator

$\nu_\mu \to \nu_\tau$

(Opera)

$$\to \begin{cases} \Delta m^2_{\text{A}} \simeq 2.4 \times 10^{-3}\,\text{eV}^2 \\ \sin^2\vartheta_{\text{A}} \simeq 0.50 \end{cases}$$

LBL Accelerator

$\nu_\mu \to \nu_e$

(T2K, MINOS)

LBL Reactor

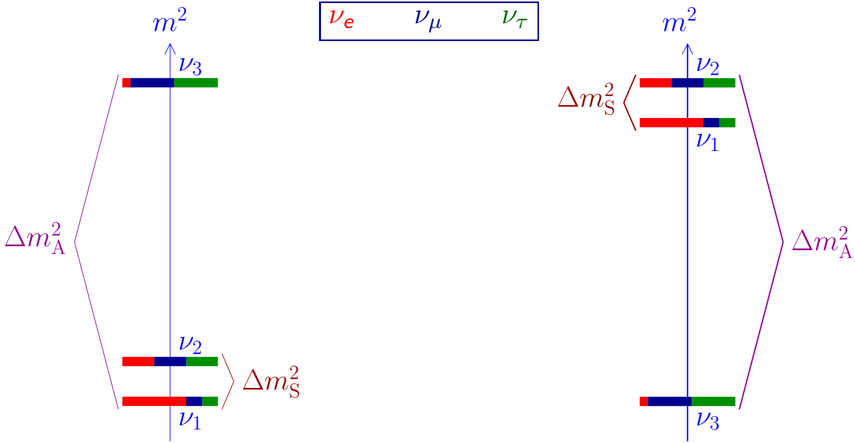
$\bar{\nu}_e$ disappearance

(Daya Bay, RENO

Double Chooz)

$$\to \begin{cases} \Delta m^2_{\text{A}} \\ \sin^2\vartheta_{13} \simeq 0.023 \end{cases}$$

Three-Neutrino Mixing Paradigm



Normal Spectrum

Inverted Spectrum

$\Delta m^2_{\rm S} = \Delta m^2_{21} = 7.50 \pm 0.20 \times 10^{-5}\,{\rm eV}^2$ uncertainty $\simeq 2.6\%$

$\Delta m^2_{\rm A} = |\Delta m^2_{31}| \simeq |\Delta m^2_{32}| = 2.32^{+0.12}_{-0.08} \times 10^{-3}\,{\rm eV}^2$ uncertainty $\simeq 5\%$

$$\nu_\alpha = \sum_{k=1}^{3} U_{\alpha k} \nu_k \qquad (\alpha = e, \mu, \tau)$$

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_2} & 0 \\ 0 & 0 & e^{i\lambda_3} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\vartheta_{23} = \vartheta_{\mathrm{A}}} \quad \underbrace{\begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix}}_{\text{Chooz, Palo Verde}} \quad \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\vartheta_{12} = \vartheta_{\mathrm{S}}} \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_2} & 0 \\ 0 & 0 & e^{i\lambda_3} \end{pmatrix}}_{\beta\beta_{0\nu}}$$

$\sin^2\vartheta_{23} \simeq 0.4 - 0.6$

T2K, MINOS

$\sin^2\vartheta_{12} = 0.30 \pm 0.01$

Daya Bay, RENO

$\sin^2\vartheta_{13} = 0.023 \pm 0.002$

$$\frac{\delta \sin^2\vartheta_{23}}{\sin^2\vartheta_{23}} \simeq 40\% \qquad \frac{\delta \sin^2\vartheta_{13}}{\sin^2\vartheta_{13}} \simeq 10\% \qquad \frac{\delta \sin^2\vartheta_{12}}{\sin^2\vartheta_{12}} \simeq 5\%$$

$\delta_{13} \neq 0, \pi \quad \Longrightarrow \quad$ CP violation in ν osc.

$$P_{\nu_\alpha \to \nu_\beta} \neq P_{\bar\nu_\alpha \to \bar\nu_\beta} \qquad (\alpha \neq \beta)$$

Effective VLBL ν_e Survival Probability

$$P_{\nu_e\to\nu_e} = \left| \sum_{k=1}^{3} |U_{ek}|^2 e^{-i m_k^2 L/2E} \right|^2$$

$$|U_{e3}|^2 \ll |U_{e1}|^2, |U_{e2}|^2 \quad \Longrightarrow \quad |U_{e1}|^2 \simeq \cos\vartheta_{12}\,,\ |U_{e2}|^2 \simeq \sin\vartheta_{12}$$

$$\begin{aligned} P_{\nu_e\to\nu_e} &\simeq \left| \sum_{k=1}^{2} |U_{ek}|^2 e^{-i m_k^2 L/2E} \right|^2 \\ &\simeq \left| \cos^2\vartheta_{12} e^{-i m_1^2 L/2E} + \sin^2\vartheta_{12} e^{-i m_2^2 L/2E} \right|^2 \\ &= \cos^4\vartheta_{12} + \sin^4\vartheta_{12} + 2\cos^2\vartheta_{12}\cos^2\vartheta_{12}\cos\left(\frac{\Delta m_{21}^2 L}{2E}\right) \\ &= 1 - \sin^2 2\vartheta_{12} \sin^2\left(\frac{\Delta m_{21}^2 L}{4E}\right) \end{aligned}$$

Solar $\nu_e \to \nu_\mu, \nu_\tau$

$$\boxed{|U_{e3}|^2 \ll 1}$$

$$U \simeq \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12}c_{23} & c_{12}c_{23} & s_{23} \\ s_{12}s_{23} & -c_{12}s_{23} & c_{23} \end{pmatrix} \Longrightarrow \left\{ \begin{array}{rl} \nu_e = & c_{12}\nu_1 + s_{12}\nu_2 \\ \nu_a^{(\mathrm{S})} = & -s_{12}\nu_1 + c_{12}\nu_2 \\ = & c_{23}\nu_\mu - s_{23}\nu_\tau \end{array} \right.$$

$$\sin^2 2\vartheta_{23} \simeq 1 \Longrightarrow \vartheta_{23} \simeq \frac{\pi}{4} \Longrightarrow U \simeq \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12}/\sqrt{2} & c_{12}/\sqrt{2} & 1/\sqrt{2} \\ s_{12}/\sqrt{2} & -c_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\text{Solar } \nu_e \to \nu_a^{(\mathrm{S})} \simeq \frac{1}{\sqrt{2}}\left(\nu_\mu - \nu_\tau\right)$$

$$\frac{\Phi_{\mathrm{CC}}^{\mathrm{SNO}}}{\Phi_{\nu_e}^{\mathrm{SSM}}} \simeq \frac{1}{3} \Longrightarrow \Phi_{\nu_e} \simeq \Phi_{\nu_\mu} \simeq \Phi_{\nu_\tau} \text{ for } E \gtrsim 6\,\mathrm{MeV}$$

$$\sin^2\vartheta_{\mathrm{S}} \simeq \frac{1}{3} \Longrightarrow U \simeq \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

Tri-Bimaximal Mixing

[Harrison, Perkins, Scott, hep-ph/0202074]

Effective ATM and LBL Oscillation Probabilities

$$P_{\nu_\alpha\to\nu_\beta} = \left|\sum_{k=1}^{3} U_{\alpha k}^* U_{\beta k} e^{-im_k^2 L/2E}\right|^2 * \left|e^{im_1^2 L/2E}\right|^2$$
$$= \left|\sum_{k=1}^{3} U_{\alpha k}^* U_{\beta k} \exp\left(-i\frac{\Delta m_{k1}^2 L}{2E}\right)\right|^2$$

$$\frac{\Delta m_{21}^2 L}{2E} \ll 1$$

$$P_{\nu_\alpha\to\nu_\beta} = \left|U_{\alpha 1}^* U_{\beta 1} + U_{\alpha 2}^* U_{\beta 2} + U_{\alpha 3}^* U_{\beta 3} \exp\left(-i\frac{\Delta m_{31}^2 L}{2E}\right)\right|^2$$

$$U_{\alpha 1}^* U_{\beta 1} + U_{\alpha 2}^* U_{\beta 2} = \delta_{\alpha\beta} - U_{\alpha 3}^* U_{\beta 3}$$

$$
\begin{aligned}
P_{\nu_\alpha \to \nu_\beta} &= \left| \delta_{\alpha\beta} - U_{\alpha 3}^{*} U_{\beta 3} \left[1 - \exp\left(-i \frac{\Delta m_{31}^2 L}{2E} \right) \right] \right|^2 \\
&= \delta_{\alpha\beta} + |U_{\alpha 3}|^2 |U_{\beta 3}|^2 \left(2 - 2\cos\frac{\Delta m_{31}^2 L}{2E} \right) \\
&\qquad - 2\delta_{\alpha\beta} |U_{\alpha 3}|^2 \left(1 - \cos\frac{\Delta m_{31}^2 L}{2E} \right) \\
&= \delta_{\alpha\beta} - 2|U_{\alpha 3}|^2 \left(\delta_{\alpha\beta} - |U_{\beta 3}|^2 \right) \left(1 - \cos\frac{\Delta m_{31}^2 L}{2E} \right) \\
&= \delta_{\alpha\beta} - 4|U_{\alpha 3}|^2 \left(\delta_{\alpha\beta} - |U_{\beta 3}|^2 \right) \sin^2 \frac{\Delta m_{31}^2 L}{4E}
\end{aligned}
$$

$$\alpha \neq \beta \quad \Longrightarrow \quad P_{\nu_\alpha \to \nu_\beta} = 4|U_{\alpha 3}|^2 |U_{\beta 3}|^2 \sin^2 \left(\frac{\Delta m_{31}^2 L}{4E} \right)$$

$$\alpha = \beta \quad \Longrightarrow \quad P_{\nu_\alpha \to \nu_\alpha} = 1 - 4|U_{\alpha 3}|^2 \left(1 - |U_{\alpha 3}|^2 \right) \sin^2 \left(\frac{\Delta m_{31}^2 L}{4E} \right)$$

$$P_{\nu_\alpha \to \nu_\beta} = \sin^2 2\vartheta_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{31} L}{4E}\right) \quad (\alpha \neq \beta)$$

$$\sin^2 2\vartheta_{\alpha\beta} = 4|U_{\alpha 3}|^2 |U_{\beta 3}|^2$$

$$P_{\nu_\alpha \to \nu_\alpha} = 1 - \sin^2 2\vartheta_{\alpha\alpha} \sin^2\left(\frac{\Delta m^2_{31} L}{4E}\right)$$

$$\sin^2 2\vartheta_{\alpha\alpha} = 4|U_{\alpha 3}|^2 \left(1 - |U_{\alpha 3}|^2\right)$$

$$U = \left(\begin{array}{cc|c|} \cline{3-3} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \\ \cline{3-3} \end{array}\right)$$

