Neutrino Physics

Part I: Theory of Neutrino Masses and Mixing

Carlo Giunti

INFN, Torino, Italy

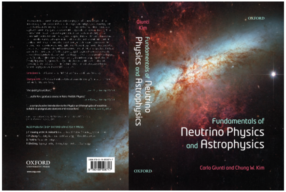
giunti@to.infn.it

Neutrino Unbound: http://www.nu.to.infn.it

Torino Graduate School in Physics and Astrophysics

Torino, 30 November - 4 December 2020

http://personalpages.to.infn.it/~giunti/slides/2020/giunti-201130-phdto1.pdf

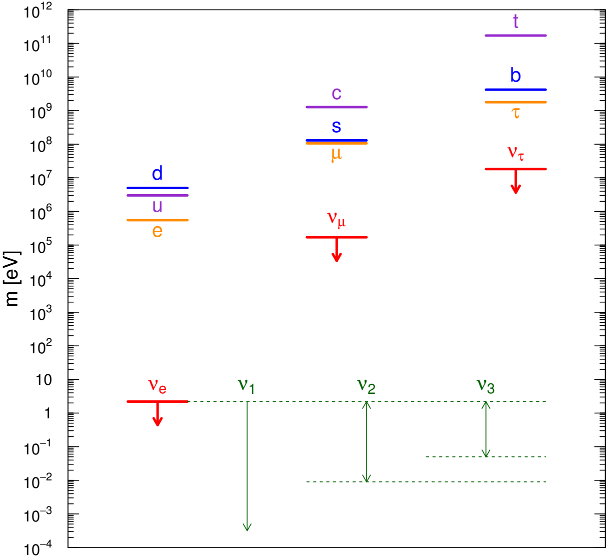


C. Giunti and C.W. Kim
Fundamentals of Neutrino Physics and Astrophysics
Oxford University Press
15 March 2007 – 728 pages

Why Neutrino Physics

- Neutrinos are special among the known particles (fermions):
  - Neutrinos are the only neutral fermions.
  - Neutrinos interact very weakly, only with left-handed weak interactions (and of course gravitational interactions).
  - Their mass is extremely small.
- In the Standard Model neutrinos are assumed to be left-handed and massless.
- The observation of neutrino oscillations implies that neutrinos are massive.
- Neutrino masses are so far the only certain phenomenon Beyond the Standard Model (BSM).
- Neutrino masses and non-standard neutrino properties and interactions are windows on the physics Beyond the Standard Model.
- Neutrinos are powerful astrophysical messengers (from Sun, supernovae, AGNs, ...) thanks to their extremely weak interactions and neutrality.

Fermion Mass Spectrum



Part I: Theory of Neutrino Masses and Mixing

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
- Sterile Neutrinos

Dirac Neutrino Masses and Mixing

- Dirac Neutrino Masses and Mixing
  - Higgs Mechanism in SM
  - SM Extension: Dirac Neutrino Masses
  - Three-Generations Dirac Neutrino Masses
  - Mixing
  - CP Violation
  - Jarlskog Invariant
  - Lepton Numbers Violating Processes
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
- Sterile Neutrinos

Dirac Mass

- Dirac Equation: $(i\partial\!\!\!/ - m)\,\nu(x) = 0$ $\qquad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$
- Dirac Lagrangian: $\mathscr{L}_\mathrm{D}(x) = \overline{\nu}(x)\,(i\partial\!\!\!/ - m)\,\nu(x)$ $\qquad \overline{\nu} \equiv \nu^\dagger \gamma^0$
- Chiral decomposition: $\nu_L \equiv P_L \nu\,, \qquad \nu_R \equiv P_R \nu\,, \qquad \nu = \nu_L + \nu_R$

  Left and Right-handed Projectors: $\quad P_L \equiv \dfrac{1-\gamma^5}{2}\,, \quad P_R \equiv \dfrac{1+\gamma^5}{2}$

$$P_L^2 = P_L\,, \quad P_R^2 = P_R\,, \quad P_L + P_R = 1\,, \quad P_L P_R = P_R P_L = 0$$

$$\mathscr{L} = \overline{\nu_L} i\partial\!\!\!/ \nu_L + \overline{\nu_R} i\partial\!\!\!/ \nu_R - m\,(\overline{\nu_L}\nu_R + \overline{\nu_R}\nu_L)$$

