

Neutrino Physics

Part I: Theory of Neutrino Masses and Mixing

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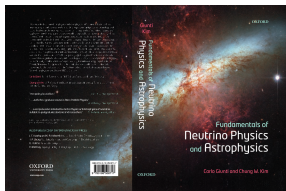
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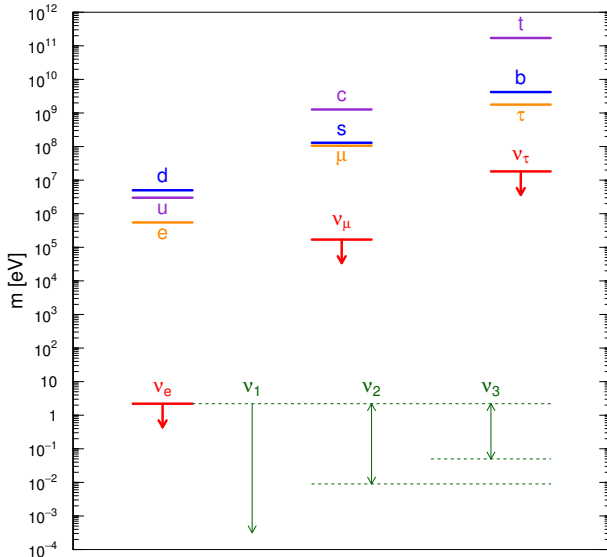


C. Giunti and C.W. Kim
Fundamentals of Neutrino Physics and
Astrophysics
Oxford University Press
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Part I: Theory of Neutrino Masses and Mixing

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
- Sterile Neutrinos

Fermion Mass Spectrum



Dirac Neutrino Masses and Mixing

- Dirac Neutrino Masses and Mixing
 - Higgs Mechanism in SM
 - SM Extension: Dirac Neutrino Masses
 - Three-Generations Dirac Neutrino Masses
 - Mixing
 - CP Violation
 - Jarlskog Invariant
 - Lepton Numbers Violating Processes
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
- Sterile Neutrinos

Dirac Mass

▶ Dirac Equation: $(i\partial - m)\nu(x) = 0$ ($\partial \equiv \gamma^\mu \partial_\mu$)

▶ Dirac Lagrangian: $\mathcal{L}_D(x) = \bar{\nu}(x)(i\partial - m)\nu(x)$

▶ Chiral decomposition: $\nu_L \equiv P_L \nu$, $\nu_R \equiv P_R \nu$, $\nu = \nu_L + \nu_R$

Left and Right-handed Projectors: $P_L \equiv \frac{1 - \gamma^5}{2}$, $P_R \equiv \frac{1 + \gamma^5}{2}$

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L + P_R = 1, \quad P_L P_R = P_R P_L = 0$$

$$\mathcal{L} = \bar{\nu}_L i\partial \nu_L + \bar{\nu}_R i\partial \nu_R - m(\bar{\nu}_L \nu_R + \bar{\nu}_R \nu_L)$$

▶ In SM only ν_L by assumption \implies no neutrino mass

Note that all the other elementary fermion fields (charged leptons and quarks) have both left and right-handed components

▶ Oscillation experiments have shown that **neutrinos are massive**

▶ Simplest and natural extension of the SM: consider also ν_R as for all the other elementary fermion fields

Higgs Mechanism in SM

▶ Higgs Doublet: $\Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}$ $|\Phi|^2 = \Phi^\dagger \Phi = \phi_+^\dagger \phi_+ + \phi_0^\dagger \phi_0$

▶ Higgs Lagrangian: $\mathcal{L}_{\text{Higgs}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(|\Phi|^2)$

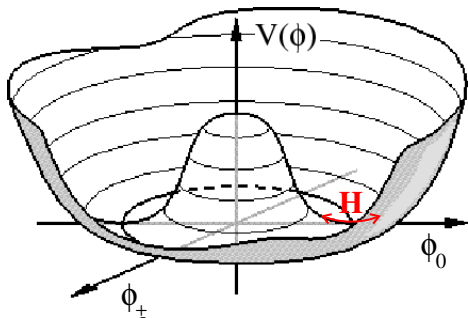
▶ Higgs Potential: $V(|\Phi|^2) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4$

▶ $\mu^2 < 0$ and $\lambda > 0 \implies V(|\Phi|^2) = \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2$

$$v \equiv \sqrt{-\frac{\mu^2}{\lambda}} = \left(\sqrt{2} G_F \right)^{-1/2} \simeq 246 \text{ GeV}$$

▶ Vacuum: V_{\min} for $|\Phi|^2 = \frac{v^2}{2} \implies \langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$

▶ Spontaneous Symmetry Breaking: $SU(2)_L \times U(1)_Y \rightarrow U(1)_Q$



► Unitary Gauge: $\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \Rightarrow |\Phi|^2 = \frac{v^2}{2} + vH + \frac{1}{2} H^2$

► $V = \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2 = \lambda v^2 H^2 + \lambda v H^3 + \frac{\lambda}{4} H^4$

$$m_H = \sqrt{2\lambda v^2} = \sqrt{-2\mu^2} \simeq 126 \text{ GeV}$$

$$-\mu^2 \simeq (89 \text{ GeV})^2 \quad \lambda = -\frac{\mu^2}{v^2} \simeq 0.13$$

SM Extension: Dirac Neutrino Masses

$$L_L \equiv \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} \quad \ell_R \quad \nu_R$$

Lepton-Higgs Yukawa Lagrangian

$$\mathcal{L}_{H,L} = -y^\ell \bar{L}_L \Phi \ell_R - y^\nu \bar{L}_L \tilde{\Phi} \nu_R + \text{H.c.}$$

Spontaneous Symmetry Breaking

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \quad \tilde{\Phi} = i\sigma_2 \Phi^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mathcal{L}_{H,L} = & -\frac{y^\ell}{\sqrt{2}} (\bar{\nu}_L \quad \bar{\ell}_L) \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \ell_R \\ & -\frac{y^\nu}{\sqrt{2}} (\bar{\nu}_L \quad \bar{\ell}_L) \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix} \nu_R + \text{H.c.} \end{aligned}$$

$$\mathcal{L}_{H,L} = -y^\ell \frac{v}{\sqrt{2}} \bar{\ell}_L \ell_R - y^\nu \frac{v}{\sqrt{2}} \bar{\nu}_L \nu_R$$

$$- \frac{y^\ell}{\sqrt{2}} \bar{\ell}_L \ell_R H - \frac{y^\nu}{\sqrt{2}} \bar{\nu}_L \nu_R H + \text{H.c.}$$

$$m_\ell = y^\ell \frac{v}{\sqrt{2}}$$

$$m_\nu = y^\nu \frac{v}{\sqrt{2}}$$

$$g_{\ell H} = \frac{y^\ell}{\sqrt{2}} = \frac{m_\ell}{v}$$

$$g_{\nu H} = \frac{y^\nu}{\sqrt{2}} = \frac{m_\nu}{v}$$

$$v = \left(\sqrt{2} G_F \right)^{-1/2} = 246 \text{ GeV}$$

PROBLEM: $y^\nu \lesssim 10^{-11} \ll y^e \sim 10^{-6}$

Three-Generations Dirac Neutrino Masses

$L'_{eL} \equiv \begin{pmatrix} \nu'_{eL} \\ \ell'_{eL} \equiv e'_L \end{pmatrix}$	$L'_{\mu L} \equiv \begin{pmatrix} \nu'_{\mu L} \\ \ell'_{\mu L} \equiv \mu'_L \end{pmatrix}$	$L'_{\tau L} \equiv \begin{pmatrix} \nu'_{\tau L} \\ \ell'_{\tau L} \equiv \tau'_L \end{pmatrix}$
$\ell'_{eR} \equiv e'_R$	$\ell'_{\mu R} \equiv \mu'_R$	$\ell'_{\tau R} \equiv \tau'_R$
ν'_{eR}	$\nu'_{\mu R}$	$\nu'_{\tau R}$

Lepton-Higgs Yukawa Lagrangian

$$\mathcal{L}_{H,L} = - \sum_{\alpha,\beta=e,\mu,\tau} \left[Y_{\alpha\beta}^{\prime\ell} \overline{L'_{\alpha L}} \Phi \ell'_{\beta R} + Y_{\alpha\beta}^{\prime\nu} \overline{L'_{\alpha L}} \tilde{\Phi} \nu'_{\beta R} \right] + \text{H.c.}$$

Spontaneous Symmetry Breaking

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \quad \tilde{\Phi} = i\sigma_2 \Phi^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

$$\mathcal{L}_{H,L} = - \left(\frac{v+H}{\sqrt{2}} \right) \sum_{\alpha,\beta=e,\mu,\tau} \left[Y_{\alpha\beta}^{l\ell} \overline{\ell'_{\alpha L}} \ell'_{\beta R} + Y_{\alpha\beta}^{l\nu} \overline{\nu'_{\alpha L}} \nu'_{\beta R} \right] + \text{H.c.}$$

$$\mathcal{L}_{H,L} = - \left(\frac{v+H}{\sqrt{2}} \right) \left[\overline{\ell'_L} Y^{l\ell} \ell'_R + \overline{\nu'_L} Y^{l\nu} \nu'_R \right] + \text{H.c.}$$

$$\ell'_L \equiv \begin{pmatrix} e'_L \\ \mu'_L \\ \tau'_L \end{pmatrix} \quad \ell'_R \equiv \begin{pmatrix} e'_R \\ \mu'_R \\ \tau'_R \end{pmatrix} \quad \nu'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \quad \nu'_R \equiv \begin{pmatrix} \nu'_{eR} \\ \nu'_{\mu R} \\ \nu'_{\tau R} \end{pmatrix}$$

$$Y^{l\ell} \equiv \begin{pmatrix} Y_{ee}^{l\ell} & Y_{e\mu}^{l\ell} & Y_{e\tau}^{l\ell} \\ Y_{\mu e}^{l\ell} & Y_{\mu\mu}^{l\ell} & Y_{\mu\tau}^{l\ell} \\ Y_{\tau e}^{l\ell} & Y_{\tau\mu}^{l\ell} & Y_{\tau\tau}^{l\ell} \end{pmatrix}$$

$$Y^{l\nu} \equiv \begin{pmatrix} Y_{ee}^{l\nu} & Y_{e\mu}^{l\nu} & Y_{e\tau}^{l\nu} \\ Y_{\mu e}^{l\nu} & Y_{\mu\mu}^{l\nu} & Y_{\mu\tau}^{l\nu} \\ Y_{\tau e}^{l\nu} & Y_{\tau\mu}^{l\nu} & Y_{\tau\tau}^{l\nu} \end{pmatrix}$$

$$M^{l\ell} = \frac{v}{\sqrt{2}} Y^{l\ell}$$

$$M^{l\nu} = \frac{v}{\sqrt{2}} Y^{l\nu}$$

$$\mathcal{L}_{H,L} = - \left(\frac{v+H}{\sqrt{2}} \right) \left[\overline{\ell}'_L Y^{\ell\ell} \ell'_R + \overline{\nu}'_L Y^{\nu\nu} \nu'_R \right] + \text{H.c.}$$

Diagonalization of $Y^{\ell\ell}$ and $Y^{\nu\nu}$ with **unitary** V_L^ℓ , V_R^ℓ , V_L^ν , V_R^ν

$$\ell'_L = V_L^\ell \ell_L \quad \ell'_R = V_R^\ell \ell_R \quad \nu'_L = V_L^\nu \nu_L \quad \nu'_R = V_R^\nu \nu_R$$

Important general remark: unitary transformations are allowed because they leave invariant the kinetic terms in the Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= \overline{\ell}'_L i \not{\partial} \ell'_L + \overline{\ell}'_R i \not{\partial} \ell'_R + \overline{\nu}'_L i \not{\partial} \nu'_L + \overline{\nu}'_R i \not{\partial} \nu'_R \\ &= \overline{\ell}_L V_L^{\ell\dagger} i \not{\partial} V_L^\ell \ell_L + \dots \\ &= \overline{\ell}_L i \not{\partial} \ell_L + \overline{\ell}_R i \not{\partial} \ell_R + \overline{\nu}_L i \not{\partial} \nu_L + \overline{\nu}_R i \not{\partial} \nu_R \end{aligned}$$

$$\mathcal{L}_{H,L} = - \left(\frac{\nu + H}{\sqrt{2}} \right) \left[\overline{\ell}'_L Y^{\ell e} \ell'_R + \overline{\nu}'_L Y^{\nu \nu} \nu'_R \right] + \text{H.c.}$$

Diagonalization of $Y^{\ell e}$ and $Y^{\nu \nu}$ with unitary $V_L^\ell, V_R^\ell, V_L^\nu, V_R^\nu$

$$\ell'_L = V_L^\ell \ell_L \quad \ell'_R = V_R^\ell \ell_R \quad \nu'_L = V_L^\nu n_L \quad \nu'_R = V_R^\nu n_R$$

$$\mathcal{L}_{H,L} = - \left(\frac{\nu + H}{\sqrt{2}} \right) \left[\overline{\ell}_L V_L^{\ell \dagger} Y^{\ell e} V_R^\ell \ell_R + \overline{n}_L V_L^{\nu \dagger} Y^{\nu \nu} V_R^\nu n_R \right] + \text{H.c.}$$

$$V_L^{\ell \dagger} Y^{\ell e} V_R^\ell = Y^\ell \quad Y_{\alpha\beta}^\ell = y_\alpha^\ell \delta_{\alpha\beta} \quad (\alpha, \beta = e, \mu, \tau)$$

$$V_L^{\nu \dagger} Y^{\nu \nu} V_R^\nu = Y^\nu \quad Y_{kj}^\nu = y_k^\nu \delta_{kj} \quad (k, j = 1, 2, 3)$$

Real and Positive y_α^ℓ, y_k^ν

$$V_L^{\dagger} Y' V_R = Y$$

$$\begin{matrix} 9 & 18 & 9 & 3 \end{matrix}$$

▶ Consider the Hermitian matrix $Y'Y'^{\dagger}$

▶ It has real eigenvalues and orthonormal eigenvectors:

$$Y'Y'^{\dagger}v_k = \lambda_k v_k \quad \Leftrightarrow \quad \sum_{\beta} (Y'Y'^{\dagger})_{\alpha\beta} (v_k)_{\beta} = \lambda_k (v_k)_{\alpha}$$

▶ Unitary diagonalizing matrix: $(V_L)_{\beta k} = (v_k)_{\beta}$

$$Y'Y'^{\dagger}V_L = \Lambda V_L \quad \Rightarrow \quad V_L^{\dagger}Y'Y'^{\dagger}V_L = \Lambda \quad \text{with} \quad \Lambda_{kj} = \lambda_k \delta_{kj}$$

▶ The real eigenvalues λ_k are positive:

$$\begin{aligned} \lambda_k &= \sum_{\alpha} (V_L^{\dagger}Y')_{k\alpha} (Y'^{\dagger}V_L)_{\alpha k} = \sum_{\alpha} (V_L^{\dagger}Y')_{k\alpha} (V_L^{\dagger}Y')_{\alpha k}^{\dagger} \\ &= \sum_{\alpha} (V_L^{\dagger}Y')_{k\alpha} (V_L^{\dagger}Y')_{k\alpha}^* = \sum_{\alpha} |(V_L^{\dagger}Y')_{k\alpha}|^2 \geq 0 \end{aligned}$$

▶ Then, we can write $V_L^{\dagger}Y'Y'^{\dagger}V_L = Y^2$ with $(Y)_{kj} = y_k \delta_{kj}$

real and positive $y_k = \sqrt{\lambda_k}$

▶ Let us write Y' as $Y' = V_L Y V_R^\dagger$

▶ This is the diagonalizing equation if V_R is unitary.