$\uparrow$

LBL

$$\sin^2 2\vartheta_{ee} \ll 1$$

$$\Downarrow$$

$$|U_{e3}|^2 \simeq \frac{\sin^2 2\vartheta_{ee}}{4}$$

Effective ATM and LBL Oscillation Amplitudes

- ν_e disappearance experiments:

$$\sin^2 2\vartheta_{ee} = 4|U_{e3}|^2 \left(1 - |U_{e3}|^2\right) = \sin^2 2\vartheta_{13} \simeq 0.090$$

Chooz, Palo Verde, Daya Bay, RENO

- ν_μ disappearance experiments:

$$\sin^2 2\vartheta_{\mu\mu} = 4|U_{\mu3}|^2 \left(1 - |U_{\mu3}|^2\right) \simeq (1 - \sin^2 \vartheta_{13}) \sin^2 2\vartheta_{23} \simeq 1 - \epsilon$$

$$|U_{\mu3}|^2 = \frac{1}{2}\left(1 \pm \sqrt{1 - \sin^2 2\vartheta_{\mu\mu}}\right) = \frac{1}{2}\left(1 \pm \sqrt{\epsilon}\right)$$

ATM, K2K, MINOS

- $\nu_\mu \to \nu_e$ experiments:

$$\sin^2 2\vartheta_{\mu e} = 4|U_{e3}|^2|U_{\mu3}|^2 = \sin^2 2\vartheta_{13} \sin^2 \vartheta_{23} \simeq 0.045$$

T2K, MINOS

- $\nu_\mu \to \nu_\tau$ experiments:

$$\sin^2 2\vartheta_{\mu\tau} = 4|U_{\mu3}|^2|U_{\tau3}|^2 = (1 - \sin^2 \vartheta_{13})^2 \sin^2 2\vartheta_{23} \simeq 0.95$$

OPERA

CP Violation?

- In this approximation there is no observable CP-violation effect!

- CP-violation can be observed only with sensitivity to Δm^2_{21}: in vacuum

$$\begin{aligned} A^{\text{CP}}_{\alpha\beta} &= P_{\nu_\alpha \to \nu_\beta} - P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta} \\ &= -16 J_{\alpha\beta} \sin\left(\frac{\Delta m^2_{21} L}{4E}\right) \sin\left(\frac{\Delta m^2_{31} L}{4E}\right) \sin\left(\frac{\Delta m^2_{32} L}{4E}\right) \end{aligned}$$

$$J_{\alpha\beta} = \text{Im}(U_{\alpha 1} U^*_{\alpha 2} U^*_{\beta 1} U_{\beta 2}) = \pm J$$

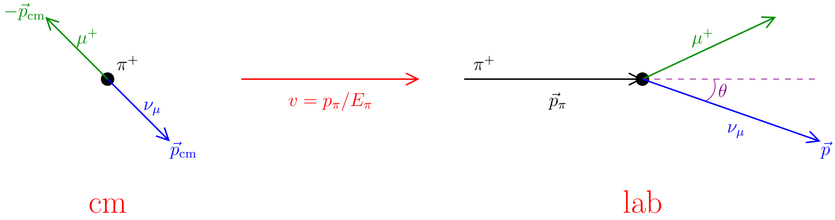
$$J = s_{12} c_{12} s_{23} c_{23} s_{13} c^2_{13} \sin \delta_{13}$$

- Necessary conditions for observation of CP violation:
  - Sensitivity to all mixing angles, including small ϑ_{13}
  - Sensitivity to oscillations due to Δm^2_{21} and Δm^2_{31}

- exercise: Derive this expression of $A^{\text{CP}}_{\alpha\beta}$

Off-Axis Experiments

high-intensity WB beam
detector shifted by a small angle from axis of beam
almost monochromatic neutrino energy



$$E_{\rm cm} = p_{\rm cm} = \frac{m_\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 29.79\,\text{MeV}$$

$$\gamma = \left(1 - v^2\right)^{-1/2} = E_\pi / m_\pi \gg 1$$

$$\begin{cases} E = \gamma\left(E_{\rm cm} + v\, p^z_{\rm cm}\right) \\ p^z = \gamma\left(v\, E_{\rm cm} + p^z_{\rm cm}\right) \end{cases}$$

$$p^z = p\,\cos\theta \qquad \Longrightarrow \qquad E = \frac{E_{\rm cm}}{\gamma\left(1 - v\,\cos\theta\right)}$$

$\cos\theta \simeq 1 - \theta^2/2$ and $v \simeq 1$

$$E = \frac{E_{\text{cm}}}{\gamma\,(1 - v\cos\theta)} \simeq \frac{\gamma\,(1+v)}{1+\gamma^2\theta^2 v\,(1+v)\,/2}\,E_{\text{cm}} \simeq \frac{2\gamma}{1+\gamma^2\theta^2}\,E_{\text{cm}}$$

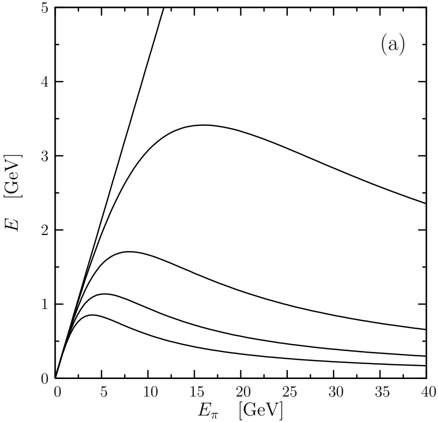
$$E \simeq \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)\frac{E_\pi}{1+\gamma^2\,\theta^2} = \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)\frac{E_\pi\,m_\pi^2}{m_\pi^2 + E_\pi^2\,\theta^2}$$

- $\theta = 0 \quad \Longrightarrow \quad E \propto E_\pi$ WB beam
- $E_\pi\theta \gg m_\pi \quad \Longrightarrow \quad E \propto \dfrac{m_\pi^2}{E_\pi\,\theta^2}$ high-energy π^+ give low-energy ν_μ

$$\frac{dE}{dE_\pi} \simeq \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)\frac{1-\gamma^2\,\theta^2}{\left(1+\gamma^2\,\theta^2\right)^2}$$

$$\frac{dE}{dE_\pi} \simeq 0 \quad \text{for} \quad \theta = \gamma^{-1} = \frac{m_\pi}{E_\pi} \quad \Longrightarrow \quad E \simeq \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)\frac{m_\pi}{2\theta} \simeq \frac{29.79\,\text{MeV}}{\theta}$$

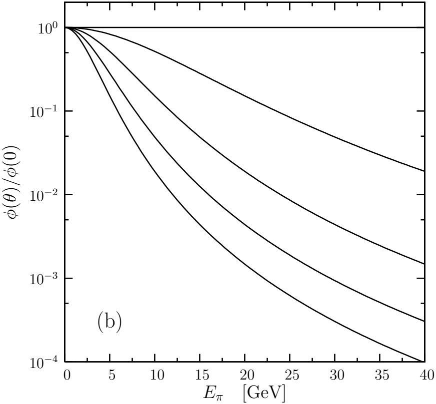
off-axis angle $\theta \simeq m_\pi / \langle E_\pi \rangle \implies E \simeq \frac{29.79\,\text{MeV}}{\theta}$



$\theta = 0.0°, 0.5°, 1.0°, 1.5°, 2.0°$

- E can be tuned on oscillation peak $E_\text{peak} = \Delta m^2 L / 2\pi$
- small $E \implies$ short $L_\text{osc} = \frac{4\pi E}{\Delta m^2} \implies$ sensitivity to small values of Δm^2

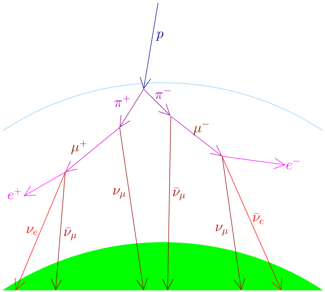
$$\frac{\phi(\theta)}{\phi(0)} = \frac{1}{4}\left(\frac{2}{1+\gamma^2\,\theta^2}\right)^2$$