$$V_R^\dagger = Y^{-1} V_L^\dagger Y' \quad V_R = Y'^\dagger V_L Y^{-1} \quad \text{with} \quad Y^\dagger = Y$$

▶ $V_R^\dagger V_R = Y^{-1} V_L^\dagger Y' Y'^\dagger V_L Y^{-1} = Y^{-1} Y^2 Y^{-1} = \mathbb{1}$

▶ $V_R V_R^\dagger = Y'^\dagger V_L Y^{-1} Y^{-1} V_L^\dagger Y' = Y'^\dagger V_L Y^{-2} V_L^\dagger Y'$
 $Y^{-2} = V_L^\dagger (Y'^\dagger)^{-1} (Y')^{-1} V_L$

$$V_R V_R^\dagger = Y'^\dagger V_L V_L^\dagger (Y'^\dagger)^{-1} (Y')^{-1} V_L V_L^\dagger Y' = Y'^\dagger (Y'^\dagger)^{-1} (Y')^{-1} Y' = \mathbb{1}$$

▶ In conclusion: $V_L^\dagger Y' V_R = Y$ with unitary V_L and V_R

$$(Y)_{kj} = y_k \delta_{kj} \quad \text{with real and positive } y_k$$

Massive Chiral Lepton Fields

$V_L^{\ell\dagger} \ell'_L = \ell_L \equiv \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}$	$V_R^{\ell\dagger} \ell'_R = \ell_R \equiv \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}$
$V_L^{\nu\dagger} \nu'_L = \mathbf{n}_L \equiv \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$	$V_R^{\nu\dagger} \nu'_R = \mathbf{n}_R \equiv \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix}$

$$\begin{aligned} \mathcal{L}_{H,L} &= - \left(\frac{v+H}{\sqrt{2}} \right) \left[\overline{\ell}_L Y^\ell \ell_R + \overline{\mathbf{n}}_L Y^\nu \mathbf{n}_R \right] + \text{H.c.} \\ &= - \left(\frac{v+H}{\sqrt{2}} \right) \left[\sum_{\alpha=e,\mu,\tau} y_\alpha^\ell \overline{\ell}_{\alpha L} \ell_{\alpha R} + \sum_{k=1}^3 y_k^\nu \overline{\nu}_{kL} \nu_{kR} \right] + \text{H.c.} \end{aligned}$$

Massive Dirac Lepton Fields

$$l_\alpha \equiv l_{\alpha L} + l_{\alpha R} \quad (\alpha = e, \mu, \tau)$$

$$\nu_k = \nu_{kL} + \nu_{kR} \quad (k = 1, 2, 3)$$

$$\begin{aligned} \mathcal{L}_{H,L} = & - \sum_{\alpha=e,\mu,\tau} \frac{y_\alpha^l v}{\sqrt{2}} \bar{l}_\alpha l_\alpha - \sum_{k=1}^3 \frac{y_k^\nu v}{\sqrt{2}} \bar{\nu}_k \nu_k && \text{Mass Terms} \\ & - \sum_{\alpha=e,\mu,\tau} \frac{y_\alpha^l}{\sqrt{2}} \bar{l}_\alpha l_\alpha H - \sum_{k=1}^3 \frac{y_k^\nu}{\sqrt{2}} \bar{\nu}_k \nu_k H && \text{Lepton-Higgs Couplings} \end{aligned}$$

Charged Lepton and Neutrino Masses

$$m_\alpha = \frac{y_\alpha^l v}{\sqrt{2}} \quad (\alpha = e, \mu, \tau) \quad m_k = \frac{y_k^\nu v}{\sqrt{2}} \quad (k = 1, 2, 3)$$

Lepton-Higgs coupling \propto Lepton Mass

Quantization

$$\nu_k(x) = \int \frac{d^3 p}{(2\pi)^3 2E_k} \sum_{h=\pm 1} \left[a_k^{(h)}(p) u_k^{(h)}(p) e^{-ip \cdot x} + b_k^{(h)\dagger}(p) v_k^{(h)}(p) e^{ip \cdot x} \right]$$

$$p^0 = E_k = \sqrt{\vec{p}^2 + m_k^2} \quad \begin{aligned} (\not{p} - m_k) u_k^{(h)}(p) &= 0 \\ (\not{p} + m_k) v_k^{(h)}(p) &= 0 \end{aligned}$$

$$\frac{\vec{p} \cdot \vec{\Sigma}}{|\vec{p}|} u_k^{(h)}(p) = h u_k^{(h)}(p)$$

$$\frac{\vec{p} \cdot \vec{\Sigma}}{|\vec{p}|} v_k^{(h)}(p) = -h v_k^{(h)}(p)$$

$$\{a_k^{(h)}(p), a_k^{(h')\dagger}(p')\} = \{b_k^{(h)}(p), b_k^{(h')\dagger}(p')\} = (2\pi)^3 2E_k \delta^3(\vec{p} - \vec{p}') \delta_{hh'}$$

$$\{a_k^{(h)}(p), a_k^{(h')}(p')\} = \{a_k^{(h)\dagger}(p), a_k^{(h')\dagger}(p')\} = 0$$

$$\{b_k^{(h)}(p), b_k^{(h')}(p')\} = \{b_k^{(h)\dagger}(p), b_k^{(h')\dagger}(p')\} = 0$$

$$\{a_k^{(h)}(p), b_k^{(h')}(p')\} = \{a_k^{(h)\dagger}(p), b_k^{(h')\dagger}(p')\} = 0$$

$$\{a_k^{(h)}(p), b_k^{(h')\dagger}(p')\} = \{a_k^{(h)\dagger}(p), b_k^{(h')}(p')\} = 0$$

Mixing

Charged-Current Weak Interaction Lagrangian

$$\mathcal{L}_1^{(\text{CC})} = -\frac{g}{2\sqrt{2}} j_W^\rho W_\rho + \text{H.c.}$$

Weak Charged Current: $j_W^\rho = j_{W,L}^\rho + j_{W,Q}^\rho$

Leptonic Weak Charged Current

$$j_{W,L}^{\rho\dagger} = 2 \sum_{\alpha=e,\mu,\tau} \overline{\ell'_{\alpha L}} \gamma^\rho \nu'_{\alpha L} = 2 \overline{\ell'_L} \gamma^\rho \nu'_L$$

$$\underline{\ell'_L} = V_L^\ell \ell_L \qquad \underline{\nu'_L} = V_L^\nu n_L$$

$$j_{W,L}^{\rho\dagger} = 2 \overline{\ell'_L} V_L^{\ell\dagger} \gamma^\rho V_L^\nu n_L = 2 \overline{\ell'_L} \gamma^\rho V_L^{\ell\dagger} V_L^\nu n_L = 2 \overline{\ell'_L} \gamma^\rho U n_L$$

Mixing Matrix:

$$U = V_L^{\ell\dagger} V_L^\nu$$

► **Definition:** Left-Handed Flavor Neutrino Fields

$$\nu_L = U \mathbf{n}_L = V_L^{\ell\dagger} V_L^\nu \mathbf{n}_L = V_L^{\ell\dagger} \nu'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$$

► They allow us to write the **Leptonic Weak Charged Current** as in the SM:

$$j_{W,L}^{\rho\dagger} = 2 \bar{\ell}_L \gamma^\rho \nu_L = 2 \sum_{\alpha=e,\mu,\tau} \bar{\ell}_{\alpha L} \gamma^\rho \nu_{\alpha L}$$

► Each **left-handed flavor neutrino field** is associated with the corresponding **charged lepton field** which describes a massive charged lepton:

$$j_{W,L}^{\rho\dagger} = 2 (\bar{e}_L \gamma^\rho \nu_{eL} + \bar{\mu}_L \gamma^\rho \nu_{\mu L} + \bar{\tau}_L \gamma^\rho \nu_{\tau L})$$

► In practice **left-handed flavor neutrino fields** are useful for calculations in the SM approximation of massless neutrinos (**interactions**).

► If neutrino masses must be taken into account, it is necessary to use

$$j_{W,L}^{\rho\dagger} = 2 \bar{\ell}_L \gamma^\rho U \mathbf{n}_L = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^3 \bar{\ell}_{\alpha L} \gamma^\rho U_{\alpha k} \nu_{kL}$$

Flavor Lepton Numbers

Flavor Neutrino Fields are useful for defining
Flavor Lepton Numbers
as in the SM

	L_e	L_μ	L_τ		L_e	L_μ	L_τ
(ν_e, e^-)	+1	0	0	(ν_e^c, e^+)	-1	0	0
(ν_μ, μ^-)	0	+1	0	(ν_μ^c, μ^+)	0	-1	0
(ν_τ, τ^-)	0	0	+1	(ν_τ^c, τ^+)	0	0	-1

$$L = L_e + L_\mu + L_\tau$$

Standard Model: Lepton numbers are conserved

▶ L_e, L_μ, L_τ are conserved in the Standard Model with massless neutrinos

▶ Dirac mass term:

$$\mathcal{L}^D = - \begin{pmatrix} \overline{\nu_{eL}} & \overline{\nu_{\mu L}} & \overline{\nu_{\tau L}} \end{pmatrix} \begin{pmatrix} m_{ee}^D & m_{e\mu}^D & m_{e\tau}^D \\ m_{\mu e}^D & m_{\mu\mu}^D & m_{\mu\tau}^D \\ m_{\tau e}^D & m_{\tau\mu}^D & m_{\tau\tau}^D \end{pmatrix} \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix} + \text{H.c.}$$

L_e, L_μ, L_τ are not conserved

▶ L is conserved: $L(\nu_{\alpha R}) = L(\nu_{\beta L}) \implies |\Delta L| = 0$

- ▶ **Leptonic Weak Charged Current** is invariant under the global U(1) gauge transformations

$$\ell_{\alpha L} \rightarrow e^{i\varphi_\alpha} \ell_{\alpha L} \quad \nu_{\alpha L} \rightarrow e^{i\varphi_\alpha} \nu_{\alpha L} \quad (\alpha = e, \mu, \tau)$$

- ▶ If neutrinos are massless (SM), Noether's theorem implies that there is, for each flavor, a conserved current:

$$j_\alpha^\rho = \overline{\nu_{\alpha L}} \gamma^\rho \nu_{\alpha L} + \overline{\ell_\alpha} \gamma^\rho \ell_\alpha \quad \partial_\rho j_\alpha^\rho = 0$$

and a conserved charge:

$$L_\alpha = \int d^3x j_\alpha^0(x) \quad \partial_0 L_\alpha = 0$$

$$\begin{aligned} :L_\alpha: &= \int \frac{d^3p}{(2\pi)^3 2E} \left[a_{\nu_\alpha}^{(-)\dagger}(p) a_{\nu_\alpha}^{(-)}(p) - b_{\nu_\alpha}^{(+)\dagger}(p) b_{\nu_\alpha}^{(+)}(p) \right] \\ &+ \int \frac{d^3p}{(2\pi)^3 2E} \sum_{h=\pm 1} \left[a_{\ell_\alpha}^{(h)\dagger}(p) a_{\ell_\alpha}^{(h)}(p) - b_{\ell_\alpha}^{(h)\dagger}(p) b_{\ell_\alpha}^{(h)}(p) \right] \end{aligned}$$

▶ Lepton-Higgs Yukawa Lagrangian:

$$\mathcal{L}_{H,L} = - \left(\frac{v+H}{\sqrt{2}} \right) \left[\sum_{\alpha=e,\mu,\tau} y_{\alpha}^{\ell} \overline{l_{\alpha L}} l_{\alpha R} + \sum_{k=1}^3 y_k^{\nu} \overline{\nu_{kL}} \nu_{kR} \right] + \text{H.c.}$$

▶ Mixing: $\nu_{\alpha L} = \sum_{k=1}^3 U_{\alpha k} \nu_{kL} \iff \nu_{kL} = \sum_{\alpha=e,\mu,\tau} U_{\alpha k}^* \nu_{\alpha L}$

$$\mathcal{L}_{H,L} = - \left(\frac{v+H}{\sqrt{2}} \right) \sum_{\alpha=e,\mu,\tau} \left[y_{\alpha}^{\ell} \overline{l_{\alpha L}} l_{\alpha R} + \overline{\nu_{\alpha L}} \sum_{k=1}^3 U_{\alpha k} y_k^{\nu} \nu_{kR} \right] + \text{H.c.}$$

▶ Invariant for

$$l_{\alpha L} \rightarrow e^{i\varphi_{\alpha}} l_{\alpha L}, \quad \nu_{\alpha L} \rightarrow e^{i\varphi_{\alpha}} \nu_{\alpha L}$$

$$l_{\alpha R} \rightarrow e^{i\varphi_{\alpha}} l_{\alpha R}, \quad \sum_{k=1}^3 U_{\alpha k} y_k^{\nu} \nu_{kR} \rightarrow e^{i\varphi_{\alpha}} \sum_{k=1}^3 U_{\alpha k} y_k^{\nu} \nu_{kR}$$

▶ But kinetic part of neutrino Lagrangian is not invariant

$$\mathcal{L}_{\text{kinetic}}^{(\nu)} = \sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}} i \not{\partial} \nu_{\alpha L} + \sum_{k=1}^3 \overline{\nu_{kR}} i \not{\partial} \nu_{kR}$$

because $\sum_{k=1}^3 U_{\alpha k} y_k^{\nu} \nu_{kR}$ is not a unitary combination of the ν_{kR} 's

Total Lepton Number

- ▶ Dirac neutrino masses violate conservation of Flavor Lepton Numbers
- ▶ Total Lepton Number is conserved, because Lagrangian is invariant under the global U(1) gauge transformations

$$\begin{aligned} \nu_{kL} &\rightarrow e^{i\varphi} \nu_{kL}, & \nu_{kR} &\rightarrow e^{i\varphi} \nu_{kR} & (k = 1, 2, 3) \\ \ell_{\alpha L} &\rightarrow e^{i\varphi} \ell_{\alpha L}, & \ell_{\alpha R} &\rightarrow e^{i\varphi} \ell_{\alpha R} & (\alpha = e, \mu, \tau) \end{aligned}$$

- ▶ From Noether's theorem:

$$j^\rho = \sum_{k=1}^3 \bar{\nu}_k \gamma^\rho \nu_k + \sum_{\alpha=e,\mu,\tau} \bar{\ell}_\alpha \gamma^\rho \ell_\alpha \quad \partial_\rho j^\rho = 0$$

$$\text{Conserved charge: } L_\alpha = \int d^3x j_\alpha^0(x) \quad \partial_0 L_\alpha = 0$$

$$\begin{aligned} :L: &= \sum_{k=1}^3 \int \frac{d^3p}{(2\pi)^3 2E} \sum_{h=\pm 1} \left[a_{\nu_k}^{(h)\dagger}(p) a_{\nu_k}^{(h)}(p) - b_{\nu_k}^{(h)\dagger}(p) b_{\nu_k}^{(h)}(p) \right] \\ &+ \sum_{\alpha=e,\mu,\tau} \int \frac{d^3p}{(2\pi)^3 2E} \sum_{h=\pm 1} \left[a_{\ell_\alpha}^{(h)\dagger}(p) a_{\ell_\alpha}^{(h)}(p) - b_{\ell_\alpha}^{(h)\dagger}(p) b_{\ell_\alpha}^{(h)}(p) \right] \end{aligned}$$

Mixing Matrix

$$\blacktriangleright U = V_L^{\ell\dagger} V_L^\nu = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}$$

- \blacktriangleright A unitary $N \times N$ matrix depends on N^2 independent real parameters:

$$N = 3 \quad \Rightarrow \quad \frac{N(N-1)}{2} = 3 \quad \text{Mixing Angles}$$
$$\frac{N(N+1)}{2} = 6 \quad \text{Phases}$$

- \blacktriangleright Not all phases are physical observables!