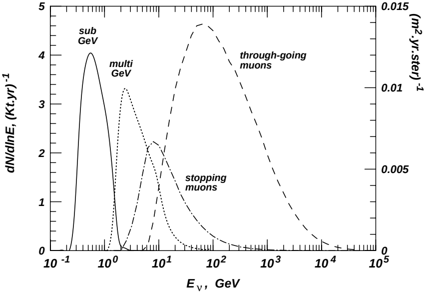


$\theta = 0.0^\circ, 0.5^\circ, 1.0^\circ, 1.5^\circ, 2.0^\circ$

flux suppression requires superbeam

Atmospheric Neutrinos





$$\frac{N(\nu_\mu + \bar{\nu}_\mu)}{N(\nu_e + \bar{\nu}_e)} \simeq 2 \quad \text{at } E \lesssim 1\,\text{GeV}$$

uncertainty on ratios: $\sim 5\%$

uncertainty on fluxes: $\sim 30\%$

ratio of ratios

$$R \equiv \frac{[N(\nu_\mu + \bar{\nu}_\mu)/N(\nu_e + \bar{\nu}_e)]_{\text{data}}}{[N(\nu_\mu + \bar{\nu}_\mu)/N(\nu_e + \bar{\nu}_e)]_{\text{MC}}}$$

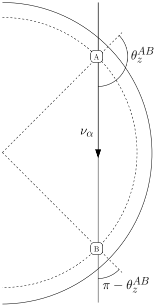
$$R^{\text{K}}_{\text{sub-GeV}} = 0.60 \pm 0.07 \pm 0.05$$

[Kamiokande, PLB 280 (1992) 146]

$$R^{\text{K}}_{\text{multi-GeV}} = 0.57 \pm 0.08 \pm 0.07$$

[Kamiokande, PLB 335 (1994) 237]

Super-Kamiokande Up-Down Asymmetry



$E_\nu \gtrsim 1\,\text{GeV} \Rightarrow$ isotropic flux of cosmic rays

$$\phi^{(A)}_{\nu_\alpha}(\theta_z^{AB}) = \phi^{(B)}_{\nu_\alpha}(\pi - \theta_z^{AB}) \qquad \phi^{(A)}_{\nu_\alpha}(\theta_z^{AB}) = \phi^{(B)}_{\nu_\alpha}(\theta_z^{AB})$$

$$\Downarrow$$

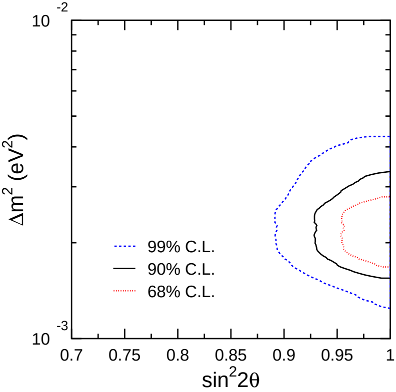
$$\phi^{(A)}_{\nu_\alpha}(\theta_z) = \phi^{(A)}_{\nu_\alpha}(\pi - \theta_z)$$

$$\boxed{A^{\text{up-down}}_{\nu_\mu}(\text{SK}) = \left(\frac{N^{\text{up}}_{\nu_\mu} - N^{\text{down}}_{\nu_\mu}}{N^{\text{up}}_{\nu_\mu} + N^{\text{down}}_{\nu_\mu}} \right) = -0.296 \pm 0.048 \pm 0.01}$$

[Super-Kamiokande, Phys. Rev. Lett. 81 (1998) 1562, hep-ex/9807003]

6σ MODEL INDEPENDENT EVIDENCE OF ν_μ DISAPPEARANCE!

Fit of Super-Kamiokande Atmospheric Data



$\underline{\nu_\mu \to \nu_\tau}$

Best Fit: $\begin{cases} \Delta m^2 = 2.1 \times 10^{-3}\,\text{eV}^2 \\ \sin^2 2\theta = 1.0 \end{cases}$

1489.2 live-days (Apr 1996 – Jul 2001)

[Super-Kamiokande, PRD 71 (2005) 112005, hep-ex/0501064]

Measure of ν_τ CC Int. is Difficult:

- $E_{\text{th}} = 3.5\,\text{GeV} \Longrightarrow \sim 20\text{events/yr}$
- τ-Decay $\Longrightarrow$ Many Final States

ν_τ-Enriched Sample

$N_{\nu_\tau}^{\text{the}} = 78 \pm 26$ @ $\Delta m^2 = 2.4 \times 10^{-3}\,\text{eV}^2$

$$\boxed{N_{\nu_\tau}^{\text{exp}} = 138^{+50}_{-58}}$$

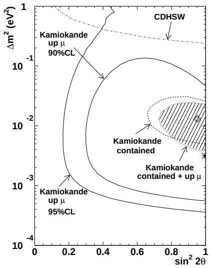
$N_{\nu_\tau} > 0$ @ 2.4σ

[Super-Kamiokande, PRL 97(2006) 171801, hep-ex/0607059]

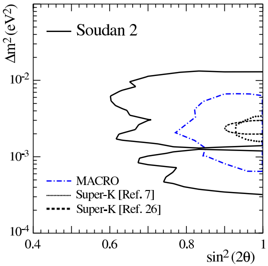
Check: OPERA ($\nu_\mu \to \nu_\tau$)
CERN to Gran Sasso (CNGS)
$L \simeq 732\,\text{km}$ $\langle E \rangle \simeq 18\,\text{GeV}$

[NJP 8 (2006) 303, hep-ex/0611023]

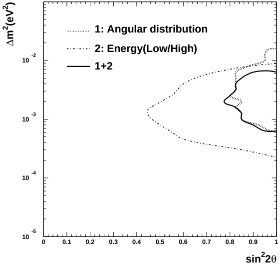
Kamiokande, Soudan-2, MACRO and MINOS



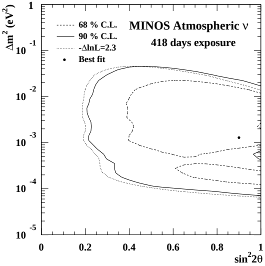
[Kamiokande, hep-ex/9806038]



[Soudan 2, hep-ex/0507068]



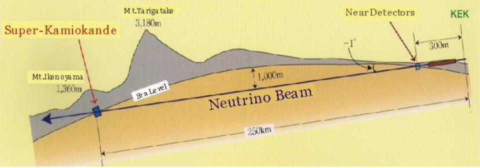
[MACRO, hep-ex/0304037]



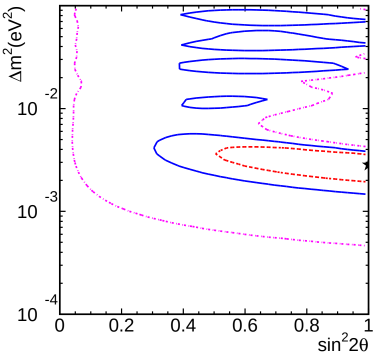
[MINOS, hep-ex/0512036]

K2K

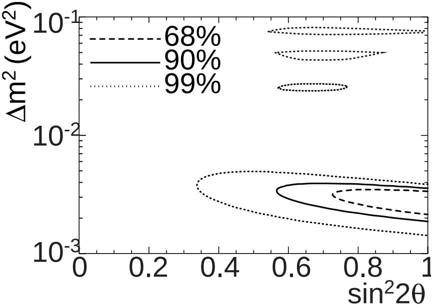
confirmation of atmospheric allowed region (June 2002)



KEK to Kamioka
(Super-Kamiokande)
250 km
$\nu_\mu \to \nu_\mu$

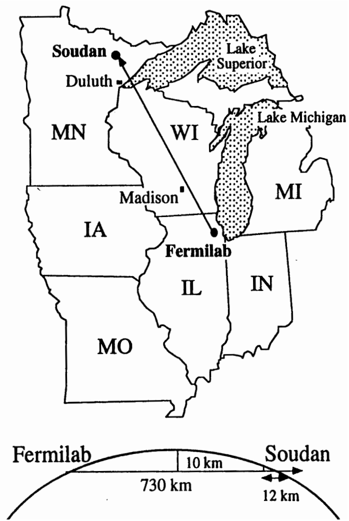


[K2K, Phys. Rev. Lett. 90 (2003) 041801]

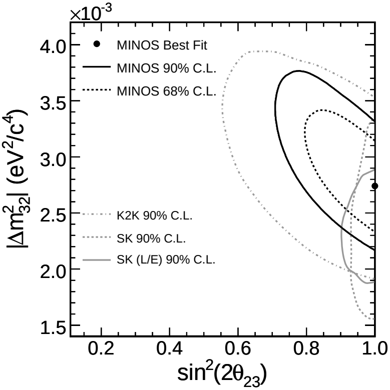


[K2K, PRL 94 (2005) 081802, hep-ex/0411038]

MINOS



Near Detector: 1 km

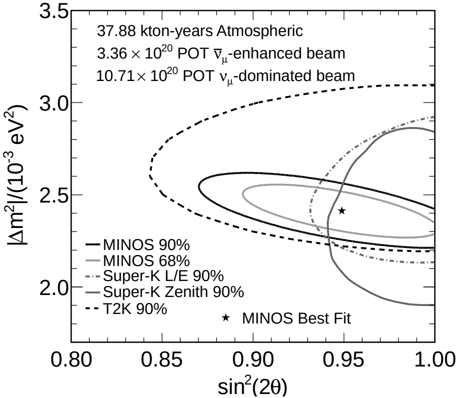


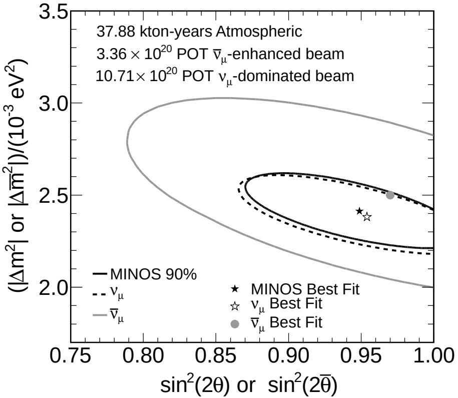
$\nu_\mu \to \nu_\mu$

$$\Delta m^2 = 2.74^{+0.44}_{-0.26} \times 10^{-3}\,\text{eV}^2$$

$$\sin^2 2\vartheta > 0.87 \text{ @ } 68\%\,CL$$

[MINOS, PRL 97 (2006) 191801, hep-ex/0607088]





$$|\Delta m^2_{31}| = 2.41^{+0.09}_{-0.10} \times 10^{-3}\,\mathrm{eV}^2 \qquad |\Delta m^2_{31}|_{\bar{\nu}} = 2.50^{+0.23}_{-0.25} \times 10^{-3}\,\mathrm{eV}^2$$