- \blacktriangleright Neutrino Lagrangian:

kinetic terms + mass terms + weak interactions

- \blacktriangleright Mixing is due to the diagonalization of the mass terms.
- \blacktriangleright The kinetic terms are invariant under unitary transformations of the fermion fields.
- \blacktriangleright What is the effect of mixing in weak interactions?

▶ Weak Charged Current:
$$j_{W,L}^{\rho\dagger} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^3 \overline{\ell_{\alpha L}} \gamma^\rho U_{\alpha k} \nu_{kL}$$

- ▶ Apart from the Weak Charged Current, the Lagrangian is invariant under the global phase transformations (6 arbitrary phases)

$$\ell_\alpha \rightarrow e^{i\varphi_\alpha} \ell_\alpha \quad (\alpha = e, \mu, \tau), \quad \nu_k \rightarrow e^{i\varphi_k} \nu_k \quad (k = 1, 2, 3)$$

- ▶ Performing this transformation, the Weak Charged Current becomes

$$j_{W,L}^{\rho\dagger} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^3 \overline{\ell_{\alpha L}} e^{-i\varphi_\alpha} \gamma^\rho U_{\alpha k} e^{i\varphi_k} \nu_{kL}$$

$$j_{W,L}^{\rho\dagger} = 2 \underbrace{e^{-i(\varphi_e - \varphi_1)}}_1 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^3 \overline{\ell_{\alpha L}} \underbrace{e^{-i(\varphi_\alpha - \varphi_e)}}_2 \gamma^\rho U_{\alpha k} \underbrace{e^{i(\varphi_k - \varphi_1)}}_2 \nu_{kL}$$

- ▶ There are 5 independent combinations of the phases of the fields that can be chosen to eliminate 5 of the 6 phases of the mixing matrix
- ▶ 5 and not 6 phases of the mixing matrix can be eliminated because a common rephasing of all the lepton fields leaves the Weak Charged Current invariant \iff conservation of Total Lepton Number.

- ▶ The mixing matrix contains 1 Physical Phase.
- ▶ It is convenient to express the 3×3 unitary mixing matrix only in terms of the four physical parameters:

3 Mixing Angles and 1 Phase

Standard Parameterization of Mixing Matrix

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}$$

$$c_{ab} \equiv \cos \vartheta_{ab} \quad s_{ab} \equiv \sin \vartheta_{ab} \quad 0 \leq \vartheta_{ab} \leq \frac{\pi}{2} \quad 0 \leq \delta_{13} < 2\pi$$

3 Mixing Angles ϑ_{12} , ϑ_{23} , ϑ_{13} and 1 Phase δ_{13}

Standard Parameterization

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of Different Phase Convention

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{i\delta_{23}} \\ 0 & -s_{23}e^{-i\delta_{13}} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of Different Parameterization

$$U = \begin{pmatrix} c'_{12} & s'_{12}e^{-i\delta'_{12}} & 0 \\ -s'_{12}e^{i\delta'_{12}} & c'_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c'_{23} & s'_{23} \\ 0 & -s'_{23} & c'_{23} \end{pmatrix} \begin{pmatrix} c'_{13} & 0 & s'_{13} \\ 0 & 1 & 0 \\ -s'_{13} & 0 & c'_{13} \end{pmatrix}$$

CP Violation

- ▶ $U \neq U^* \implies$ CP Violation (CPV)
- ▶ General conditions for CP violation (14 conditions):
 1. No charged leptons or neutrinos are degenerate in mass (6 conditions)
 2. No mixing angle is equal to 0 or $\pi/2$ (6 conditions)
 3. The physical phase is different from 0 or π (2 conditions)

- ▶ These 14 conditions are combined into the single condition

$$\det C \neq 0 \quad \text{with} \quad C = -i [M^{\nu\nu} M^{\nu\nu\dagger}, M^{\ell\ell} M^{\ell\ell\dagger}]$$

$$\det C = -2 J (m_{\nu_2}^2 - m_{\nu_1}^2) (m_{\nu_3}^2 - m_{\nu_1}^2) (m_{\nu_3}^2 - m_{\nu_2}^2) \\ (m_{\mu}^2 - m_e^2) (m_{\tau}^2 - m_e^2) (m_{\tau}^2 - m_{\mu}^2) \neq 0$$

- ▶ Jarlskog invariant: $J = \text{Im} [U_{e2} U_{e3}^* U_{\mu 2}^* U_{\mu 3}]$

[C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039, Z. Phys. C 29 (1985) 491]

[O. W. Greenberg, Phys. Rev. D 32 (1985) 1841]

[I. Dunietz, O. W. Greenberg, Dan-di Wu, Phys. Rev. Lett. 55 (1985) 2935]

Example: $\vartheta_{12} = 0$

$$U = R_{23}R_{13}W_{12}$$

$$W_{12} = \begin{pmatrix} \cos \vartheta_{12} & \sin \vartheta_{12} e^{-i\delta_{12}} & 0 \\ -\sin \vartheta_{12} e^{-i\delta_{12}} & \cos \vartheta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vartheta_{12} = 0 \quad \Rightarrow \quad W_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}$$

Real Mixing Matrix: $U = R_{23}R_{13}$

Example: $\vartheta_{13} = \pi/2$

$$U = R_{23} W_{13} R_{12}$$

$$W_{13} = \begin{pmatrix} \cos \vartheta_{13} & 0 & \sin \vartheta_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -\sin \vartheta_{13} e^{i\delta_{13}} & 0 & \cos \vartheta_{13} \end{pmatrix}$$

$$\vartheta_{13} = \pi/2 \quad \Rightarrow \quad W_{13} = \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -e^{i\delta_{13}} & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}e^{i\delta_{13}} & 0 \\ s_{12}s_{23} - c_{12}c_{23}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}e^{i\delta_{13}} & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ |U_{\mu 1}| e^{i\lambda_{\mu 1}} & |U_{\mu 2}| e^{i\lambda_{\mu 2}} & 0 \\ |U_{\tau 1}| e^{i\lambda_{\tau 1}} & |U_{\tau 2}| e^{i\lambda_{\tau 2}} & 0 \end{pmatrix}$$

$$\lambda_{\mu 1} - \lambda_{\mu 2} = \lambda_{\tau 1} - \lambda_{\tau 2} \pm \pi \quad \lambda_{\tau 1} - \lambda_{\mu 1} = \lambda_{\tau 2} - \lambda_{\mu 2} \pm \pi$$

$$\nu_k \rightarrow e^{i\varphi_k} \nu_k \quad (k = 1, 2, 3), \quad \ell_\alpha \rightarrow e^{i\varphi_\alpha} \ell_\alpha \quad (\alpha = e, \mu, \tau)$$

$$U \rightarrow \begin{pmatrix} e^{-i\varphi_e} & 0 & 0 \\ 0 & e^{-i\varphi_\mu} & 0 \\ 0 & 0 & e^{-i\varphi_\tau} \end{pmatrix} \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ |U_{\mu 1}| e^{i\lambda_{\mu 1}} & |U_{\mu 2}| e^{i\lambda_{\mu 2}} & 0 \\ |U_{\tau 1}| e^{i\lambda_{\tau 1}} & |U_{\tau 2}| e^{i\lambda_{\tau 2}} & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{i\varphi_3} \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & e^{i(-\delta_{13} - \varphi_e + \varphi_3)} \\ |U_{\mu 1}| e^{i(\lambda_{\mu 1} - \varphi_\mu + \varphi_1)} & |U_{\mu 2}| e^{i(\lambda_{\mu 2} - \varphi_\mu + \varphi_2)} & 0 \\ |U_{\tau 1}| e^{i(\lambda_{\tau 1} - \varphi_\tau + \varphi_1)} & |U_{\tau 2}| e^{i(\lambda_{\tau 2} - \varphi_\tau + \varphi_2)} & 0 \end{pmatrix}$$

$$\varphi_1 = 0 \quad \varphi_\mu = \lambda_{\mu 1} \quad \varphi_\tau = \lambda_{\tau 1} \quad \varphi_2 = \varphi_\mu - \lambda_{\mu 2} = \lambda_{\mu 1} - \lambda_{\mu 2}$$

$$\varphi_2 = \varphi_\tau - \lambda_{\tau 2} \pm \pi = \lambda_{\tau 1} - \lambda_{\tau 2} \pm \pi = \lambda_{\mu 1} - \lambda_{\mu 2}$$

$$\text{Real Mixing Matrix: } U = \begin{pmatrix} 0 & 0 & \pm 1 \\ |U_{\mu 1}| & |U_{\mu 2}| & 0 \\ |U_{\tau 1}| & -|U_{\tau 2}| & 0 \end{pmatrix}$$

Example: $m_{\nu_2} = m_{\nu_3}$

$$j_{W,L}^{\rho\dagger} = 2 \bar{\ell}_L \gamma^\rho U \mathbf{n}_L$$

$$U = R_{12} R_{13} W_{23} \quad \Rightarrow \quad j_{W,L}^{\rho\dagger} = 2 \bar{\ell}_L \gamma^\rho R_{12} R_{13} W_{23} \mathbf{n}_L$$

$$W_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta_{23} & \sin \vartheta_{23} e^{-i\delta_{23}} \\ 0 & -\sin \vartheta_{23} e^{-i\delta_{23}} & \cos \vartheta_{23} \end{pmatrix}$$

$$W_{23} \mathbf{n}_L = \mathbf{n}'_L \quad R_{12} R_{13} = U' \quad \Rightarrow \quad j_{W,L}^{\rho\dagger} = 2 \bar{\ell}_L \gamma^\rho U' \mathbf{n}'_L$$

ν_2 and ν_3 are indistinguishable if they have the same mass!

\mathbf{n}'_L is equivalent to \mathbf{n}_L

$$\text{Drop the prime} \quad \Rightarrow \quad j_{W,L}^{\rho\dagger} = 2 \bar{\ell}_L \gamma^\rho U \mathbf{n}_L$$

$$\text{With the Real Mixing Matrix} \quad U = R_{12} R_{13}$$

Jarlskog Invariant

- ▶ Since physics is invariant under reparameterizations of the mixing matrix, all physical quantities can be expressed in terms of reparameterization-invariant quantities.
- ▶ Simplest invariants: $|U_{\alpha k}|^2 = U_{\alpha k} U_{\alpha k}^*$, $U_{\alpha k} U_{\alpha j}^* U_{\beta k}^* U_{\beta j}$
- ▶ Simplest CPV invariants: $\text{Im}[U_{\alpha k} U_{\alpha j}^* U_{\beta k}^* U_{\beta j}] = \pm J$

Jarlskog invariant: $J = \text{Im}[U_{e2} U_{e3}^* U_{\mu 2}^* U_{\mu 3}] = \text{Im} \begin{pmatrix} \cdot & \circ & \times \\ \cdot & \times & \circ \\ \cdot & \cdot & \cdot \end{pmatrix}$

- ▶ In standard parameterization:

$$\begin{aligned} J &= c_{12} s_{12} c_{23} s_{23} c_{13}^2 s_{13} \sin \delta_{13} \\ &= \frac{1}{8} \sin 2\vartheta_{12} \sin 2\vartheta_{23} \cos \vartheta_{13} \sin 2\vartheta_{13} \sin \delta_{13} \end{aligned}$$

- ▶ For CPV all mixing angles must be different from 0 and $\pi/2$!
- ▶ The Jarlskog invariant is useful for quantifying CPV in a parameterization-independent way.
- ▶ All measurable CPV effects depend on J .

Maximal CP Violation

- ▶ Maximal CP violation is defined as the case in which $|J|$ has its maximum possible value

$$|J|_{\max} = \text{Max} \left| \underbrace{c_{12}s_{12}}_{\frac{1}{2}} \underbrace{c_{23}s_{23}}_{\frac{1}{2}} \underbrace{c_{13}^2 s_{13}}_{\frac{2}{3\sqrt{3}}} \underbrace{\sin \delta_{13}}_1 \right| = \frac{1}{6\sqrt{3}}$$

- ▶ In the standard parameterization it is obtained for

$$\vartheta_{12} = \vartheta_{23} = \pi/4, \quad s_{13} = 1/\sqrt{3}, \quad \sin \delta_{13} = \pm 1$$

- ▶ This case is called **Trimaximal Mixing**. All the absolute values of the elements of the mixing matrix are equal to $1/\sqrt{3}$:

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \mp \frac{i}{\sqrt{3}} \\ -\frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2} \mp \frac{i}{2\sqrt{3}} & -\frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & \mp i \\ -e^{\pm i\pi/6} & e^{\mp i\pi/6} & 1 \\ e^{\mp i\pi/6} & -e^{\pm i\pi/6} & 1 \end{pmatrix}$$

GIM Mechanism

[S.L. Glashow, J. Iliopoulos, L. Maiani, Phys. Rev. D 2 (1970) 1285]

- ▶ Neutral-Current Weak Interaction Lagrangian:

$$\mathcal{L}_I^{(\text{NC})} = -\frac{g}{2 \cos \vartheta_W} j_Z^\rho Z_\rho \quad j_Z^\rho = j_{Z,L}^\rho + j_{Z,Q}^\rho$$

- ▶ Leptonic Weak Neutral Current: ($g_L^\nu = \frac{1}{2}$, $g_L^\ell = -\frac{1}{2} + \sin^2 \vartheta_W$, $g_R^\ell = \sin^2 \vartheta_W$)

$$j_{Z,L}^\rho = 2g_L^\nu \bar{\nu}'_L \gamma^\rho \nu'_L + 2g_L^\ell \bar{\ell}'_L \gamma^\rho \ell'_L + 2g_R^\ell \bar{\ell}'_R \gamma^\rho \ell'_R$$

- ▶ Invariant under mixing transformations with unitary V_L^ℓ , V_R^ℓ , V_L^ν :

$$\begin{aligned} j_{Z,L}^\rho &= 2g_L^\nu \bar{\mathbf{n}}_L V_L^{\nu\dagger} \gamma^\rho V_L^\nu \mathbf{n}_L + 2g_L^\ell \bar{\ell}_L V_L^{\ell\dagger} \gamma^\rho V_L^\ell \ell_L + 2g_R^\ell \bar{\ell}_R V_R^{\ell\dagger} \gamma^\rho V_R^\ell \ell_R \\ &= 2g_L^\nu \bar{\mathbf{n}}_L \gamma^\rho \mathbf{n}_L + 2g_L^\ell \bar{\ell}_L \gamma^\rho \ell_L + 2g_R^\ell \bar{\ell}_R \gamma^\rho \ell_R \end{aligned}$$

- ▶ Invariant also under the mixing transformation $\nu_L = U \mathbf{n}_L$ which defines the flavor neutrino fields:

$$\begin{aligned} j_{Z,L}^\rho &= 2g_L^\nu \bar{\nu}_L U \gamma^\rho U^\dagger \nu_L + 2g_L^\ell \bar{\ell}_L \gamma^\rho \ell_L + 2g_R^\ell \bar{\ell}_R \gamma^\rho \ell_R \\ &= 2g_L^\nu \bar{\nu}_L \gamma^\rho \nu_L + 2g_L^\ell \bar{\ell}_L \gamma^\rho \ell_L + 2g_R^\ell \bar{\ell}_R \gamma^\rho \ell_R \end{aligned}$$

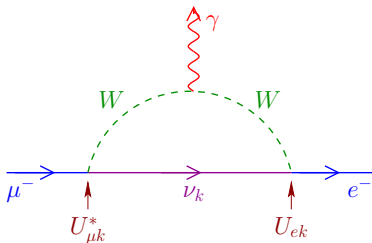
- ▶ Mixing has no effect in neutral-current weak interactions.