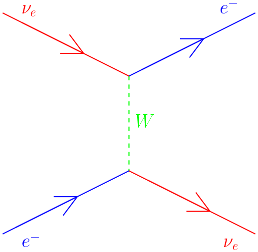
$$\sin^2 2\vartheta_{23} = 0.950^{+0.035}_{-0.036} \qquad \sin^2 2\vartheta^{\bar{\nu}}_{23} = 0.97^{+0.03}_{-0.08}$$

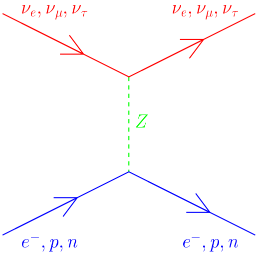
$$|\Delta m^2_{31}|_{\bar{\nu}} - |\Delta m^2_{31}|_{\nu} = 0.12^{+0.24}_{-0.26} \times 10^{-3}\,\mathrm{eV}^2$$

[MINOS, arXiv:1304.6335]

Neutrino Oscillations in Matter

Effective Potentials in Matter





$$\boxed{V_{\text{CC}} = \sqrt{2} G_{\text{F}} N_e} \qquad V_{\text{NC}}^{(e^-)} = -V_{\text{NC}}^{(p)} \quad \Rightarrow \quad \boxed{V_{\text{NC}} = V_{\text{NC}}^{(n)} = -\frac{\sqrt{2}}{2} G_{\text{F}} N_n}$$

$$V_e = V_{\text{CC}} + V_{\text{NC}} \qquad V_\mu = V_\tau = V_{\text{NC}}$$

only $V_{\text{CC}} = V_e - V_\mu = V_e - V_\tau$ is important for flavor transitions

antineutrinos: $\overline{V}_{\text{CC}} = -V_{\text{CC}}$ $\overline{V}_{\text{NC}} = -V_{\text{NC}}$

Matter Effects

a flavor neutrino ν_α with momentum p is described by

$$|\nu_\alpha(p)\rangle = \sum_k U_{\alpha k}^* \, |\nu_k(p)\rangle$$

$$\mathcal{H}_0 \, |\nu_k(p)\rangle = E_k \, |\nu_k(p)\rangle \qquad E_k = \sqrt{p^2 + m_k^2}$$

in matter $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ $\qquad \mathcal{H}_I \, |\nu_\alpha(p)\rangle = V_\alpha \, |\nu_\alpha(p)\rangle$

$V_\alpha =$ effective potential due to coherent interactions with the medium

forward elastic CC and NC scattering

Evolution of Neutrino Flavors in Matter

Schrödinger picture: $i\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\nu(p,t)\rangle = \mathcal{H}|\nu(p,t)\rangle\,, \qquad |\nu(p,0)\rangle = |\nu_\alpha(p)\rangle$

flavor transition amplitudes: $\varphi_\beta(p,t) = \langle\nu_\beta(p)|\nu(p,t)\rangle\,, \qquad \varphi_\beta(p,0) = \delta_{\alpha\beta}$

$$i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi_\beta(p,t) = \langle\nu_\beta(p)|\mathcal{H}|\nu(p,t)\rangle = \langle\nu_\beta(p)|\mathcal{H}_0|\nu(p,t)\rangle + \langle\nu_\beta(p)|\mathcal{H}_I|\nu(p,t)\rangle$$

$$\begin{aligned}\langle\nu_\beta(p)|\mathcal{H}_0|\nu(p,t)\rangle &= \sum_\rho \langle\nu_\beta(p)|\mathcal{H}_0|\nu_\rho(p)\rangle\,\underbrace{\langle\nu_\rho(p)|\nu(p,t)\rangle}_{\varphi_\rho(p,t)} \\ &= \sum_\rho \sum_{k,j} U_{\beta k}\,\underbrace{\langle\nu_k(p)|\mathcal{H}_0|\nu_j(p)\rangle}_{\delta_{kj}E_k}\,U^*_{\rho j}\,\varphi_\rho(p,t)\end{aligned}$$

$$\langle\nu_\beta(p)|\mathcal{H}_I|\nu(p,t)\rangle = \sum_\rho \underbrace{\langle\nu_\beta(p)|\mathcal{H}_I|\nu_\rho(p)\rangle}_{\delta_{\beta\rho}V_\beta}\,\varphi_\rho(p,t) = V_\beta\,\varphi_\beta(p,t)$$

$$\boxed{i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi_\beta = \sum_\rho \left(\sum_k U_{\beta k}\,E_k\,U^*_{\rho k} + \delta_{\beta\rho}\,V_\beta \right) \varphi_\rho}$$

ultrarelativistic neutrinos: $E_k = p + \frac{m_k^2}{2E}$ $\quad E = p \quad t = x$

$$V_e = V_{\text{CC}} + V_{\text{NC}} \qquad V_\mu = V_\tau = V_{\text{NC}}$$

$$i \frac{\mathrm{d}}{\mathrm{d}x} \varphi_\beta(p,x) = (p + V_{\text{NC}}) \varphi_\beta(p,x) + \sum_\rho \left(\sum_k U_{\beta k} \frac{m_k^2}{2E} U_{\rho k}^* + \delta_{\beta e} \delta_{\rho e} V_{\text{CC}} \right) \varphi_\rho(p,x)$$

$$\psi_\beta(p,x) = \varphi_\beta(p,x)\, e^{ipx + i \int_0^x V_{\text{NC}}(x')\, \mathrm{d}x'}$$

$$\Downarrow$$

$$i \frac{\mathrm{d}}{\mathrm{d}x} \psi_\beta = e^{ipx + i \int_0^x V_{\text{NC}}(x')\, \mathrm{d}x'} \left(-p - V_{\text{NC}} + i \frac{\mathrm{d}}{\mathrm{d}x} \right) \varphi_\beta$$

$$\boxed{i \frac{\mathrm{d}}{\mathrm{d}x} \psi_\beta = \sum_\rho \left(\sum_k U_{\beta k} \frac{m_k^2}{2E} U_{\rho k}^* + \delta_{\beta e} \delta_{\rho e} V_{\text{CC}} \right) \psi_\rho}$$

$$\boxed{P_{\nu_\alpha \to \nu_\beta} = |\varphi_\beta|^2 = |\psi_\beta|^2}$$

evolution of flavor transition amplitudes in matrix form

$$\boxed{i \, \frac{\mathrm{d}}{\mathrm{d}x} \, \Psi_\alpha = \frac{1}{2E} \left(U \, \mathbb{M}^2 \, U^\dagger + \mathbb{A} \right) \Psi_\alpha}$$

$$\Psi_\alpha = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \qquad \mathbb{M}^2 = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} \qquad \mathbb{A} = \begin{pmatrix} A_{\mathrm{CC}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_{\mathrm{CC}} = 2 E V_{\mathrm{CC}} = 2\sqrt{2} E G_{\mathrm{F}} N_e$$

effective mass-squared matrix in vacuum

$$\mathbb{M}^2_{\mathrm{VAC}} = U \, \mathbb{M}^2 \, U^\dagger \xrightarrow{\text{matter}} U \, \mathbb{M}^2 \, U^\dagger + 2 \, E \, \underset{\uparrow}{\mathbb{V}} = \mathbb{M}^2_{\mathrm{MAT}}$$

effective mass-squared matrix in matter

potential due to coherent forward elastic scattering

Two-Neutrino Mixing

$\nu_e \to \nu_{\mu(\tau)}$ transitions with $U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$

$$U\,\mathbb{M}^2\,U^\dagger = \begin{pmatrix} \cos^2\vartheta\, m_1^2 + \sin^2\vartheta\, m_2^2 & \cos\vartheta\,\sin\vartheta\,\left(m_2^2 - m_1^2\right) \\ \cos\vartheta\,\sin\vartheta\,\left(m_2^2 - m_1^2\right) & \sin^2\vartheta\, m_1^2 + \cos^2\vartheta\, m_2^2 \end{pmatrix}$$

$$= \frac{1}{2}\,\underset{\substack{\uparrow \\ \text{irrelevant common phase}}}{\Sigma m^2} + \frac{1}{2}\begin{pmatrix} -\Delta m^2\cos2\vartheta & \Delta m^2\sin2\vartheta \\ \Delta m^2\sin2\vartheta & \Delta m^2\cos2\vartheta \end{pmatrix}$$

$$\Sigma m^2 \equiv m_1^2 + m_2^2 \qquad\qquad \Delta m^2 \equiv m_2^2 - m_1^2$$

$$\boxed{i\,\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi_e\\\psi_\mu\end{pmatrix}=\frac{1}{4E}\begin{pmatrix}-\Delta m^2\cos2\vartheta+2A_{\mathrm{CC}} & \Delta m^2\sin2\vartheta\\\Delta m^2\sin2\vartheta & \Delta m^2\cos2\vartheta\end{pmatrix}\begin{pmatrix}\psi_e\\\psi_\mu\end{pmatrix}}$$

initial ν_e $\Longrightarrow$ $\begin{pmatrix}\psi_e(0)\\\psi_\mu(0)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

$$\boxed{\begin{array}{l}P_{\nu_e\to\nu_\mu}(x)=|\psi_\mu(x)|^2\\P_{\nu_e\to\nu_e}(x)=|\psi_e(x)|^2=1-P_{\nu_e\to\nu_\mu}(x)\end{array}}$$