Lepton Numbers Violating Processes

Dirac mass term allows L_e, L_μ, L_τ violating processes

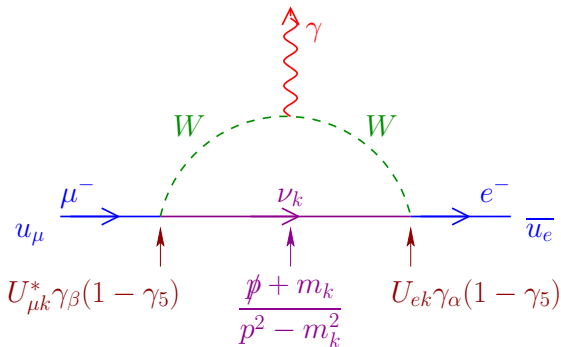
Example: $\mu^\pm \rightarrow e^\pm + \gamma, \quad \mu^\pm \rightarrow e^\pm + e^+ + e^-$

$$\mu^- \rightarrow e^- + \gamma$$



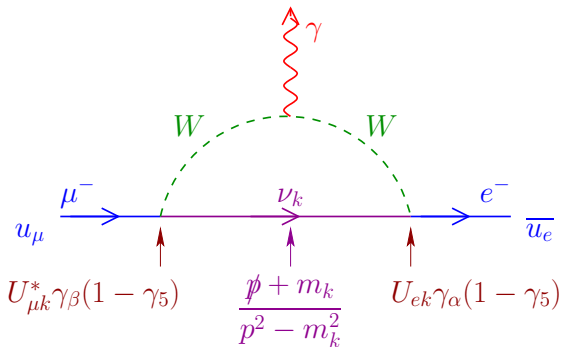
$$\sum_k U_{\mu k}^* U_{ek} = 0 \quad \Rightarrow \quad \text{GIM suppression:} \quad A \propto \sum_k U_{\mu k}^* U_{ek} f(m_k)$$

$$\begin{aligned} \mathcal{L}_1^{(\text{CC})} &= -\frac{g}{2\sqrt{2}} W^\alpha [\bar{\nu}_e \gamma_\alpha (1 - \gamma_5) e + \bar{\nu}_\mu \gamma_\alpha (1 - \gamma_5) \mu + \dots] \\ &= -\frac{g}{2\sqrt{2}} W^\alpha \sum_k [\bar{\nu}_k U_{ek}^* \gamma_\alpha (1 - \gamma_5) e + \bar{\nu}_k U_{\mu k}^* \gamma_\alpha (1 - \gamma_5) \mu + \dots] \end{aligned}$$



$$A \propto \sum_k \bar{u}_e U_{ek} \gamma_\alpha (1 - \gamma_5) \frac{\not{p} + m_k}{p^2 - m_k^2} U_{\mu k}^* \gamma_\beta (1 - \gamma_5) u_\mu$$

$$\begin{aligned} \mathcal{L}_1^{(\text{CC})} &= -\frac{g}{2\sqrt{2}} W^\alpha [\bar{\nu}_e \gamma_\alpha (1 - \gamma_5) e + \bar{\nu}_\mu \gamma_\alpha (1 - \gamma_5) \mu + \dots] \\ &= -\frac{g}{2\sqrt{2}} W^\alpha \sum_k [\bar{\nu}_k U_{ek}^* \gamma_\alpha (1 - \gamma_5) e + \bar{\nu}_k U_{\mu k}^* \gamma_\alpha (1 - \gamma_5) \mu + \dots] \end{aligned}$$



$$A \propto \sum_k \bar{u}_e U_{ek} \gamma_\alpha (1 - \gamma_5) \frac{\not{p} + \cancel{m_k}}{p^2 - m_k^2} (1 + \gamma_5) \gamma_\beta U_{\mu k}^* u_\mu$$

$$\frac{1}{p^2 - m_k^2} = p^{-2} \left(1 - \frac{m_k^2}{p^2}\right)^{-1} \simeq p^{-2} \left(1 + \frac{m_k^2}{p^2}\right)$$

$$A \propto \sum_k U_{ek} U_{\mu k}^* \left(1 + \frac{m_k^2}{p^2}\right) = \sum_k U_{ek} U_{\mu k}^* \frac{m_k^2}{p^2} \rightarrow \sum_k U_{ek} U_{\mu k}^* \frac{m_k^2}{m_W^2}$$

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3} \underbrace{\frac{3\alpha}{32\pi} \left| \sum_k U_{ek} U_{\mu k}^* \frac{m_k^2}{m_W^2} \right|^2}_{\text{BR}}$$

[Petcov, SJNP 25 (1977) 340; Bilenky, Petcov, Pontecorvo, PLB 67 (1977) 309]

[Lee, Shrock, PRD 16 (1977) 1444]

Suppression factor: $\frac{m_k}{m_W} \lesssim 10^{-11}$ for $m_k \lesssim 1 \text{ eV}$

$$(\text{BR})_{\text{the}} \lesssim 10^{-47}$$

$$(\text{BR})_{\text{exp}} \lesssim 10^{-11}$$

Majorana Neutrino Masses and Mixing

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
 - Two-Component Theory of a Massless Neutrino
 - Majorana Equation
 - CP Symmetry
 - Effective Majorana Mass
 - Mixing of Three Majorana Neutrinos
- Dirac-Majorana Mass Term
- Sterile Neutrinos

Two-Component Theory of a Massless Neutrino

[Landau, NP 3 (1957) 127; Lee, Yang, PR 105 (1957) 1671; Salam, NC 5 (1957) 299]

▶ Dirac Equation: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

▶ Chiral decomposition of a Fermion Field: $\psi = \psi_L + \psi_R$

▶ Equations for the Chiral components are coupled by mass:

$$i\gamma^\mu \partial_\mu \psi_L = m \psi_R$$

$$i\gamma^\mu \partial_\mu \psi_R = m \psi_L$$

▶ They are decoupled for a massless fermion: **Weyl Equations** (1929)

$$i\gamma^\mu \partial_\mu \psi_L = 0$$

$$i\gamma^\mu \partial_\mu \psi_R = 0$$

▶ A massless fermion can be described by a single chiral field ψ_L or ψ_R (**Weyl Spinor**), which has only **two independent components** (half the number of degrees of freedom of a Dirac field, which has four independent components).

- Chiral representation of γ matrices:

$$\gamma^0 = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$P_L = \frac{1 - \gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad P_R = \frac{1 + \gamma^5}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$

- Four-components Dirac spinor: $\psi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$

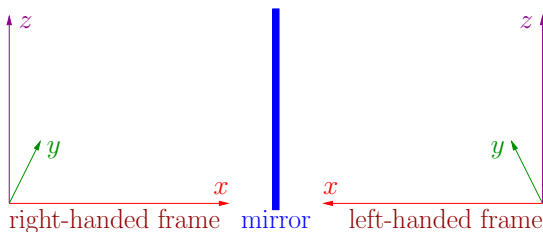
- The Weyl spinors ψ_L and ψ_R have only two components:

$$\psi_L = P_L \psi = \begin{pmatrix} 0 \\ \chi_L \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix} \quad \psi_R = P_R \psi = \begin{pmatrix} \chi_R \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \\ 0 \\ 0 \end{pmatrix}$$

- ▶ The possibility to describe a physical particle with a Weyl spinor was rejected by Pauli in 1933 because it leads to **parity violation**:

$$\psi_L \xrightarrow{P} (\psi^P)_R$$

- ▶ Parity is the symmetry of **space inversion** (mirror transformation)



- ▶ Parity was considered to be an exact symmetry of nature
- ▶ **1956**: Lee and Yang understand that Parity can be violated in **Weak Interactions** (1957 Physics Nobel Prize)
- ▶ **1957**: Wu et al. discover Parity violation in β -decay of ^{60}Co

▶ Parity: $x^\mu = (x^0, \vec{x}) \xrightarrow{P} x_P^\mu = (x^0, -\vec{x}) = x_\mu$

▶ The transformation of a fermion field $\psi(x)$ under parity is determined from the invariance of the theory under parity.

▶ Dirac Lagrangian:

$$\mathcal{L}_D(x) = \bar{\psi}(x) (i\not{\partial} - m) \psi(x) = \bar{\psi}(x) \left(i\gamma^0 \partial_0 + i\gamma^k \partial_k - m \right) \psi(x)$$

↓ P

$$\bar{\psi}^P(x_P) \left(i\gamma^0 \partial_0 - i\gamma^k \partial_k - m \right) \psi^P(x_P)$$

▶ It is equal to $\mathcal{L}_D(x_P)$ if $\psi^P(x_P) = \xi_P \gamma^0 \psi(x)$

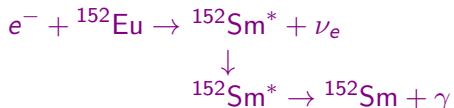
▶ Invariance is obtained from the action because $\left| \frac{\partial x_P}{\partial x} \right| = 1$:

$$I_D = \int d^4x \mathcal{L}_D(x) = \int d^4x_P \mathcal{L}_D(x_P)$$

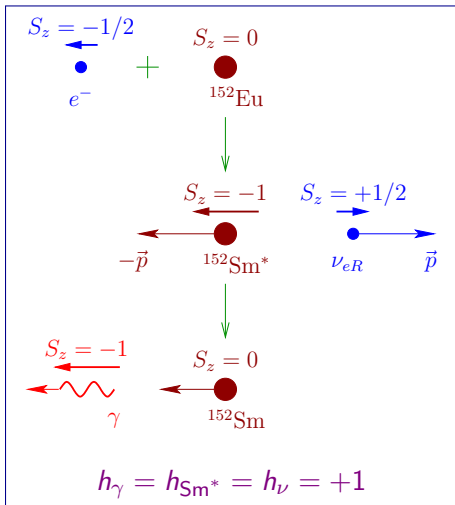
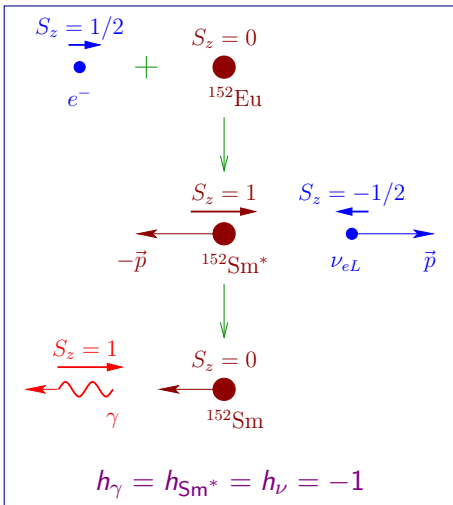
- ▶ $\psi(x) \xrightarrow{P} \psi^P(x_P) = \xi_P \gamma^0 \psi(x)$
- ▶ $\psi_L(x) \xrightarrow{P} (\psi_L)^P(x_P) = \xi_P \gamma^0 \psi_L(x)$
- ▶ $P_L(\psi_L)^P = \xi_P \frac{1 - \gamma^5}{2} \gamma^0 \psi_L = \xi_P \gamma^0 \frac{1 + \gamma^5}{2} \psi_L = 0$
- ▶ $P_R(\psi_L)^P = \xi_P \frac{1 + \gamma^5}{2} \gamma^0 \psi_L = \xi_P \gamma^0 \frac{1 - \gamma^5}{2} \psi_L = (\psi_L)^P$
- ▶ Therefore $(\psi_L)^P$ is right-handed: $\psi_L \xrightarrow{P} (\psi^P)_R$
- ▶ Explicit swap of left and right components in the chiral representation:

$$(\psi_L)^P = \xi_P \gamma^0 \psi_L = \xi_P \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \chi_L \end{pmatrix} = -\xi_P \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}$$

- ▶ The discovery of **parity violation** in 1956-57 invalidated Pauli's reasoning, opening the possibility to describe massless particles with Weyl spinor fields \implies **Two-component Theory of a Massless Neutrino (1957)**
- ▶ **1958: Goldhaber, Grodzins and Sunyar** measure the neutrino helicity with the electron capture process



The neutrino helicity is the same as the measurable helicity of the photon when it is emitted in the same direction of the ${}^{152}\text{Sm}^*$ recoil.



$h_\gamma = -0.91 \pm 0.19 \implies$ NEUTRINOS ARE LEFT-HANDED: ν_L

[Goldhaber, Grodzins and Sunyar, PR 109 (1958) 1015]

V - A Weak Interactions

[Feynman, Gell-Mann, PR 109 (1958) 193; Sudarshan, Marshak, PR 109 (1958) 1860; Sakurai, NC 7 (1958) 649]

- ▶ The Fermi Hamiltonian (1934) $H_\beta = g (\bar{p}\gamma^\alpha n) (\bar{e}\gamma^\alpha \nu) + \text{H.c.}$ explained only nuclear decays with $\Delta J = 0$.
- ▶ 1936: Gamow and Teller extension to describe observed nuclear decays with $|\Delta J| = 1$:

[PR 49 (1936) 895]

$$H_\beta = \sum_{j=1}^5 [g_j (\bar{p}\Omega^j n) (\bar{e}\Omega_j \nu_e) + g'_j (\bar{p}\Omega^j n) (\bar{e}\Omega_j \gamma_5 \nu_e)] + \text{H.c.}$$

$$\text{with } \Omega^1 = 1, \Omega^2 = \gamma^\alpha, \Omega^3 = \sigma^{\alpha\beta}, \Omega^4 = \gamma^\alpha \gamma^5, \Omega^5 = \gamma^5$$

- ▶ 1958: Using simplicity arguments, Feynman and Gell-Mann, Sudarshan and Marshak, Sakurai propose the universal theory of parity-violating V - A Weak Interactions:

$$H_W = \frac{G_F}{\sqrt{2}} \left\{ [\bar{p}\gamma^\alpha (1 - \gamma^5) n] [\bar{e}\gamma^\alpha (1 - \gamma^5) \nu] + [\bar{\nu}\gamma^\alpha (1 - \gamma^5) \mu] [\bar{e}\gamma^\alpha (1 - \gamma^5) \nu] \right\} + \text{H.c.}$$

in agreement with $\nu_L = \frac{1 - \gamma^5}{2} \nu$

Quantization

$$\nu(x) = \int \frac{d^3p}{(2\pi)^3 2E} \sum_{h=\pm 1} \left[a^{(h)}(p) u^{(h)}(p) e^{-ip \cdot x} + b^{(h)\dagger}(p) v^{(h)}(p) e^{ip \cdot x} \right]$$

$$p^0 = E = \sqrt{\vec{p}^2 + m^2} \quad \begin{aligned} (\not{p} - m) u^{(h)}(p) &= 0 \\ (\not{p} + m) v^{(h)}(p) &= 0 \end{aligned}$$

$$\frac{\vec{p} \cdot \vec{\Sigma}}{|\vec{p}|} u^{(h)}(p) = h u^{(h)}(p)$$

$$\frac{\vec{p} \cdot \vec{\Sigma}}{|\vec{p}|} v^{(h)}(p) = -h v^{(h)}(p)$$

$$\{a^{(h)}(p), a^{(h')\dagger}(p')\} = \{b^{(h)}(p), b^{(h')\dagger}(p')\} = (2\pi)^3 2E \delta^3(\vec{p} - \vec{p}') \delta_{hh'}$$

$$\{a^{(h)}(p), a^{(h')}(p')\} = \{a^{(h)\dagger}(p), a^{(h')\dagger}(p')\} = 0$$

$$\{b^{(h)}(p), b^{(h')}(p')\} = \{b^{(h)\dagger}(p), b^{(h')\dagger}(p')\} = 0$$

$$\{a^{(h)}(p), b^{(h')}(p')\} = \{a^{(h)\dagger}(p), b^{(h')\dagger}(p')\} = 0$$

$$\{a^{(h)}(p), b^{(h')\dagger}(p')\} = \{a^{(h)\dagger}(p), b^{(h')}(p')\} = 0$$

Left-handed neutrino: $|\nu_L(p)\rangle = |\nu(p, h = -1)\rangle = a^{(-)\dagger}(p)|0\rangle$