Constant Matter Density

$$\boxed{i\,\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix} = \frac{1}{4E}\begin{pmatrix}-\Delta m^2\cos2\vartheta + 2A_{\mathrm{CC}} & \Delta m^2\sin2\vartheta\\ \Delta m^2\sin2\vartheta & \Delta m^2\cos2\vartheta\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix}}$$

$$\frac{\mathrm{d}A_{\mathrm{CC}}}{\mathrm{d}x} = 0$$

Diagonalization of Effective Hamiltonian

$$\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix} = \begin{pmatrix}\cos\vartheta_{\mathrm{M}} & \sin\vartheta_{\mathrm{M}}\\ -\sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$i\,\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \left[\underset{\uparrow}{\frac{A_{\mathrm{CC}}}{4E}} + \frac{1}{4E}\begin{pmatrix}-\Delta m^2_{\mathrm{M}} & 0\\ 0 & \Delta m^2_{\mathrm{M}}\end{pmatrix}\right]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

irrelevant common phase

Effective Mixing Angle in Matter

$$\tan 2\vartheta_{\mathrm{M}} = \frac{\tan 2\vartheta}{1 - \dfrac{A_{\mathrm{CC}}}{\Delta m^2 \cos 2\vartheta}}$$

Effective Squared-Mass Difference

$$\Delta m_{\mathrm{M}}^2 = \sqrt{\left(\Delta m^2 \cos 2\vartheta - A_{\mathrm{CC}}\right)^2 + \left(\Delta m^2 \sin 2\vartheta\right)^2}$$

Resonance ($\vartheta_{\mathrm{M}} = \pi/4$)

$$A_{\mathrm{CC}}^{\mathrm{R}} = \Delta m^2 \cos 2\vartheta \quad \Longrightarrow \quad N_e^{\mathrm{R}} = \frac{\Delta m^2 \cos 2\vartheta}{2\sqrt{2} E G_{\mathrm{F}}}$$

$$i \frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\Delta m^2_{\mathrm{M}} & 0 \\ 0 & \Delta m^2_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\vartheta_{\mathrm{M}} & \sin\vartheta_{\mathrm{M}} \\ -\sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos\vartheta_{\mathrm{M}} & -\sin\vartheta_{\mathrm{M}} \\ \sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\nu_e \to \nu_\mu \quad \Longrightarrow \quad \begin{pmatrix} \psi_e(0) \\ \psi_\mu(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} = \begin{pmatrix} \cos\vartheta_{\mathrm{M}} \\ \sin\vartheta_{\mathrm{M}} \end{pmatrix}$$

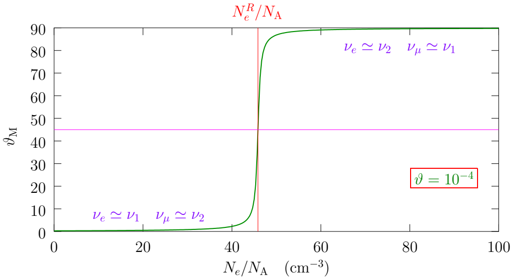
$$\psi_1(x) = \cos\vartheta_{\mathrm{M}} \exp\left(i \frac{\Delta m^2_{\mathrm{M}} x}{4E} \right)$$

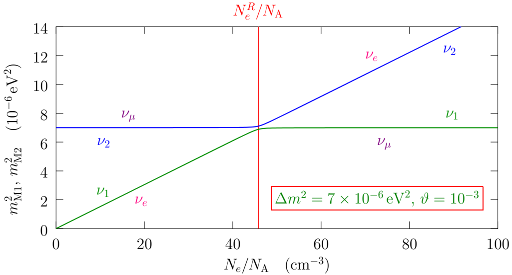
$$\psi_2(x) = \sin\vartheta_{\mathrm{M}} \exp\left(-i \frac{\Delta m^2_{\mathrm{M}} x}{4E} \right)$$

$$P_{\nu_e \to \nu_\mu}(x) = |\psi_\mu(x)|^2 = |-\sin\vartheta_{\mathrm{M}} \psi_1(x) + \cos\vartheta_{\mathrm{M}} \psi_2(x)|^2$$

$$\boxed{P_{\nu_e \to \nu_\mu}(x) = \sin^2 2\vartheta_{\mathrm{M}} \, \sin^2 \left(\frac{\Delta m^2_{\mathrm{M}} x}{4E} \right)}$$

MSW Effect (Resonant Transitions in Matter)





$$\nu_e = \cos\vartheta_M\,\nu_1 + \sin\vartheta_M\,\nu_2$$
$$\nu_\mu = -\sin\vartheta_M\,\nu_1 + \cos\vartheta_M\,\nu_2$$

$$\tan 2\vartheta_M = \frac{\tan 2\vartheta}{1 - \dfrac{A_{CC}}{\Delta m^2 \cos 2\vartheta}}$$

$$\Delta m^2_M = \left[\left(\Delta m^2 \cos 2\vartheta - A_{CC}\right)^2 + \left(\Delta m^2 \sin 2\vartheta\right)^2 \right]^{1/2}$$

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\vartheta_{\mathrm{M}} & \sin\vartheta_{\mathrm{M}} \\ -\sin\vartheta_{\mathrm{M}} & \cos\vartheta_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$i\,\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \left[\underset{\substack{\uparrow \\ \text{irrelevant common phase}}}{\frac{A_{\mathrm{CC}}}{4E}} + \frac{1}{4E} \begin{pmatrix} -\Delta m^2_{\mathrm{M}} & 0 \\ 0 & \Delta m^2_{\mathrm{M}} \end{pmatrix} + \underset{\substack{\uparrow \\ \text{maximum near resonance}}}{\begin{pmatrix} 0 & -i\,\dfrac{\mathrm{d}\vartheta_{\mathrm{M}}}{\mathrm{d}x} \\ i\,\dfrac{\mathrm{d}\vartheta_{\mathrm{M}}}{\mathrm{d}x} & 0 \end{pmatrix}} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} = \begin{pmatrix} \cos\vartheta^0_{\mathrm{M}} & -\sin\vartheta^0_{\mathrm{M}} \\ \sin\vartheta^0_{\mathrm{M}} & \cos\vartheta^0_{\mathrm{M}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\vartheta^0_{\mathrm{M}} \\ \sin\vartheta^0_{\mathrm{M}} \end{pmatrix}$$

$$\begin{aligned}
\psi_1(x) \;\simeq\; & \left[\cos\vartheta^0_{\mathrm{M}} \exp\left(i \int_0^{x_{\mathrm{R}}} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E}\,\mathrm{d}x' \right) \mathcal{A}^{\mathrm{R}}_{11} + \sin\vartheta^0_{\mathrm{M}} \exp\left(-i \int_0^{x_{\mathrm{R}}} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E}\,\mathrm{d}x' \right) \mathcal{A}^{\mathrm{R}}_{21} \right] \\
& \times \exp\left(i \int_{x_{\mathrm{R}}}^{x} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E}\,\mathrm{d}x' \right) \\
\psi_2(x) \;\simeq\; & \left[\cos\vartheta^0_{\mathrm{M}} \exp\left(i \int_0^{x_{\mathrm{R}}} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E}\,\mathrm{d}x' \right) \mathcal{A}^{\mathrm{R}}_{12} + \sin\vartheta^0_{\mathrm{M}} \exp\left(-i \int_0^{x_{\mathrm{R}}} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E}\,\mathrm{d}x' \right) \mathcal{A}^{\mathrm{R}}_{22} \right] \\
& \times \exp\left(-i \int_{x_{\mathrm{R}}}^{x} \frac{\Delta m^2_{\mathrm{M}}(x')}{4E}\,\mathrm{d}x' \right)
\end{aligned}$$

Averaged ν_e Survival Probability on Earth

$$\psi_e(x) = \cos\vartheta\,\psi_1(x) + \sin\vartheta\,\psi_2(x)$$

neglect interference (averaged over energy spectrum)

$$\begin{aligned}\overline{P}_{\nu_e\to\nu_e}(x) = |\langle\psi_e(x)\rangle|^2 = {} & \cos^2\vartheta\,\cos^2\vartheta^0_{\mathrm{M}}\,|\mathcal{A}^{\mathrm{R}}_{11}|^2 + \cos^2\vartheta\,\sin^2\vartheta^0_{\mathrm{M}}\,|\mathcal{A}^{\mathrm{R}}_{21}|^2 \\ & + \sin^2\vartheta\,\cos^2\vartheta^0_{\mathrm{M}}\,|\mathcal{A}^{\mathrm{R}}_{12}|^2 + \sin^2\vartheta\,\sin^2\vartheta^0_{\mathrm{M}}\,|\mathcal{A}^{\mathrm{R}}_{22}|^2\end{aligned}$$

conservation of probability (unitarity)

$$|\mathcal{A}^{\mathrm{R}}_{12}|^2 = |\mathcal{A}^{\mathrm{R}}_{21}|^2 = P_{\mathrm{c}} \qquad |\mathcal{A}^{\mathrm{R}}_{11}|^2 = |\mathcal{A}^{\mathrm{R}}_{22}|^2 = 1 - P_{\mathrm{c}}$$

$P_{\mathrm{c}} \equiv$ crossing probability

$$\boxed{\overline{P}_{\nu_e\to\nu_e}(x) = \frac{1}{2} + \left(\frac{1}{2} - P_{\mathrm{c}}\right)\cos2\vartheta^0_{\mathrm{M}}\,\cos2\vartheta}$$

[Parke, PRL 57 (1986) 1275]