Right-handed neutrino: $|\nu_R(p)\rangle = |\nu(p, h = +1)\rangle = a^{(+)\dagger}(p)|0\rangle$

Left-handed antineutrino: $|\bar{\nu}_L(p)\rangle = |\bar{\nu}(p, h = -1)\rangle = b^{(-)\dagger}(p)|0\rangle$

Right-handed antineutrino: $|\bar{\nu}_R(p)\rangle = |\bar{\nu}(p, h = +1)\rangle = b^{(+)\dagger}(p)|0\rangle$

Helicity and Chirality

$$\nu_L(x) = \int \frac{d^3p}{(2\pi)^3 2E} \sum_{h=\pm 1} \left[a^{(h)}(p) u_L^{(h)}(p) e^{-ip \cdot x} + b^{(h)\dagger}(p) v_L^{(h)}(p) e^{ip \cdot x} \right]$$

$$u^{(h)\dagger}(p) u^{(h)}(p) = 2E$$

$$u^{(h)\dagger}(p) \gamma^5 u^{(h)}(p) = 2h|\vec{p}|$$

$$v^{(h)\dagger}(p) v^{(h)}(p) = 2E$$

$$v^{(h)\dagger}(p) \gamma^5 v^{(h)}(p) = -2h|\vec{p}|$$

$$u_L^{(h)\dagger}(p) u_L^{(h)}(p) = u^{(h)\dagger}(p) \left(\frac{1 - \gamma^5}{2} \right) u^{(h)}(p) = E - h|\vec{p}|$$

$$u_L^{(-)\dagger}(p) u_L^{(-)}(p) = E + |\vec{p}| \simeq 2E - \frac{m^2}{2E} \quad \text{left-handed neutrinos}$$

$$u_L^{(+)\dagger}(p) u_L^{(+)}(p) = E - |\vec{p}| \simeq \frac{m^2}{2E} \quad \text{suppressed right-handed neutrinos}$$

$$v_L^{(h)\dagger}(p) v_L^{(h)}(p) = v^{(h)\dagger}(p) \left(\frac{1 - \gamma^5}{2} \right) v^{(h)}(p) = E + h|\vec{p}|$$

$$v_L^{(-)\dagger}(p) v_L^{(-)}(p) = E - |\vec{p}| \simeq \frac{m^2}{2E} \quad \text{right-handed antineutrinos}$$

$$v_L^{(+)\dagger}(p) v_L^{(+)}(p) = E + |\vec{p}| \simeq 2E - \frac{m^2}{2E} \quad \text{suppressed left-handed antineutrino}$$

Massless Left-Handed Neutrino Field

$$\nu_L(x) = \int \frac{d^3p}{(2\pi)^3 2E} \left[a^{(-)}(p) u_L^{(-)}(p) e^{-ip \cdot x} + b^{(+)\dagger}(p) v_L^{(+)}(p) e^{ip \cdot x} \right]$$

Left-handed neutrino: $|\nu_L(p)\rangle = |\nu(p, h = -1)\rangle = a^{(-)\dagger}(p)|0\rangle$

Right-handed antineutrino: $|\bar{\nu}_R(p)\rangle = |\bar{\nu}(p, h = +1)\rangle = b^{(+)\dagger}(p)|0\rangle$

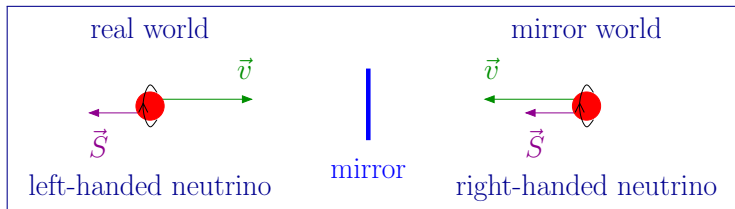
Standard Model

- ▶ Glashow (1961), Weinberg (1967) and Salam (1968) formulate the Standard Model of ElectroWeak Interactions (1979 Physics Nobel Prize) assuming that neutrinos are massless and left-handed

- ▶ Universal $V - A$ Weak Interactions

- ▶ Quantum Field Theory: $\nu_L \Rightarrow |\nu(h = -1)\rangle$ and $|\bar{\nu}(h = +1)\rangle$

- ▶ Parity is violated: $|\nu(h = -1)\rangle \xrightarrow{P} \cancel{|\nu(h = +1)\rangle}$



- ▶ Particle-Antiparticle symmetry (Charge Conjugation) is violated:

$$|\nu(h = -1)\rangle \xrightarrow{C} \cancel{|\bar{\nu}(h = +1)\rangle}$$

Majorana Equation

- ▶ Can a two-component spinor describe a massive fermion?

Yes! (E. Majorana, 1937)

- ▶ Trick: ν_R and ν_L are not independent: $\nu_R = \nu_L^c = C \bar{\nu}_L^T$
charge-conjugation matrix: $C \gamma_\mu^T C^{-1} = -\gamma_\mu$

- ▶ The relation between ν_R and ν_L must satisfy two requirements:

- ▶ It must have the correct chirality.

This is satisfied, because ν_L^c is right-handed: $P_R \nu_L^c = \nu_L^c$ $P_L \nu_L^c = 0$

- ▶ It must be compatible with the chiral Dirac equations

$$i\gamma^\mu \partial_\mu \nu_L = m \nu_R$$

$$i\gamma^\mu \partial_\mu \nu_R = m \nu_L$$

Check:

$$\begin{aligned} i\gamma^\mu \partial_\mu \nu_R &= i\gamma^\mu \partial_\mu C \bar{\nu}_L^T = i C C^{-1} \gamma^\mu C \partial_\mu \bar{\nu}_L^T = -i C (\gamma^\mu)^T \partial_\mu \bar{\nu}_L^T \\ &= -i C (\partial_\mu \bar{\nu}_L \gamma^\mu)^T = m C \bar{\nu}_R^T = m \nu_L \quad \text{OK} \end{aligned}$$

- ▶ Other relations between ψ_R and ψ_L do not satisfy the two requirements.
- ▶ For example $\psi_R = \psi_L^P = \gamma^0 \psi_L$ satisfies the chirality requirements, because $\psi_L \xrightarrow{P} \psi_R$, but

$$i\gamma^\mu \partial_\mu \psi_R = i\gamma^\mu \partial_\mu \gamma^0 \psi_L = i\gamma^0 (\gamma^\mu)^\dagger \partial_\mu \psi_L \neq i\gamma^0 \gamma^\mu \partial_\mu \psi_L = m \gamma^0 \psi_R = m \psi_L$$

- ▶ There are several relations which satisfy only the chirality requirements, for example $\psi_R = \gamma^\mu \psi_L$ for $\mu = 0, 1, 2, 3$
- ▶ There is only one adequate relation ($\psi_R = C \overline{\psi_L}^T$) that can be derived from the chiral Dirac equations: consider $i\gamma^\mu \partial_\mu \psi_R = m \psi_L$

$$\text{Hermitian conj.} \times \gamma^0 \implies -i \partial_\mu \overline{\psi_R}^\dagger (\gamma^\mu)^\dagger \gamma^0 = m \overline{\psi_L}$$

$$\gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu \implies -i \partial_\mu \overline{\psi_R} \gamma^\mu = m \overline{\psi_L}$$

$$C \times \text{transpose} \implies -i C (\gamma^\mu)^T \partial_\mu \overline{\psi_R}^T = m C \overline{\psi_L}^T$$

$$C (\gamma^\mu)^T C^{-1} = -\gamma^\mu \implies i \gamma^\mu \partial_\mu C \overline{\psi_R}^T = m C \overline{\psi_L}^T$$

Identical to $i\gamma^\mu \partial_\mu \psi_L = m \psi_R$ for $\psi_R = C \overline{\psi_L}^T \iff \psi_L = C \overline{\psi_R}^T$ (Majorana)

▶ $i\gamma^\mu \partial_\mu \nu_L = m \nu_R \quad \rightarrow \quad \boxed{i\gamma^\mu \partial_\mu \nu_L = m \nu_L^c}$ Majorana equation

▶ Majorana field: $\nu = \nu_L + \nu_R = \nu_L + \nu_L^c$

$\boxed{\nu = \nu^c}$ Majorana condition

▶ $\nu = \nu^c$ implies the equality of particle and antiparticle

▶ Only neutral fermions can be Majorana particles

▶ For a Majorana field, the electromagnetic current vanishes identically:

$$\bar{\nu} \gamma^\mu \nu = \bar{\nu}^c \gamma^\mu \nu^c = -\nu^T C^\dagger \gamma^\mu C \bar{\nu}^T = \bar{\nu} C \gamma^{\mu T} C^\dagger \nu = -\bar{\nu} \gamma^\mu \nu = 0$$

▶ Only two independent components: in the chiral representation

$$\nu = \begin{pmatrix} i\sigma^2 \chi_L^* \\ \chi_L \end{pmatrix} = \begin{pmatrix} \chi_{L2}^* \\ -\chi_{L1}^* \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$$

Majorana Lagrangian

Dirac Lagrangian

$$\begin{aligned}\mathcal{L}^D &= \bar{\nu} (i\partial - m) \nu \\ &= \bar{\nu}_L i\partial \nu_L + \bar{\nu}_R i\partial \nu_R - m(\bar{\nu}_R \nu_L + \bar{\nu}_L \nu_R)\end{aligned}$$

$$\nu_R \rightarrow \nu_L^c = C \bar{\nu}_L^T$$

$$\frac{1}{2} \mathcal{L}^D \rightarrow \bar{\nu}_L i\partial \nu_L - \frac{m}{2} \left(-\nu_L^T C^\dagger \nu_L + \bar{\nu}_L C \bar{\nu}_L^T \right)$$

Majorana Lagrangian

$$\begin{aligned}\mathcal{L}^M &= \bar{\nu}_L i\partial \nu_L - \frac{m}{2} \left(-\nu_L^T C^\dagger \nu_L + \bar{\nu}_L C \bar{\nu}_L^T \right) \\ &= \bar{\nu}_L i\partial \nu_L - \frac{m}{2} (\bar{\nu}_L^c \nu_L + \bar{\nu}_L \nu_L^c)\end{aligned}$$

▶ Majorana Lagrangian: $\mathcal{L}^M = \frac{1}{2} \bar{\nu} (i\partial - m) \nu|_{\nu=\nu^c}$

▶ Quantized Dirac Neutrino Field:

$$\nu(x) = \int \frac{d^3p}{(2\pi)^3 2E} \sum_{h=\pm 1} \left[a^{(h)}(p) u^{(h)}(p) e^{-ip \cdot x} + b^{(h)\dagger}(p) v^{(h)}(p) e^{ip \cdot x} \right]$$

▶ Quantized Majorana Neutrino Field: $b^{(h)}(p) = a^{(h)}(p)$

$$\nu(x) = \int \frac{d^3p}{(2\pi)^3 2E} \sum_{h=\pm 1} \left[a^{(h)}(p) u^{(h)}(p) e^{-ip \cdot x} + a^{(h)\dagger}(p) v^{(h)}(p) e^{ip \cdot x} \right]$$

▶ A Majorana field has half the degrees of freedom of a Dirac field

Total Lepton Number

$$\cancel{L = +1} \leftarrow \boxed{\nu = \nu^c} \rightarrow \cancel{L = -1}$$

$$\nu_L \implies L = +1$$

$$\nu_L^c \implies L = -1$$

$$\mathcal{L}^M = \bar{\nu}_L i \not{\partial} \nu_L - \frac{m}{2} (\bar{\nu}_L^c \nu_L + \bar{\nu}_L \nu_L^c)$$

Total Lepton Number is not conserved: $\boxed{\Delta L = \pm 2}$

Best process to find violation of Total Lepton Number:

Neutrinoless Double- β Decay

$$\mathcal{N}(A, Z) \rightarrow \mathcal{N}(A, Z + 2) + 2e^- + \cancel{2\nu_e} \quad (\beta\beta_{0\nu}^-)$$

$$\mathcal{N}(A, Z) \rightarrow \mathcal{N}(A, Z - 2) + 2e^+ + \cancel{2\nu_e} \quad (\beta\beta_{0\nu}^+)$$

CP Symmetry

- ▶ Under a CP transformation

$$\nu_L(x) \xrightarrow{\text{CP}} \xi_\nu^{\text{CP}} \gamma^0 \nu_L^c(x_P)$$

$$\nu_L^c(x) \xrightarrow{\text{CP}} -\xi_\nu^{\text{CP}*} \gamma^0 \nu_L(x_P)$$

$$\bar{\nu}_L(x) \xrightarrow{\text{CP}} \xi_\nu^{\text{CP}*} \bar{\nu}_L^c(x_P) \gamma^0$$

$$\bar{\nu}_L^c(x) \xrightarrow{\text{CP}} -\xi_\nu^{\text{CP}} \bar{\nu}_L(x_P) \gamma^0$$

with $|\xi_\nu^{\text{CP}}|^2 = 1$, $x^\mu = (x^0, \vec{x})$, and $x_P^\mu = (x^0, -\vec{x})$

- ▶ The theory is CP-symmetric if there are values of the phase ξ_ν^{CP} such that the Lagrangian transforms as

$$\mathcal{L}(x) \xrightarrow{\text{CP}} \mathcal{L}(x_P)$$

in order to keep invariant the action $I = \int d^4x \mathcal{L}(x)$

▶ The Majorana Mass Term

$$\mathcal{L}_{\text{mass}}^{\text{M}}(x) = -\frac{1}{2} m \left[\overline{\nu_L^c}(x) \nu_L(x) + \overline{\nu_L}(x) \nu_L^c(x) \right]$$

transforms as

$$\mathcal{L}_{\text{mass}}^{\text{M}}(x) \xrightarrow{\text{CP}} -\frac{1}{2} m \left[-(\xi_\nu^{\text{CP}})^2 \overline{\nu_L}(x_P) \nu_L^c(x_P) - (\xi_\nu^{\text{CP}*})^2 \overline{\nu_L^c}(x_P) \nu_L(x_P) \right]$$

▶ $\mathcal{L}_{\text{mass}}^{\text{M}}(x) \xrightarrow{\text{CP}} \mathcal{L}_{\text{mass}}^{\text{M}}(x_P)$ for $\xi_\nu^{\text{CP}} = \pm i$

▶ The one-generation Majorana theory is CP-symmetric

▶ The Majorana case is different from the Dirac case, in which the CP phase ξ_ν^{CP} is arbitrary

No Majorana Neutrino Mass in the SM

- ▶ Majorana Mass Term $\propto [\nu_L^T C^\dagger \nu_L - \bar{\nu}_L C \bar{\nu}_L^T]$ involves only the neutrino left-handed chiral field ν_L , which is present in the SM
- ▶ Eigenvalues of the weak isospin I , of its third component I_3 , of the hypercharge Y and of the charge Q of the lepton and Higgs multiplets:

		I	I_3	Y	$Q = I_3 + \frac{Y}{2}$
lepton doublet	$L_L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}$	1/2	1/2 -1/2	-1	0 -1
lepton singlet	ℓ_R	0	0	-2	-1
Higgs doublet	$\Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}$	1/2	1/2 -1/2	+1	1 0

- ▶ $\nu_L^T C^\dagger \nu_L$ has $I_3 = 1$ and $Y = -2 \implies$ needed $Y = 2$ Higgs triplet ($I = 1$, $I_3 = -1$)
- ▶ Compare with Dirac Mass Term $\propto \bar{\nu}_R \nu_L$ with $I_3 = 1/2$ and $Y = -1$ balanced by $\phi_0 \rightarrow \nu$ with $I_3 = -1/2$ and $Y = +1$