Crossing Probability

$$P_{\mathrm{c}} = \frac{\exp\left(-\frac{\pi}{2}\gamma F\right) - \exp\left(-\frac{\pi}{2}\gamma \frac{F}{\sin^2\vartheta}\right)}{1 - \exp\left(-\frac{\pi}{2}\gamma \frac{F}{\sin^2\vartheta}\right)}$$

[Kuo, Pantaleone, PRD 39 (1989) 1930]

adiabaticity parameter: $$\gamma = \left.\frac{\Delta m^2_{\mathrm{M}}/2E}{2\left|\mathrm{d}\vartheta_{\mathrm{M}}/\mathrm{d}x\right|}\right|_{\mathrm{R}} = \frac{\Delta m^2 \sin^2 2\vartheta}{2E\cos 2\vartheta \left|\frac{\mathrm{d}\ln A_{\mathrm{CC}}}{\mathrm{d}x}\right|_{\mathrm{R}}}$$

$A \propto x$ $F = 1$ (Landau-Zener approximation) [Parke, PRL 57 (1986) 1275]

$A \propto 1/x$ $F = \left(1 - \tan^2\vartheta\right)^2 / \left(1 + \tan^2\vartheta\right)$ [Kuo, Pantaleone, PRD 39 (1989) 1930]

$A \propto \exp(-x)$ $F = 1 - \tan^2\vartheta$

[Pizzochero, PRD 36 (1987) 2293]

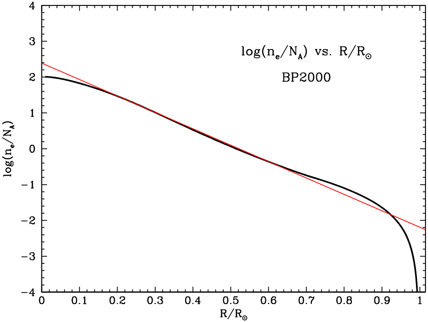
[Toshev, PLB 196 (1987) 170]

[Petcov, PLB 200 (1988) 373]

Review: [Kuo, Pantaleone, RMP 61 (1989) 937]

Solar Neutrinos

SUN: $N_e(x) \simeq N_e^{\rm c} \exp\left(-\frac{x}{x_0}\right)$ $N_e^{\rm c} = 245\, N_{\rm A}/{\rm cm}^3$ $x_0 = \frac{R_\odot}{10.54}$



$$\overline{P}^{\rm sun}_{\nu_e\to\nu_e} = \frac{1}{2} + \left(\frac{1}{2} - P_{\rm c}\right) \cos 2\vartheta^0_{\rm M}\, \cos 2\vartheta$$

$$P_{\rm c} = \frac{\exp\left(-\frac{\pi}{2}\gamma F\right) - \exp\left(-\frac{\pi}{2}\gamma\frac{F}{\sin^2\vartheta}\right)}{1 - \exp\left(-\frac{\pi}{2}\gamma\frac{F}{\sin^2\vartheta}\right)}$$

$$\gamma = \frac{\Delta m^2 \sin^2 2\vartheta}{2E\cos 2\vartheta \left|\frac{{\rm d}\ln A_{\rm CC}}{{\rm d}x}\right|_{\rm R}}$$

$$F = 1 - \tan^2\vartheta$$

$$A_{\rm CC} = 2\sqrt{2} E G_{\rm F} N_e$$

practical prescription:

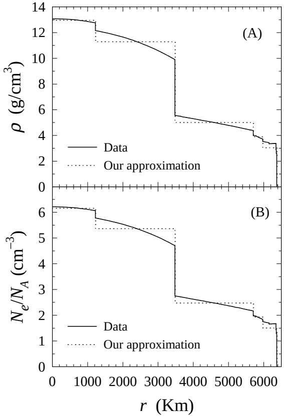
[Lisi et al., PRD 63 (2001) 093002]

$$\begin{cases} \text{numerical } |{\rm d}\ln A_{\rm CC}/{\rm d}x|_{\rm R} & \text{for } x \leq 0.904 R_\odot \\ |{\rm d}\ln A_{\rm CC}/{\rm d}x|_{\rm R} \to \dfrac{18.9}{R_\odot} & \text{for } x > 0.904 R_\odot \end{cases}$$

Electron Neutrino Regeneration in the Earth

$$P^{\text{sun+earth}}_{\nu_e\to\nu_e} = \overline{P}^{\text{sun}}_{\nu_e\to\nu_e} + \frac{\left(1 - 2\overline{P}^{\text{sun}}_{\nu_e\to\nu_e}\right)\left(P^{\text{earth}}_{\nu_2\to\nu_e} - \sin^2\vartheta\right)}{\cos 2\vartheta}$$

[Mikheev, Smirnov, Sov. Phys. Usp. 30 (1987) 759], [Baltz, Weneser, PRD 35 (1987) 528]



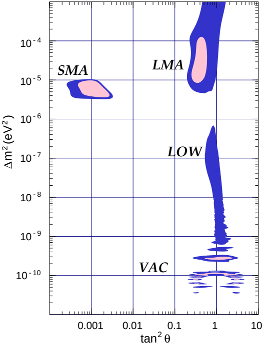
$P^{\text{earth}}_{\nu_2\to\nu_e}$ is usually calculated numerically approximating the Earth density profile with a step function.

Effective massive neutrinos propagate as plane waves in regions of constant density.

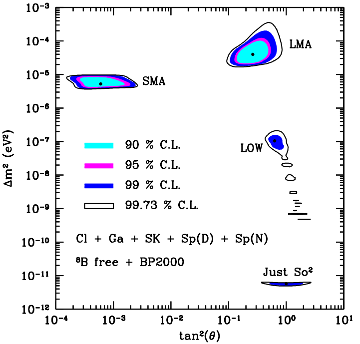
Wave functions of flavor neutrinos are joined at the boundaries of steps.

Solar Neutrino Oscillations

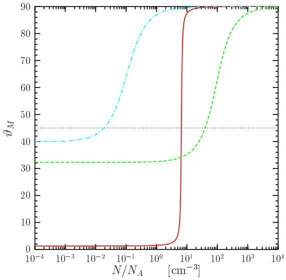
LMA (Large Mixing Angle):	$\Delta m^2 \sim 5 \times 10^{-5}\,\mathrm{eV}^2$,	$\tan^2\vartheta \sim 0.8$
LOW (LOW Δm^2):	$\Delta m^2 \sim 7 \times 10^{-8}\,\mathrm{eV}^2$,	$\tan^2\vartheta \sim 0.6$
SMA (Small Mixing Angle):	$\Delta m^2 \sim 5 \times 10^{-6}\,\mathrm{eV}^2$,	$\tan^2\vartheta \sim 10^{-3}$
QVO (Quasi-Vacuum Oscillations):	$\Delta m^2 \sim 10^{-9}\,\mathrm{eV}^2$,	$\tan^2\vartheta \sim 1$
VAC (VACuum oscillations):	$\Delta m^2 \lesssim 5 \times 10^{-10}\,\mathrm{eV}^2$,	$\tan^2\vartheta \sim 1$



[de Gouvea, Friedland, Murayama, PLB 490 (2000) 125]



[Bahcall, Krastev, Smirnov, JHEP 05 (2001) 015]



solid line: $\Delta m^2 = 5 \times 10^{-6}\,\mathrm{eV}^2$

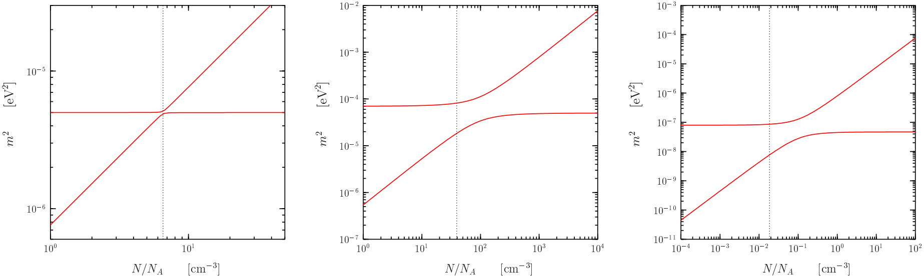
(typical SMA) $\tan^2\vartheta = 5 \times 10^{-4}$

dashed line: $\Delta m^2 = 7 \times 10^{-5}\,\mathrm{eV}^2$

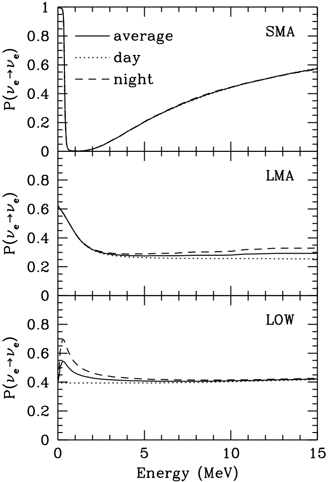
(typical LMA) $\tan^2\vartheta = 0.4$

dash-dotted line: $\Delta m^2 = 8 \times 10^{-8}\,\mathrm{eV}^2$

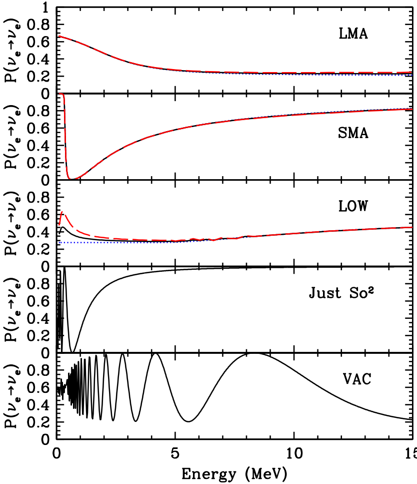
(typical LOW) $\tan^2\vartheta = 0.7$



typical SMA typical LMA typical LOW



[Bahcall, Krastev, Smirnov, PRD 58 (1998) 096016]