Confusing Majorana Antineutrino Terminology

- ▶ A Majorana neutrino is the same as a Majorana antineutrino
- ▶ Neutrino interactions are described by the CC and NC Lagrangians

$$\mathcal{L}_{l,L}^{\text{CC}} = -\frac{g}{\sqrt{2}} \left(\bar{\nu}_L \gamma^\mu \ell_L W_\mu + \bar{\ell}_L \gamma^\mu \nu_L W_\mu^\dagger \right)$$

$$\mathcal{L}_{l,\nu}^{\text{NC}} = -\frac{g}{2 \cos \vartheta_W} \bar{\nu}_L \gamma^\mu \nu_L Z_\mu$$

- ▶ Dirac: ν_L $\left\{ \begin{array}{l} \text{destroys left-handed neutrinos} \\ \text{creates right-handed antineutrinos} \end{array} \right.$
- ▶ Majorana: ν_L $\left\{ \begin{array}{l} \text{destroys left-handed neutrinos} \\ \text{creates right-handed neutrinos} \end{array} \right.$
- ▶ Common implicit definitions:
 - left-handed Majorana neutrino \equiv neutrino
 - right-handed Majorana neutrino \equiv antineutrino

Effective Majorana Mass

- ▶ Dimensional analysis: Fermion Field $\sim [E]^{3/2}$ Boson Field $\sim [E]$
- ▶ Dimensionless action: $I = \int d^4x \mathcal{L}(x) \implies \mathcal{L}(x) \sim [E]^4$
- ▶ Kinetic terms: $\bar{\psi}i\not{\partial}\psi \sim [E]^4$, $(\partial_\mu\phi)^\dagger \partial^\mu\phi \sim [E]^4$
- ▶ Mass terms: $m\bar{\psi}\psi \sim [E]^4$, $m^2\phi^\dagger\phi \sim [E]^4$
- ▶ CC weak interaction: $g\bar{\nu}_L\gamma^\rho\ell_L W_\rho \sim [E]^4$
- ▶ Yukawa couplings: $y\bar{L}_L\Phi\ell_R \sim [E]^4$
- ▶ Product of fields \mathcal{O}_d with energy dimension $d \equiv \text{dim-}d$ operator
- ▶ $\mathcal{L}(\mathcal{O}_d) = C_{(\mathcal{O}_d)}\mathcal{O}_d \implies C_{(\mathcal{O}_d)} \sim [E]^{4-d}$
- ▶ $\mathcal{O}_{d>4}$ are not renormalizable

- ▶ SM Lagrangian includes all $\mathcal{O}_{d \leq 4}$ invariant under $SU(2)_L \times U(1)_Y$
- ▶ SM cannot be considered as the final theory of everything
- ▶ SM is an effective low-energy theory
- ▶ It is likely that SM is the low-energy product of the symmetry breaking of a high-energy unified theory
- ▶ It is plausible that at low-energy there are effective non-renormalizable $\mathcal{O}_{d > 4}$ [S. Weinberg, Phys. Rev. Lett. 43 (1979) 1566]
- ▶ All \mathcal{O}_d must respect $SU(2)_L \times U(1)_Y$, because they are generated by the high-energy theory which must include the gauge symmetries of the SM in order to be effectively reduced to the SM at low energies

- ▶ $\mathcal{O}_{d>4}$ is suppressed by a coefficient \mathcal{M}^{4-d} , where \mathcal{M} is a heavy mass characteristic of the symmetry breaking scale of the high-energy unified theory:

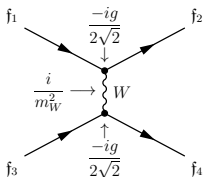
$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{g_5}{\mathcal{M}} \mathcal{O}_5 + \frac{g_6}{\mathcal{M}^2} \mathcal{O}_6 + \dots$$

- ▶ Analogy with Fermi effective low-energy theory of weak interactions:

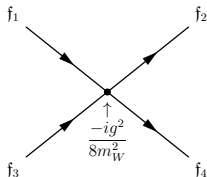
$$\mathcal{L}_{\text{eff}}^{(\text{CC})} \propto G_{\text{F}} (\bar{\nu}_{eL} \gamma^\rho e_L) (\bar{e}_L \gamma_\rho \nu_{eL}) + \dots$$

$$\mathcal{O}_6 \rightarrow (\bar{\nu}_{eL} \gamma^\rho e_L) (\bar{e}_L \gamma_\rho \nu_{eL}) + \dots \quad \frac{g_6}{\mathcal{M}^2} \rightarrow \frac{G_{\text{F}}}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

$$G_{\mu\nu}^{(W)}(p) = i \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_W^2}}{p^2 - m_W^2} \xrightarrow{p^2 \ll m_W^2} i \frac{g_{\mu\nu}}{m_W^2}$$



\Rightarrow



- ▶ \mathcal{M}^{4-d} is a strong suppression factor which limits the observability of the low-energy effects of the new physics beyond the SM
- ▶ The difficulty to observe the effects of the effective low-energy non-renormalizable operators increase rapidly with their dimensionality
- ▶ $\mathcal{O}_5 \implies$ Majorana neutrino masses (Lepton number violation)
- ▶ $\mathcal{O}_6 \implies$ Baryon number violation (proton decay)
- ▶ Majorana neutrino masses provide the most accessible low-energy window on new physics beyond the SM.
- ▶ Indeed, the existence of neutrino masses is the first and so far the only well established phenomenon beyond the SM.

- ▶ The only $SU(2)_L \times U(1)_Y$ invariant dim-5 Lagrangian term that can be constructed with SM fields:

$$\mathcal{L}_5 = -\frac{g_5}{\mathcal{M}} \left[\left(\overline{L}_L \tilde{\Phi} \right) \left(\tilde{\Phi}^T L_L^c \right) + \left(\overline{L}_L^c \tilde{\Phi}^* \right) \left(\tilde{\Phi}^\dagger L_L \right) \right]$$

- ▶ Electroweak Symmetry Breaking:

$$\tilde{\Phi} = i\sigma_2 \Phi^* \xrightarrow[\text{Breaking}]{\text{EW Symmetry}} \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

- ▶ $\mathcal{L}_5 \xrightarrow[\text{Breaking}]{\text{EW Symmetry}} \mathcal{L}_{\text{mass}}^M = -\frac{1}{2} \frac{g_5 v^2}{\mathcal{M}} (\overline{\nu}_L \nu_L^c + \overline{\nu}_L^c \nu_L)$

- ▶ Majorana neutrino mass:

$$m = \frac{g_5 v^2}{\mathcal{M}}$$

► General Seesaw Mechanism: $m \propto \frac{v^2}{\mathcal{M}} = v \frac{v}{\mathcal{M}}$

natural explanation of the strong suppression of neutrino masses with respect to the electroweak scale

► Example: $\mathcal{M} \sim 10^{15} \text{ GeV}$ (GUT scale)

$$v \sim 10^2 \text{ GeV} \implies \frac{v}{\mathcal{M}} \sim 10^{-13} \implies m \sim 10^{-2} \text{ eV}$$

Mixing of Three Majorana Neutrinos

► $\nu'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix}$

$$\mathcal{L}_{\text{mass}}^M = \frac{1}{2} \nu'^T_L C^\dagger M^L \nu'_L + \text{H.c.}$$
$$= \frac{1}{2} \sum_{\alpha, \beta=e, \mu, \tau} \nu'^T_{\alpha L} C^\dagger M^L_{\alpha\beta} \nu'_{\beta L} + \text{H.c.}$$

► In general, the matrix M^L is a complex symmetric matrix

$$\begin{aligned} \sum_{\alpha, \beta} \nu'^T_{\alpha L} C^\dagger M^L_{\alpha\beta} \nu'_{\beta L} &= \sum_{\alpha, \beta} \left(\nu'^T_{\alpha L} C^\dagger M^L_{\alpha\beta} \nu'_{\beta L} \right)^T \\ &= - \sum_{\alpha, \beta} \nu'_{\beta L} M^L_{\alpha\beta} (C^\dagger)^T \nu'_{\alpha L} = \sum_{\alpha, \beta} \nu'^T_{\beta L} C^\dagger M^L_{\alpha\beta} \nu'_{\alpha L} \\ &= \sum_{\alpha, \beta} \nu'^T_{\alpha L} C^\dagger M^L_{\beta\alpha} \nu'_{\beta L} \end{aligned}$$

$$M^L_{\alpha\beta} = M^L_{\beta\alpha} \iff M^L = M^{LT}$$

▶ $\mathcal{L}_{\text{mass}}^{\text{M}} = \frac{1}{2} \nu_L'^T C^\dagger M^L \nu_L' + \text{H.c.}$

▶ $\nu_L' = V_L^\nu \mathbf{n}_L \quad \Rightarrow \quad \mathcal{L}_{\text{mass}}^{\text{M}} = \frac{1}{2} \nu_L'^T (V_L^\nu)^T C^\dagger M^L V_L^\nu \nu_L' + \text{H.c.}$

▶ $(V_L^\nu)^T M^L V_L^\nu = M, \quad M_{kj} = m_k \delta_{kj} \quad (k, j = 1, 2, 3)$

▶ Left-handed chiral fields with definite mass: $\mathbf{n}_L = V_L^{\nu\dagger} \nu_L' = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$

$$\begin{aligned} \mathcal{L}_{\text{mass}}^{\text{M}} &= \frac{1}{2} \left(\mathbf{n}_L^T C^\dagger M \mathbf{n}_L - \overline{\mathbf{n}}_L M C \mathbf{n}_L^T \right) \\ &= \frac{1}{2} \sum_{k=1}^3 m_k \left(\nu_{kL}^T C^\dagger \nu_{kL} - \overline{\nu}_{kL} C \nu_{kL}^T \right) \end{aligned}$$

▶ Majorana fields of massive neutrinos: $\nu_k = \nu_{kL} + \nu_{kL}^c$

$\nu_k^c = \nu_k$

▶ $\mathbf{n} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \Rightarrow \mathcal{L}^{\text{M}} = \frac{1}{2} \sum_{k=1}^3 \overline{\nu}_k (i\not{\partial} - m_k) \nu_k |_{\nu_k = \nu_k^c}$

Mixing Matrix

- ▶ Leptonic Weak Charged Current:

$$j_{W,L}^{\rho\dagger} = 2 \bar{\ell}_L \gamma^\rho U \mathbf{n}_L \quad \text{with} \quad U = V_L^{\ell\dagger} V_L^\nu$$

- ▶ As in the Dirac case, we define the left-handed flavor neutrino fields as

$$\nu_L = U \mathbf{n}_L = V_L^{\ell\dagger} \nu'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$$

- ▶ In this way, as in the Dirac case, the Leptonic Weak Charged Current has the SM form

$$j_{W,L}^{\rho\dagger} = 2 \bar{\ell}_L \gamma^\rho \nu_L = 2 \sum_{\alpha=e,\mu,\tau} \bar{\ell}_{\alpha L} \gamma^\rho \nu_{\alpha L}$$

- ▶ Important difference with respect to Dirac case:
Two additional CP-violating phases: Majorana phases

- Majorana Mass Term $\mathcal{L}^M = \frac{1}{2} \sum_{k=1}^3 m_k \nu_{kL}^T C^\dagger \nu_{kL} + \text{H.c.}$ is not invariant under the global U(1) gauge transformations

$$\nu_{kL} \rightarrow e^{i\varphi_k} \nu_{kL} \quad (k = 1, 2, 3)$$

- For eliminating some of the 6 phases of the unitary mixing matrix we can use only the global phase transformations (3 arbitrary phases)

$$l_\alpha \rightarrow e^{i\varphi_\alpha} l_\alpha \quad (\alpha = e, \mu, \tau)$$

▶ Weak Charged Current:
$$j_{W,L}^{\rho\dagger} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^3 \overline{l_{\alpha L}} \gamma^\rho U_{\alpha k} \nu_{kL}$$

▶ Performing the transformation $l_\alpha \rightarrow e^{i\varphi_\alpha} l_\alpha$ we obtain

$$j_{W,L}^{\rho\dagger} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^3 \overline{l_{\alpha L}} e^{-i\varphi_\alpha} \gamma^\rho U_{\alpha k} \nu_{kL}$$

$$j_{W,L}^{\rho\dagger} = 2 \underbrace{e^{-i\varphi_e}}_1 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^3 \overline{l_{\alpha L}} \underbrace{e^{-i(\varphi_\alpha - \varphi_e)}}_2 \gamma^\rho U_{\alpha k} \nu_{kL}$$

▶ We can eliminate **3 phases** of the mixing matrix: one overall phase and two phases which can be factorized on the left.

▶ In the Dirac case we could eliminate also two phases which can be factorized on the right.

- ▶ In the Majorana case there are two additional physical Majorana phases which can be factorized on the right of the mixing matrix:

$$U = U^D D^M \quad D^M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_2} & 0 \\ 0 & 0 & e^{i\lambda_3} \end{pmatrix}$$

- ▶ U^D is a Dirac mixing matrix, with one Dirac phase
- ▶ Standard parameterization:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_2} & 0 \\ 0 & 0 & e^{i\lambda_3} \end{pmatrix}$$

▶ $D^M = \text{diag}(e^{i\lambda_1}, e^{i\lambda_2}, e^{i\lambda_3})$, but only two Majorana phases are physical

▶ All measurable quantities depend only on the differences of the Majorana phases because $e^{i(\lambda_k - \lambda_j)}$ remains constant under the allowed phase transformation

$$l_\alpha \rightarrow e^{i\varphi} l_\alpha \implies e^{i\lambda_k} \rightarrow e^{i(\lambda_k - \varphi)}$$

▶ Our convention: $\lambda_1 = 0 \implies D^M = \text{diag}(1, e^{i\lambda_2}, e^{i\lambda_3})$

▶ CP is conserved if all the elements of each column of the mixing matrix are either real or purely imaginary:

$$\delta_{13} = 0 \text{ or } \pi \quad \text{and} \quad \lambda_k = 0 \text{ or } \pi/2 \text{ or } \pi \text{ or } 3\pi/2$$

Dirac-Majorana Mass Term

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
 - One Generation Dirac-Majorana Mass Term
 - Type-I Seesaw Mechanism
 - Three-Generation Mixing
- Sterile Neutrinos

One Generation Dirac-Majorana Mass Term

If ν_R exists, the most general mass term is the

Dirac-Majorana Mass Term

$$\mathcal{L}_{\text{mass}}^{\text{D+M}} = \mathcal{L}_{\text{mass}}^{\text{D}} + \mathcal{L}_{\text{mass}}^{\text{L}} + \mathcal{L}_{\text{mass}}^{\text{R}}$$

$$\mathcal{L}_{\text{mass}}^{\text{D}} = -m_{\text{D}} \bar{\nu}_R \nu_L + \text{H.c.}$$

Standard Dirac Mass Term

$$\mathcal{L}_{\text{mass}}^{\text{L}} = \frac{1}{2} m_{\text{L}} \nu_L^T C^\dagger \nu_L + \text{H.c.}$$

ν_L Majorana Mass Term
Forbidden in the SM

$$\mathcal{L}_{\text{mass}}^{\text{R}} = \frac{1}{2} m_{\text{R}} \nu_R^T C^\dagger \nu_R + \text{H.c.}$$

ν_R Majorana Mass Term
Allowed in the SM

▶ Column matrix of left-handed chiral fields: $N_L = \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} = \begin{pmatrix} \nu_L \\ C \bar{\nu}_R^T \end{pmatrix}$

$$\mathcal{L}_{\text{mass}}^{\text{D+M}} = \frac{1}{2} N_L^T C^\dagger M N_L + \text{H.c.} \quad M = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}$$

▶ The Dirac-Majorana Mass Term has the structure of a Majorana Mass Term for two chiral neutrino fields coupled by the Dirac mass

▶ Diagonalization: $n_L = U^\dagger N_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix}$

$$U^T M U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad \text{Real } m_k \geq 0$$