[Bahcall, Krastev, Smirnov, JHEP 05 (2001) 015]

SMA: $\Delta m^2 = 5.0 \times 10^{-6}\,\text{eV}^2$ $\sin^2 2\vartheta = 3.5 \times 10^{-3}$

LMA: $\Delta m^2 = 1.6 \times 10^{-5}\,\text{eV}^2$ $\sin^2 2\vartheta = 0.57$

LOW: $\Delta m^2 = 7.9 \times 10^{-8}\,\text{eV}^2$ $\sin^2 2\vartheta = 0.95$

LMA: $\Delta m^2 = 4.2 \times 10^{-5}\,\text{eV}^2$ $\tan^2 \vartheta = 0.26$

SMA: $\Delta m^2 = 5.2 \times 10^{-6}\,\text{eV}^2$ $\tan^2 \vartheta = 5.5 \times 10^{-4}$

LOW: $\Delta m^2 = 7.6 \times 10^{-8}\,\text{eV}^2$ $\tan^2 \vartheta = 0.72$

Just So^2: $\Delta m^2 = 5.5 \times 10^{-12}\,\text{eV}^2$ $\tan^2 \vartheta = 1.0$

VAC: $\Delta m^2 = 1.4 \times 10^{-10}\,\text{eV}^2$ $\tan^2 \vartheta = 0.38$

SNO: Sudbury Neutrino Observatory

water Cherenkov detector, Sudbury, Ontario, Canada

1 kton of D_2O, 9456 20-cm PMTs

2073 m underground, 6010 m.w.e.

CC: $\nu_e + d \to p + p + e^-$
NC: $\nu + d \to p + n + \nu$
ES: $\nu + e^- \to \nu + e^-$

$$\left.\begin{array}{l} \text{CC threshold: } E_{\text{th}}^{\text{SNO}}(\text{CC}) \simeq 8.2\,\text{MeV} \\ \text{NC threshold: } E_{\text{th}}^{\text{SNO}}(\text{NC}) \simeq 2.2\,\text{MeV} \\ \text{ES threshold: } E_{\text{th}}^{\text{SNO}}(\text{ES}) \simeq 7.0\,\text{MeV} \end{array}\right\} \Longrightarrow {}^{8}\text{B}, \textit{hep}$$

D_2O phase: 1999 – 2001

$$\frac{R_{\text{CC}}^{\text{SNO}}}{R_{\text{CC}}^{\text{SSM}}} = 0.35 \pm 0.02$$
$$\frac{R_{\text{NC}}^{\text{SNO}}}{R_{\text{NC}}^{\text{SSM}}} = 1.01 \pm 0.13$$
$$\frac{R_{\text{ES}}^{\text{SNO}}}{R_{\text{ES}}^{\text{SSM}}} = 0.47 \pm 0.05$$

[PRL 89 (2002) 011301]

NaCl phase: 2001 – 2002

$$\frac{R_{\text{CC}}^{\text{SNO}}}{R_{\text{CC}}^{\text{SSM}}} = 0.31 \pm 0.02$$
$$\frac{R_{\text{NC}}^{\text{SNO}}}{R_{\text{NC}}^{\text{SSM}}} = 1.03 \pm 0.09$$
$$\frac{R_{\text{ES}}^{\text{SNO}}}{R_{\text{ES}}^{\text{SSM}}} = 0.44 \pm 0.06$$

[nucl-ex/0309004]

$$\Phi^{\text{SNO}}_{\nu_e} = 1.76 \pm 0.11 \times 10^6\,\text{cm}^{-2}\,\text{s}^{-1}$$

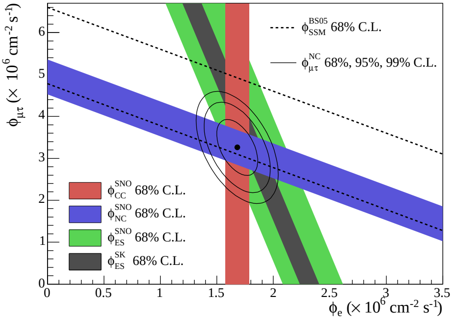
$$\Phi^{\text{SNO}}_{\nu_\mu,\nu_\tau} = 5.41 \pm 0.66 \times 10^6\,\text{cm}^{-2}\,\text{s}^{-1}$$

SNO solved
solar neutrino problem

⇓

Neutrino Physics
(April 2002)

[SNO, PRL 89 (2002) 011301, nucl-ex/0204008]



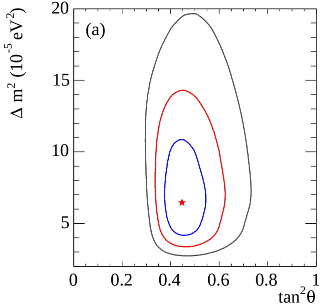
$\nu_e \to \nu_\mu, \nu_\tau$ oscillations

⇓

Large Mixing Angle solution

$$\Delta m^2 \simeq 7 \times 10^{-5}\,\text{eV}^2$$

$$\tan^2\vartheta \simeq 0.45$$



[SNO, PRC 72 (2005) 055502, nucl-ex/0502021]

KamLAND

Kamioka Liquid scintillator Anti-Neutrino Detector

long-baseline reactor $\bar{\nu}_e$ experiment

Kamioka mine (200 km west of Tokyo), 1000 m underground, 2700 m.w.e.

53 nuclear power reactors in Japan and Korea

average distance from reactors: 180 km
- 6.7% of flux from one reactor at 88 km
- 79% of flux from 26 reactors at 138–214 km
- 14.3% of flux from other reactors at $>$295 km

1 kt liquid scintillator detector: $\bar{\nu}_e + p \to e^+ + n$, energy threshold: $E_{\text{th}}^{\bar{\nu}_e p} = 1.8\,\text{MeV}$

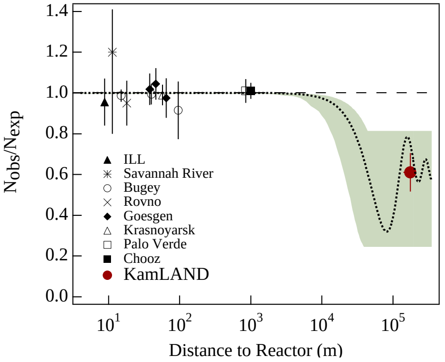
data taking: 4 March – 6 October 2002, 145.1 days (162 ton yr)

expected number of reactor neutrino events (no osc.):	$N_{\text{expected}}^{\text{KamLAND}} = 86.8 \pm 5.6$
expected number of background events:	$N_{\text{background}}^{\text{KamLAND}} = 0.95 \pm 0.99$
observed number of neutrino events:	$N_{\text{observed}}^{\text{KamLAND}} = 54$

$$\frac{N_{\text{observed}}^{\text{KamLAND}} - N_{\text{background}}^{\text{KamLAND}}}{N_{\text{expected}}^{\text{KamLAND}}} = 0.611 \pm 0.085 \pm 0.041$$

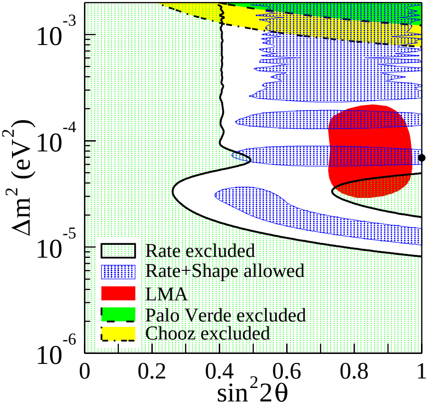
99.95% C.L. evidence
of $\bar{\nu}_e$ disappearance

confirmation of LMA (December 2002)



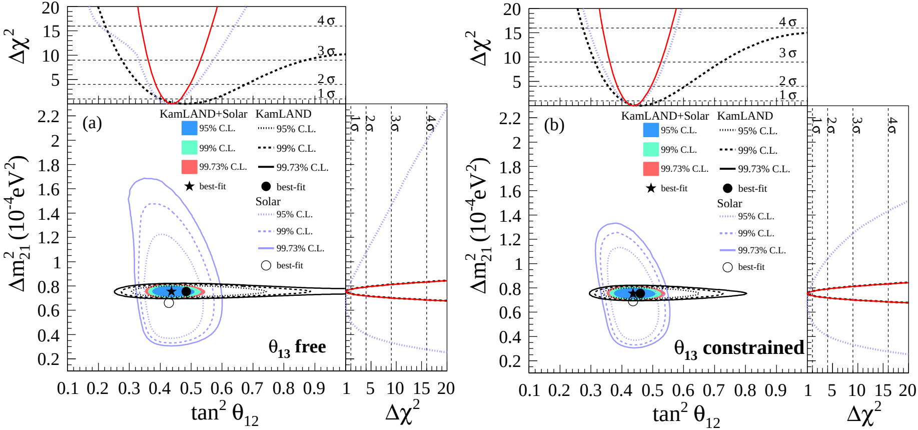
Shade: 95% C.L. LMA

Curve: $\begin{cases} \Delta m^2 = 5.5 \times 10^{-5}\,\text{eV}^2 \\ \sin^2 2\vartheta = 0.83 \end{cases}$



95% C.L.

[KamLAND, PRL 90 (2003) 021802, hep-ex/0212021]



$$\Delta m^2_{21} = 7.53^{+0.19}_{-0.18} \times 10^{-5}\,\text{eV}^2$$

$$\Delta m^2_{21} = 7.53 \pm 0.18 \times 10^{-5}\,\text{eV}^2$$

$$\tan^2\vartheta_{12} = 0.437^{+0.029}_{-0.026}$$

$$\tan^2\vartheta_{12} = 0.436^{+0.029}_{-0.025}$$

$$\sin^2\vartheta_{13} = 0.023 \pm 0.015$$

$$\sin^2\vartheta_{13} = 0.023 \pm 0.002$$

[KamLAND, arXiv:1303.4667]

LMA Solar Neutrino Oscillations

best fit of reactor + solar neutrino data: $\Delta m^2 \simeq 7 \times 10^{-5}\,\text{eV}^2$ $\tan^2\vartheta \simeq 0.4$

$$\boxed{\overline{P}^{\text{sun}}_{\nu_e\to\nu_e} = \frac{1}{2} + \left(\frac{1}{2} - P_{\text{c}}\right)\cos 2\vartheta^0_{\text{M}}\,\cos 2\vartheta}$$

$$P_{\text{c}} = \frac{\exp\left(-\frac{\pi}{2}\gamma F\right) - \exp\left(-\frac{\pi}{2}\gamma\frac{F}{\sin^2\vartheta}\right)}{1 - \exp\left(-\frac{\pi}{2}\gamma\frac{F}{\sin^2\vartheta}\right)} \qquad \gamma = \frac{\Delta m^2 \sin^2 2\vartheta}{2E\cos 2\vartheta\left|\frac{\mathrm{d}\ln A}{\mathrm{d}x}\right|_{\text{R}}} \qquad F = 1 - \tan^2\vartheta$$

$$A_{\text{CC}} \simeq 2\sqrt{2}EG_{\text{F}}N_e^{\text{c}}\exp\left(-\frac{x}{x_0}\right) \quad\Longrightarrow\quad \left|\frac{\mathrm{d}\ln A}{\mathrm{d}x}\right| \simeq \frac{1}{x_0} = \frac{10.54}{R_\odot} \simeq 3\times 10^{-15}\,\text{eV}$$

$$\tan^2\vartheta \simeq 0.4 \quad\Longrightarrow\quad \sin^2 2\vartheta \simeq 0.82\,,\ \cos 2\vartheta \simeq 0.43 \qquad\qquad \gamma \simeq 2\times 10^4\left(\frac{E}{\text{MeV}}\right)^{-1}$$

$$\gamma \gg 1 \quad\Longrightarrow\quad P_{\text{c}} \ll 1 \quad\Longrightarrow\quad \boxed{\overline{P}^{\text{sun,LMA}}_{\nu_e\to\nu_e} \simeq \frac{1}{2} + \frac{1}{2}\cos 2\vartheta^0_{\text{M}}\,\cos 2\vartheta}$$

$$\boxed{\cos 2\vartheta_{\mathrm{M}}^{0} = \frac{\Delta m^2 \cos 2\vartheta - A_{\mathrm{CC}}^{0}}{\sqrt{\left(\Delta m^2 \cos 2\vartheta - A_{\mathrm{CC}}^{0}\right)^2 + \left(\Delta m^2 \sin 2\vartheta\right)^2}}}$$

critical parameter [Bahcall, Peña-Garay, hep-ph/0305159]

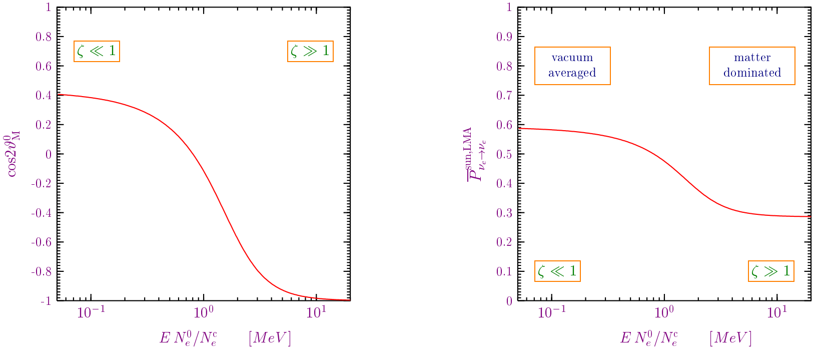
$$\zeta = \frac{A_{\mathrm{CC}}^{0}}{\Delta m^2 \cos 2\vartheta} = \frac{2\sqrt{2} E G_{\mathrm{F}} N_e^0}{\Delta m^2 \cos 2\vartheta} \simeq 1.2 \left(\frac{E}{\mathrm{MeV}}\right) \left(\frac{N_e^0}{N_e^{\mathrm{c}}}\right)$$

$$\zeta \ll 1 \Longrightarrow \vartheta_{\mathrm{M}}^{0} \simeq \vartheta \quad \Longrightarrow \overline{P}_{\nu_e\to\nu_e}^{\mathrm{sun}} \simeq 1 - \tfrac{1}{2} \sin^2 2\vartheta$$

vacuum averaged survival probability

$$\zeta \gg 1 \Longrightarrow \vartheta_{\mathrm{M}}^{0} \simeq \pi/2 \Longrightarrow \overline{P}_{\nu_e\to\nu_e}^{\mathrm{sun}} \simeq \sin^2\vartheta$$

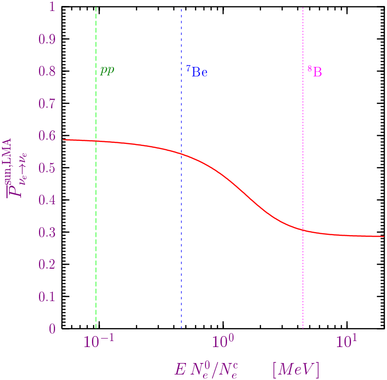
matter dominated survival probability



$$\langle E \rangle_{pp} \simeq 0.27\,\text{MeV}\,,\ \langle r_0 \rangle_{pp} \simeq 0.1\,R_\odot \quad \Longrightarrow \quad \langle E\,N_e^0/N_e^c \rangle_{pp} \simeq 0.094\,\text{MeV}$$

$$E_{^7\text{Be}} \simeq 0.86\,\text{MeV}\,,\ \langle r_0 \rangle_{^7\text{Be}} \simeq 0.06\,R_\odot \quad \Longrightarrow \quad \langle E\,N_e^0/N_e^c \rangle_{^7\text{Be}} \simeq 0.46\,\text{MeV}$$

$$\langle E \rangle_{^8\text{B}} \simeq 6.7\,\text{MeV}\,,\ \langle r_0 \rangle_{^8\text{B}} \simeq 0.04\,R_\odot \quad \Longrightarrow \quad \langle E\,N_e^0/N_e^c \rangle_{^8\text{B}} \simeq 4.4\,\text{MeV}$$



each neutrino experiment is mainly sensitive to one flux

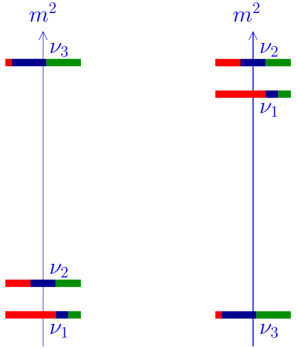
each neutrino experiment is mainly sensitive to ϑ

accurate *pp* experiment can improve determination of ϑ

[Bahcall, Peña-Garay, hep-ph/0305159]

Mass Hierarchy

1. Matter Effect (Atmospheric, Long-Baseline, Supernova Experiments):
   - $\nu_e \leftrightarrows \nu_\mu$ MSW resonance: $V = \frac{\Delta m^2_{13} \cos 2\vartheta_{13}}{2E} \Leftrightarrow \Delta m^2_{13} > 0$ NH
   - $\bar{\nu}_e \leftrightarrows \bar{\nu}_\mu$ MSW resonance: $V = -\frac{\Delta m^2_{13} \cos 2\vartheta_{13}}{2E} \Leftrightarrow \Delta m^2_{13} < 0$ IH
2. Phase Difference (Reactor $\bar{\nu}_e \to \bar{\nu}_e$):



Normal Hierarchy

$$|\Delta m^2_{31}| = |\Delta m^2_{32}| + |\Delta m^2_{21}|$$

$$|\Delta m^2_{31}| > |\Delta m^2_{32}|$$

Inverted Hierarchy

$$|\Delta m^2_{31}| = |\Delta m^2_{32}| - |\Delta m^2_{21}|$$

$$|\Delta m^2_{31}| < |\Delta m^2_{32}|$$

In Neutrino Oscillations Dirac = Majorana

[Bilenky, Hosek, Petcov, PLB 94 (1980) 495; Doi, Kotani, Nishiura, Okuda, Takasugi, PLB 102 (1981) 323]

[Langacker, Petcov, Steigman, Toshev, NPB 282 (1987) 589]

Evolution of Amplitudes: $i \frac{\mathrm{d}\psi_\alpha}{\mathrm{d}x} = \frac{1}{2E} \sum_\beta \left(U M^2 U^\dagger + 2 E V \right)_{\alpha\beta} \psi_\beta$

difference: $\begin{cases} \text{Dirac:} & U^{(\mathrm{D})} \\ \text{Majorana:} & U^{(\mathrm{M})} = U^{(\mathrm{D})} D(\lambda) \end{cases}$

$$D(\lambda) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\lambda_{21}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\lambda_{N1}} \end{pmatrix} \quad \Rightarrow \quad D^\dagger = D^{-1}$$

$$M^2 = \begin{pmatrix} m_1^2 & 0 & \cdots & 0 \\ 0 & m_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_N^2 \end{pmatrix} \quad \Longrightarrow \quad D M^2 = M^2 D \quad \Longrightarrow \quad D M^2 D^\dagger = M^2$$

$$U^{(\mathrm{M})} M^2 (U^{(\mathrm{M})})^\dagger = U^{(\mathrm{D})} D M^2 D^\dagger (U^{(\mathrm{D})})^\dagger = U^{(\mathrm{D})} M^2 (U^{(\mathrm{D})})^\dagger$$