▶ $\mathcal{L}_{\text{mass}}^{\text{D+M}} = \frac{1}{2} \sum_{k=1,2} m_k \nu_{kL}^T C^\dagger \nu_{kL} + \text{H.c.} = -\frac{1}{2} \sum_{k=1,2} m_k \bar{\nu}_k \nu_k$

$$\nu_k = \nu_{kL} + \nu_{kL}^c$$

▶ Massive neutrinos are Majorana! $\nu_k = \nu_k^c$

Real Mass Matrix

▶ CP is conserved if the mass matrix is real: $M = M^*$

▶ $M = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}$ we consider real and positive m_R and m_D and real m_L

▶ A real symmetric mass matrix can be diagonalized with $U = \mathcal{O} \rho$

$$\mathcal{O} = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix} \quad \rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \quad \rho_k^2 = \pm 1$$

▶ $\mathcal{O}^T M \mathcal{O} = \begin{pmatrix} m'_1 & 0 \\ 0 & m'_2 \end{pmatrix}$ $\tan 2\vartheta = \frac{2m_D}{m_R - m_L}$

$$m'_{2,1} = \frac{1}{2} \left[m_L + m_R \pm \sqrt{(m_L - m_R)^2 + 4m_D^2} \right]$$

▶ m'_1 is negative if $m_L m_R < m_D^2$

$$U^T M U = \rho^T \mathcal{O}^T M \mathcal{O} \rho = \begin{pmatrix} \rho_1^2 m'_1 & 0 \\ 0 & \rho_2^2 m'_2 \end{pmatrix} \implies \boxed{m_k = \rho_k^2 m'_k}$$

- ▶ m'_2 is always positive:

$$m_2 = m'_2 = \frac{1}{2} \left[m_L + m_R + \sqrt{(m_L - m_R)^2 + 4 m_D^2} \right]$$

- ▶ If $m_L m_R \geq m_D^2$, then $m'_1 \geq 0$ and $\rho_1^2 = 1$

$$m_1 = \frac{1}{2} \left[m_L + m_R - \sqrt{(m_L - m_R)^2 + 4 m_D^2} \right]$$

$$\rho_1 = 1 \text{ and } \rho_2 = 1 \implies U = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix}$$

- ▶ If $m_L m_R < m_D^2$, then $m'_1 < 0$ and $\rho_1^2 = -1$

$$m_1 = \frac{1}{2} \left[\sqrt{(m_L - m_R)^2 + 4 m_D^2} - (m_L + m_R) \right]$$

$$\rho_1 = i \text{ and } \rho_2 = 1 \implies U = \begin{pmatrix} i \cos \vartheta & \sin \vartheta \\ -i \sin \vartheta & \cos \vartheta \end{pmatrix}$$

- ▶ If Δm^2 is small, there are oscillations between active ν_a generated by ν_L and sterile ν_s generated by ν_R^c :

$$P_{\nu_a \rightarrow \nu_s}(L, E) = \sin^2 2\vartheta \sin^2 \left(\frac{\Delta m^2 L}{4 E} \right)$$

$$\Delta m^2 = m_2^2 - m_1^2 = (m_L + m_R) \sqrt{(m_L - m_R)^2 + 4 m_D^2}$$

- ▶ It can be shown that the CP parity of ν_k is $\xi_k^{\text{CP}} = i \rho_k^2$:

$$\nu_k(x) \xrightarrow{\text{CP}} \xi_k^{\text{CP}} \gamma^0 \overline{\nu_k}^T(x_P) \quad \xi_1^{\text{CP}} = i \rho_1^2 \quad \xi_2^{\text{CP}} = i$$

- ▶ Special cases:

- ▶ $m_L = m_R \implies$ Maximal Mixing
- ▶ $m_L = m_R = 0 \implies$ Dirac Limit
- ▶ $|m_L|, m_R \ll m_D \implies$ Pseudo-Dirac Neutrinos
- ▶ $m_L = 0 \quad m_D \ll m_R \implies$ Seesaw Mechanism

Maximal Mixing

$$m_L = m_R$$

$$\tan 2\vartheta = \frac{2m_D}{m_R - m_L} \implies \vartheta = \pi/4$$

$$m'_{2,1} = m_L \pm m_D$$

$$\left\{ \begin{array}{ll} \rho_1^2 = +1, & m_1 = m_L - m_D \quad \text{if } m_L \geq m_D \\ \rho_1^2 = -1, & m_1 = m_D - m_L \quad \text{if } m_L < m_D \end{array} \right.$$
$$m_2 = m_L + m_D$$

$$\underline{m_L < m_D}$$

$$\left\{ \begin{array}{l} \nu_{1L} = \frac{-i}{\sqrt{2}} (\nu_L - \nu_R^c) \\ \nu_{2L} = \frac{1}{\sqrt{2}} (\nu_L + \nu_R^c) \end{array} \right.$$

$$\left\{ \begin{array}{l} \nu_1 = \nu_{1L} + \nu_{1L}^c = \frac{-i}{\sqrt{2}} [(\nu_L + \nu_R) - (\nu_L^c + \nu_R^c)] \\ \nu_2 = \nu_{2L} + \nu_{2L}^c = \frac{1}{\sqrt{2}} [(\nu_L + \nu_R) + (\nu_L^c + \nu_R^c)] \end{array} \right.$$

Dirac Limit

$$m_L = m_R = 0$$

$$\blacktriangleright m'_{2,1} = \pm m_D \implies \begin{cases} \rho_1^2 = -1 & m_1 = m_D \\ \rho_2^2 = +1 & m_2 = m_D \end{cases} \quad \begin{cases} \xi_1^{\text{CP}} = -i \\ \xi_2^{\text{CP}} = i \end{cases}$$

- \blacktriangleright The two Majorana fields ν_1 and ν_2 can be combined to give one Dirac field:

$$\nu = \frac{1}{\sqrt{2}} (i\nu_1 + \nu_2) = \nu_L + \nu_R$$

- \blacktriangleright A Dirac field ν can always be split in two Majorana fields:

$$\begin{aligned} \nu &= \frac{1}{2} [(\nu - \nu^c) + (\nu + \nu^c)] \\ &= \frac{i}{\sqrt{2}} \left(-i \frac{\nu - \nu^c}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} \left(\frac{\nu + \nu^c}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} (i\nu_1 + \nu_2) \end{aligned}$$

- \blacktriangleright A Dirac field is equivalent to two Majorana fields with the same mass and opposite CP parities

Pseudo-Dirac Neutrinos

$$|m_L|, m_R \ll m_D$$

$$\blacktriangleright m'_{2,1} \simeq \frac{m_L + m_R}{2} \pm m_D$$

$$\blacktriangleright m'_1 < 0 \implies \rho_1^2 = -1 \implies m_{2,1} \simeq m_D \pm \frac{m_L + m_R}{2}$$

\blacktriangleright The two massive Majorana neutrinos are almost degenerate in mass and have opposite CP parities ($\xi_1^{\text{CP}} = -i$, $\xi_2^{\text{CP}} = i$)

\blacktriangleright The best way to reveal pseudo-Dirac neutrinos are active-sterile neutrino oscillations due to the small squared-mass difference

$$\Delta m^2 \simeq m_D (m_L + m_R)$$

\blacktriangleright The oscillations occur with practically maximal mixing:

$$\tan 2\vartheta = \frac{2m_D}{m_R - m_L} \gg 1 \implies \vartheta \simeq \pi/4$$

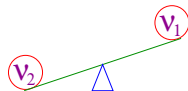
Type-I Seesaw Mechanism

[Minkowski, PLB 67 (1977) 42; Yanagida (1979); Gell-Mann, Ramond, Slansky (1979); Mohapatra, Senjanovic, PRL 44 (1980) 912]

$$m_L = 0 \quad m_D \ll m_R$$

- ▶ $\mathcal{L}_{\text{mass}}^L$ is forbidden in the SM $\implies m_L = 0$
- ▶ $m_D \lesssim v \sim 100 \text{ GeV}$ is generated by SM Higgs Mechanism (protected by SM symmetries)
- ▶ m_R is not protected by SM symmetries $\implies m_R \sim \mathcal{M}_{\text{GUT}} \gg v$

$$\left. \begin{array}{l} m'_1 \simeq -\frac{m_D^2}{m_R} \\ m'_2 \simeq m_R \end{array} \right\} \implies \left\{ \begin{array}{l} \rho_1^2 = -1, \quad m_1 \simeq \frac{m_D^2}{m_R} \\ \rho_2^2 = +1, \quad m_2 \simeq m_R \end{array} \right.$$



- ▶ Natural explanation of smallness of neutrino masses
- ▶ Mixing angle is very small: $\tan 2\vartheta = 2 \frac{m_D}{m_R} \ll 1$
- ▶ ν_1 is composed mainly of active ν_L : $\nu_{1L} \simeq -i\nu_L$
- ▶ ν_2 is composed mainly of sterile ν_R : $\nu_{2L} \simeq \nu_R^c$

Connection with Effective Lagrangian Approach

- ▶ Dirac–Majorana neutrino mass term with $m_L = 0$:

$$\mathcal{L}^{\text{D+M}} = -m_D (\bar{\nu}_R \nu_L + \bar{\nu}_L \nu_R) + \frac{1}{2} m_R (\nu_R^T C^\dagger \nu_R + \nu_R^\dagger C \nu_R^*)$$

- ▶ Above the electroweak symmetry-breaking scale:

$$\mathcal{L}^{\text{D+M}} = -y^\nu (\bar{\nu}_R \tilde{\Phi}^\dagger L_L + \bar{L}_L \tilde{\Phi} \nu_R) + \frac{1}{2} m_R (\nu_R^T C^\dagger \nu_R + \nu_R^\dagger C \nu_R^*)$$

- ▶ If $m_R \gg v \implies \nu_R$ is static \implies kinetic term in equation of motion can be neglected:

$$0 \simeq \frac{\partial \mathcal{L}^{\text{D+M}}}{\partial \nu_R} = m_R \nu_R^T C^\dagger - y^\nu \bar{L}_L \tilde{\Phi}$$

$$\nu_R \simeq -\frac{y^\nu}{m_R} \tilde{\Phi}^T C \bar{L}_L^T$$

$$\mathcal{L}^{\text{D+M}} \rightarrow \mathcal{L}_5^{\text{D+M}} \simeq -\frac{1}{2} \frac{(y^\nu)^2}{m_R} (L_L^T \sigma_2 \Phi) C^\dagger (\Phi^T \sigma_2 L_L) + \text{H.c.}$$

$$\mathcal{L}_5 = \frac{g}{\mathcal{M}} (L_L^T \sigma_2 \Phi) C^\dagger (\Phi^T \sigma_2 L_L) + \text{H.c.}$$

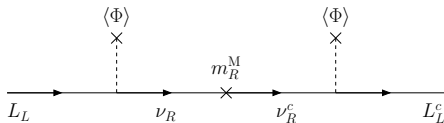
$$\mathcal{L}_5^{\text{D+M}} \simeq -\frac{1}{2} \frac{(y^\nu)^2}{m_R} (L_L^T \sigma_2 \Phi) C^\dagger (\Phi^T \sigma_2 L_L) + \text{H.c.}$$

$$g = -\frac{(y^\nu)^2}{2} \quad \mathcal{M} = m_R$$

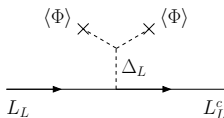
- ▶ See-saw mechanism is a particular case of the effective Lagrangian approach.
- ▶ See-saw mechanism is obtained when dimension-five operator is generated only by the presence of ν_R with $m_R \sim \mathcal{M}$.
- ▶ In general, other terms can contribute to \mathcal{L}_5 .

Three Seesaw Types

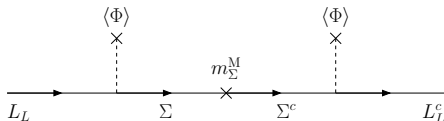
- ▶ Since combining the two doublets L_L and Φ one can form singlets and triplets, there are three types of Seesaw types that can be generated at the tree level.
- ▶ **Type-I Seesaw:** intermediate fermion singlets ν_R



- ▶ **Type-II Seesaw:** coupling with boson triplets Δ_L



- ▶ **Type-III Seesaw:** intermediate fermion triplets Σ



Generalized Seesaw Mechanism

- ▶ General effective Dirac-Majorana mass matrix:

$$M = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}$$

- ▶ m_L generated by dim-5 operator:

$$m_L \ll m_D \ll m_R$$

- ▶ Eigenvalues:

$$\begin{vmatrix} m_L - \mu & m_D \\ m_D & m_R - \mu \end{vmatrix} = 0$$

$$\mu^2 - (\cancel{m_L} + m_R)\mu + m_L m_R - m_D^2 = 0$$

$$\mu = \frac{1}{2} \left[m_R \pm \sqrt{m_R^2 - 4(m_L m_R - m_D^2)} \right]$$

$$\begin{aligned}
 \mu &= \frac{1}{2} \left[m_R \pm \sqrt{m_R^2 - 4(m_L m_R - m_D^2)} \right] \\
 &= \frac{1}{2} \left[m_R \pm m_R \left(1 - 4 \frac{m_L m_R - m_D^2}{m_R^2} \right)^{1/2} \right] \\
 &\simeq \frac{1}{2} \left[m_R \pm m_R \left(1 - 2 \frac{m_L m_R - m_D^2}{m_R^2} \right) \right]
 \end{aligned}$$

$$+ \rightarrow m_{\text{heavy}} \simeq m_R$$

$$- \rightarrow m_{\text{light}} \simeq \left| m_L - \frac{m_D^2}{m_R} \right|$$

Dominant type I seesaw: $m_L \ll \frac{m_D^2}{m_R} \implies m_{\text{light}} \simeq \frac{m_D^2}{m_R}$

Dominant type II or III seesaw: $m_L \gg \frac{m_D^2}{m_R} \implies m_{\text{light}} \simeq m_L$

Right-Handed Neutrino Mass Term

$$\mathcal{L}_R^M = -\frac{1}{2} m (\overline{\nu}_R^c \nu_R + \overline{\nu}_R \nu_R^c)$$

- ▶ \mathcal{L}_R^M respects the $SU(2)_L \times U(1)_Y$ SM symmetry
- ▶ \mathcal{L}_R^M breaks Lepton number conservation

Three possibilities:

- ▶ Lepton number can be explicitly broken
- ▶ Lepton number is spontaneously broken locally, with a massive vector boson coupled to the lepton number current
- ▶ Lepton number is spontaneously broken globally and a massless Goldstone boson appears in the theory (Majoron)

Singlet Majoron Model

[Chikashige, Mohapatra, Peccei, Phys. Lett. B98 (1981) 265, Phys. Rev. Lett. 45 (1980) 1926]

$$\mathcal{L}_\Phi = -y_d (\overline{L}_L \Phi \nu_R + \overline{\nu}_R \Phi^\dagger L_L) \xrightarrow{\langle \Phi \rangle \neq 0} -m_D (\overline{\nu}_L \nu_R + \overline{\nu}_R \nu_L)$$

$$\mathcal{L}_\eta = -y_s (\eta \overline{\nu}_R^c \nu_R + \eta^\dagger \overline{\nu}_R \nu_R^c) \xrightarrow{\langle \eta \rangle \neq 0} -\frac{1}{2} m_R (\overline{\nu}_R^c \nu_R + \overline{\nu}_R \nu_R^c)$$

$$\eta = 2^{-1/2} (\langle \eta \rangle + \rho + i\chi) \quad \mathcal{L}_{\text{mass}} = -\frac{1}{2} (\overline{\nu}_L^c \ \overline{\nu}_R) \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \text{H.c.}$$

$$m_R \gg m_D \implies \text{Type-I Seesaw: } m_1 \simeq \frac{m_D^2}{m_R}$$

scale of L violation EW scale

$\rho =$ massive scalar, $\chi =$ Majoron (massless pseudoscalar Goldstone boson)

The Majoron is weakly coupled to the light neutrino

$$\mathcal{L}_{\chi-\nu} = \frac{iy_s}{\sqrt{2}} \chi \left[\overline{\nu}_2 \gamma^5 \nu_2 - \frac{m_D}{m_R} (\overline{\nu}_2 \gamma^5 \nu_1 + \overline{\nu}_1 \gamma^5 \nu_2) + \left(\frac{m_D}{m_R} \right)^2 \overline{\nu}_1 \gamma^5 \nu_1 \right]$$

Three-Generation Mixing

$$\mathcal{L}_{\text{mass}}^{\text{D+M}} = \mathcal{L}_{\text{mass}}^{\text{D}} + \mathcal{L}_{\text{mass}}^{\text{L}} + \mathcal{L}_{\text{mass}}^{\text{R}}$$

$$\mathcal{L}_{\text{mass}}^{\text{D}} = - \sum_{s=1}^{N_S} \sum_{\alpha=e,\mu,\tau} \overline{\nu'_{sR}} M_{s\alpha}^{\text{D}} \nu'_{\alpha L} + \text{H.c.}$$

$$\mathcal{L}_{\text{mass}}^{\text{L}} = \frac{1}{2} \sum_{\alpha,\beta=e,\mu,\tau} \nu'_{\alpha L} C^\dagger M_{\alpha\beta}^{\text{L}} \nu'_{\beta L} + \text{H.c.}$$

$$\mathcal{L}_{\text{mass}}^{\text{R}} = \frac{1}{2} \sum_{s,s'=1}^{N_S} \nu'_{sR} C^\dagger M_{ss'}^{\text{R}} \nu'_{s'R} + \text{H.c.}$$

$$\mathbf{N}'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \\ \nu'_{1R} \\ \vdots \\ \nu'_{N_S R} \end{pmatrix} \quad \nu'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \quad \nu'^{\text{C}}_R \equiv \begin{pmatrix} \nu'_{1R} \\ \vdots \\ \nu'_{N_S R} \end{pmatrix}$$

$$\mathcal{L}_{\text{mass}}^{\text{D+M}} = \frac{1}{2} \mathbf{N}'_L{}^T C^\dagger M^{\text{D+M}} \mathbf{N}'_L + \text{H.c.} \quad M^{\text{D+M}} = \begin{pmatrix} M^{\text{L}} & M^{\text{D}T} \\ M^{\text{D}} & M^{\text{R}} \end{pmatrix}$$

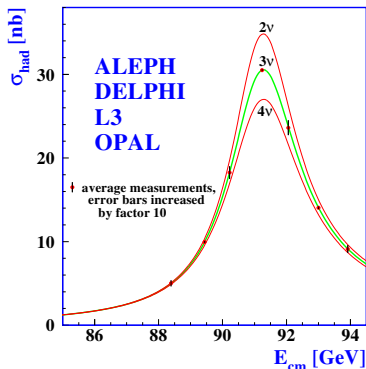
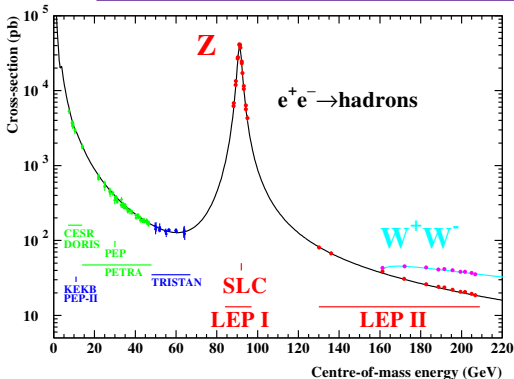
- ▶ Diagonalization of the **Dirac-Majorana Mass Term** \implies massive Majorana neutrinos
- ▶ **See-Saw Mechanism** \implies right-handed neutrinos have large Majorana masses and are decoupled from the low-energy phenomenology.
- ▶ If all right-handed neutrinos have large Majorana masses, at low energy we have an effective mixing of three Majorana neutrinos.
- ▶ It is possible that not all right-handed neutrinos have large Majorana masses: some right-handed neutrinos may correspond to low-energy Majorana particles which belong to new physics beyond the Standard Model.
- ▶ Light anti- ν_R are called **sterile neutrinos**

$$\nu_R^c \rightarrow \nu_{sL} \quad (\text{left-handed})$$

Sterile Neutrinos

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
- Sterile Neutrinos
 - Number of Flavor and Massive Neutrinos?
 - Sterile Neutrinos
 - Fundamental Fields in QFT

Number of Flavor and Massive Neutrinos?



[LEP, Phys. Rept. 427 (2006) 257, arXiv:hep-ex/0509008]

$$\Gamma_Z = \sum_{\ell=e,\mu,\tau} \Gamma_{Z \rightarrow \ell\bar{\ell}} + \sum_{q \neq t} \Gamma_{Z \rightarrow q\bar{q}} + \Gamma_{\text{inv}}$$

$$\Gamma_{\text{inv}} = N_\nu \Gamma_{Z \rightarrow \nu\bar{\nu}}$$

$$N_\nu = 2.9840 \pm 0.0082$$

Improved cross section: $N_\nu = 2.9975 \pm 0.0074$

[Janot, Jadach, arXiv:1912.02067]

$$e^+ e^- \rightarrow Z \xrightarrow{\text{invisible}} \sum_{a=\text{active}} \nu_a \bar{\nu}_a \implies \nu_e \nu_\mu \nu_\tau$$

3 light active flavor neutrinos

mixing $\implies \nu_{\alpha L} = \sum_{k=1}^N U_{\alpha k} \nu_{kL} \quad \alpha = e, \mu, \tau$ $N \geq 3$
no upper limit!

Mass Basis:	ν_1	ν_2	ν_3	ν_4	ν_5	\dots
Flavor Basis:	ν_e	ν_μ	ν_τ	ν_{s_1}	ν_{s_2}	\dots
	ACTIVE			STERILE		

$$\nu_{\alpha L} = \sum_{k=1}^N U_{\alpha k} \nu_{kL} \quad \alpha = e, \mu, \tau, s_1, s_2, \dots$$

Sterile Neutrinos

- ▶ Sterile means no standard model interactions

[Pontecorvo, Sov. Phys. JETP 26 (1968) 984]

- ▶ Obviously no electromagnetic interactions as normal active neutrinos
- ▶ Thus sterile means no standard weak interactions
- ▶ But sterile neutrinos are not absolutely sterile:
 - ▶ Gravitational Interactions
 - ▶ New non-standard interactions of the physics beyond the Standard Model which generates the masses of sterile neutrinos
- ▶ Active neutrinos (ν_e, ν_μ, ν_τ) can oscillate into sterile neutrinos (ν_s)
- ▶ Observables:
 - ▶ Disappearance of active neutrinos
 - ▶ Indirect evidence through combined fit of data
- ▶ Powerful window on new physics beyond the Standard Model

No GIM with Sterile Neutrinos

[Lee, Shrock, PRD 16 (1977) 1444; Schechter, Valle PRD 22 (1980) 2227]

- ▶ Neutrino Neutral-Current Weak Interaction Lagrangian:

$$\mathcal{L}_1^{(\text{NC})} = -\frac{g}{2 \cos \vartheta_W} Z_\rho \bar{\nu}'_L \gamma^\rho \nu'_L$$

- ▶ The transformation to active flavor neutrino fields is independent of the existence of sterile neutrinos: $\nu'_L = V_L^\ell \nu_L$

$$\mathcal{L}_1^{(\text{NC})} = -\frac{g}{2 \cos \vartheta_W} Z_\rho \bar{\nu}_L \gamma^\rho \nu_L = -\frac{g}{2 \cos \vartheta_W} Z_\rho \sum_{\alpha=e,\mu,\tau} \bar{\nu}_{\alpha L} \gamma^\rho \nu_{\alpha L}$$

- ▶ Mixing with sterile neutrinos: $\nu_{\alpha L} = \sum_{k=1}^{3+N_s} U_{\alpha k} \nu_{kL}$

- ▶ No GIM: $\mathcal{L}_1^{(\text{NC})} = -\frac{g}{2 \cos \vartheta_W} Z_\rho \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} \bar{\nu}_{jL} \gamma^\rho \nu_{kL} \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k}$

- ▶ $\sum_{\alpha=e,\mu,\tau,S_1,\dots} U_{\alpha j}^* U_{\alpha k} = \delta_{jk}$ but $\sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k} \neq \delta_{jk}$

Effect on Invisible Width of Z Boson?

- ▶ Amplitude of $Z \rightarrow \nu_j \bar{\nu}_k$ decay:

$$\begin{aligned} A(Z \rightarrow \nu_j \bar{\nu}_k) &= \langle \nu_j \bar{\nu}_k | - \int d^4x \mathcal{L}_1^{(\text{NC})}(x) | Z \rangle \\ &= \frac{g}{2 \cos \vartheta_W} \langle \nu_j \bar{\nu}_k | \int d^4x \bar{\nu}_{jL}(x) \gamma^\rho \nu_{kL}(x) Z_\rho(x) | Z \rangle \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k} \end{aligned}$$

- ▶ If $m_k \ll m_Z/2$ for all k 's, the neutrino masses are negligible in all the matrix elements and we can approximate

$$\frac{g}{2 \cos \vartheta_W} \langle \nu_j \bar{\nu}_k | \int d^4x \bar{\nu}_{jL}(x) \gamma^\rho \nu_{kL}(x) Z_\rho(x) | Z \rangle \simeq A_{\text{SM}}(Z \rightarrow \nu_\ell \bar{\nu}_\ell)$$

- ▶ $A_{\text{SM}}(Z \rightarrow \nu_\ell \bar{\nu}_\ell)$ is the Standard Model amplitude of Z decay into a massless neutrino-antineutrino pair of any flavor $\ell = e, \mu, \tau$

- ▶ $A(Z \rightarrow \nu_j \bar{\nu}_k) \simeq A_{\text{SM}}(Z \rightarrow \nu_\ell \bar{\nu}_\ell) \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k}$

- ▶ $P(Z \rightarrow \nu \bar{\nu}) = \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} |A(Z \rightarrow \nu_j \bar{\nu}_k)|^2$

▶ $P(Z \rightarrow \nu\bar{\nu}) \simeq P_{\text{SM}}(Z \rightarrow \nu_e\bar{\nu}_e) \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} \left| \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k} \right|^2$

▶ Effective number of neutrinos in Z decay:

$$N_\nu^{(Z)} = \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} \left| \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k} \right|^2$$

▶ Using the unitarity relation $\sum_{k=1}^{3+N_s} U_{\alpha k} U_{\beta k}^* = \delta_{\alpha\beta}$ we obtain

$$\begin{aligned} N_\nu^{(Z)} &= \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k} \sum_{\beta=e,\mu,\tau} U_{\beta j} U_{\beta k}^* \\ &= \sum_{\alpha=e,\mu,\tau} \sum_{\beta=e,\mu,\tau} \underbrace{\sum_{j=1}^{3+N_s} U_{\alpha j}^* U_{\beta j}}_{\delta_{\alpha\beta}} \underbrace{\sum_{k=1}^{3+N_s} U_{\alpha k} U_{\beta k}^*}_{\delta_{\alpha\beta}} = \sum_{\alpha=e,\mu,\tau} 1 = 3 \end{aligned}$$

▶ $N_\nu^{(Z)} = 3$ independently of the number of light sterile neutrinos!

Effect of Heavy Sterile Neutrinos

[Jarlskog, PLB 241 (1990) 579; Bilenky, Grimus, Neufeld, PLB 252 (1990) 119]

$$\blacktriangleright N_{\nu}^{(Z)} = \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} \left| \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k} \right|^2 R_{jk} \quad \text{with}$$

$$R_{jk} = \left(1 - \frac{m_j^2 + m_k^2}{2m_Z^2} - \frac{(m_j^2 - m_k^2)^2}{2m_Z^4} \right) \frac{\lambda(m_Z^2, m_j^2, m_k^2)}{m_Z^2} \theta(m_Z - m_j - m_k)$$

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$

$$\blacktriangleright R_{jk} \leq 1 \quad \Rightarrow \quad \boxed{N_{\nu}^{(Z)} \leq 3}$$

Fundamental Fields in QFT

- ▶ Each elementary particle is described by a field which is an irreducible representation of the Poincaré group (Lorentz group + space-time translations).
- ▶ In this way
 - ▶ Under Poincaré transformation an elementary particle remains itself.
 - ▶ Lagrangian is constructed with invariant products of elementary fields.
- ▶ Spinorial structure of a particle is determined by its representation under the restricted Lorentz group of proper and orthochronous Lorentz transformation (no space or time inversions).

▶ Restricted Lorentz group is isomorphic to $SU(2) \times SU(2)$.

▶ Classification of fundamental representations:

$(0, 0)$ scalar φ

$(1/2, 0)$ left-handed Weyl spinor χ_L (Majorana if massive)

$(0, 1/2)$ right-handed Weyl spinor χ_R (Majorana if massive)

▶ All representations are constructed combining the two fundamental Weyl spinor representations.

$(1/2, 1/2)$ four-vector v^μ (irreducible)

$(1/2, 0) + (0, 1/2)$ four-component Dirac spinor ψ (reducible)

▶ Two-component Weyl (Majorana if massive) spinor is more fundamental than four-component Dirac spinor.

- ▶ Two-component left-handed Weyl (Majorana if massive) spinor:

$$\chi_L = \begin{pmatrix} \chi_{L1} \\ \chi_{L2} \end{pmatrix}$$

- ▶ Two-component right-handed Weyl (Majorana if massive) spinor:

$$\chi_R = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix}$$

- ▶ Four-component Dirac spinor: $\psi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$

▶ Lorentz transformation: $v^\mu \rightarrow v'^\mu = \Lambda^\mu{}_\nu v^\nu$

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \quad g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

▶ Restricted Lorentz transformation: $\Lambda^\mu{}_\nu = [e^\omega]^\mu{}_\nu \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & v_1 & v_2 & v_3 \\ -v_1 & 0 & \theta_3 & -\theta_2 \\ -v_2 & -\theta_3 & 0 & \theta_1 \\ -v_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}$$

▶ 6 parameters:

▶ 3 for rotations: $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$

▶ 3 for boosts: $\vec{v} = (v_1, v_2, v_3)$

$$\chi_L \rightarrow \chi'_L = \Lambda_L \chi_L \quad \Lambda_L = e^{i(\vec{\theta} - i\vec{v}) \cdot \vec{\sigma} / 2}$$

$$\chi_R \rightarrow \chi'_R = \Lambda_R \chi_R \quad \Lambda_R = e^{i(\vec{\theta} + i\vec{v}) \cdot \vec{\sigma} / 2}$$

- ▶ Four-component form of two-component left-handed Weyl (Majorana if massive) spinor:

$$\psi_L = \begin{pmatrix} 0 \\ 0 \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$$

- ▶ Majorana mass term:

$$\mathcal{L}_{\text{mass}}^L = \underbrace{\frac{1}{2} m_L \psi_L^T C^\dagger \psi_L}_{\text{four-component form}} + \text{H.c.} = -\underbrace{\frac{1}{2} m_L \chi_L^T i\sigma^2 \chi_L}_{\text{two-component form}} + \text{H.c.}$$

$$(1/2, 0) \times (1/2, 0) = \underbrace{(1, 0)}_{\text{symmetric}} + \underbrace{(0, 0)}_{\text{antisymmetric}} \quad \sigma^2 \text{ is antisymmetric!}$$

- ▶ Anticommutativity of spinors is necessary, otherwise

$$\chi_L^T i\sigma^2 \chi_L = (\chi_L^T i\sigma^2 \chi_L)^T = -\chi_L^T i\sigma^2 \chi_L = 0$$