- In SM only ν_L by assumption $\Longrightarrow$ no neutrino mass
  Note that all the other elementary fermion fields (charged leptons and quarks) have both left and right-handed components
- Oscillation experiments have shown that neutrinos are massive
- Simplest and natural extension of the SM: consider also ν_R
  as for all the other elementary fermion fields

Higgs Mechanism in SM

- Higgs Doublet: $\Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}$ $\qquad |\Phi|^2 = \Phi^\dagger \Phi = \phi_+^\dagger \phi_+ + \phi_0^\dagger \phi_0$

- Higgs Lagrangian: $\mathscr{L}_{\text{Higgs}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(|\Phi|^2)$

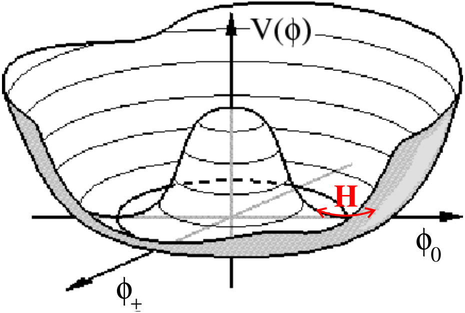
- Higgs Potential: $V(|\Phi|^2) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4$

- $\mu^2 < 0$ and $\lambda > 0 \quad \Longrightarrow \quad V(|\Phi|^2) = \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2$

$$v \equiv \sqrt{-\frac{\mu^2}{\lambda}} = \left(\sqrt{2} G_F \right)^{-1/2} \simeq 246\,\text{GeV}$$

- Vacuum: V_{min} for $|\Phi|^2 = \frac{v^2}{2} \Longrightarrow \langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$

- Spontaneous Symmetry Breaking: $SU(2)_L \times U(1)_Y \to U(1)_Q$



- Unitary Gauge: $\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \Longrightarrow |\Phi|^2 = \frac{v^2}{2} + vH + \frac{1}{2} H^2$

- $V = \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2 = \lambda v^2 H^2 + \lambda v H^3 + \frac{\lambda}{4} H^4$

$$m_H = \sqrt{2\lambda v^2} = \sqrt{-2\mu^2} \simeq 126\,\text{GeV}$$

$$-\mu^2 \simeq (89\,\text{GeV})^2 \qquad \lambda = -\frac{\mu^2}{v^2} \simeq 0.13$$

SM Extension: Dirac Neutrino Masses

		I	I_3	Y	$Q = I_3 + \frac{Y}{2}$
SM left-handed lepton doublet	$L_L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}$	$1/2$	$1/2$ $-1/2$	-1	0 -1
SM right-handed charged lepton singlet	ℓ_R	0	0	-2	-1
BSM right-handed neutrino singlet	ν_R	0	0	0	0
SM Higgs doublet	$\Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}$	$1/2$	$1/2$ $-1/2$	$+1$	1 0

Third component of weak isospin: $I_3 = \frac{\sigma_3}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$I_3 L_L = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} = \begin{pmatrix} (1/2)\,\nu_L \\ (-1/2)\,\ell_L \end{pmatrix}$$

		I	I_3	Y	$Q = I_3 + \frac{Y}{2}$
SM left-handed lepton doublet	$L_L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}$	$1/2$	$1/2$ $-1/2$	-1	0 -1
SM right-handed charged lepton singlet	ℓ_R	0	0	-2	-1
BSM right-handed neutrino singlet	ν_R	0	0	0	0
SM Higgs doublet	$\Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}$	$1/2$	$1/2$ $-1/2$	$+1$	1 0

Lepton-Higgs Yukawa Lagrangian:

$$\mathscr{L}_{H,\mathrm{L}} = -y^{\ell}\,\overline{L_L}\,\Phi\,\ell_R - y^{\nu}\,\overline{L_L}\,\widetilde{\Phi}\,\nu_R + \mathrm{H.c.}$$

Y: $+1\,+1\,-2$ $\qquad$ $+1\,-1\;\;0$

with

$$\widetilde{\Phi} = i\sigma_2\Phi^* = \begin{pmatrix} \phi_0^*(x) \\ -\phi_+^*(x) \end{pmatrix} \quad \leftarrow \quad Y = -1$$

Invariance under $\mathrm{SU}(2)_L$

- $\mathrm{SU}(2)_L$ transformation of doublets: $L_L \to U L_L$ and $\Phi \to U\Phi$ with

$$U = \exp\left(\frac{i}{2}\sum_{k=1}^{3}\theta^k\sigma_k\right)$$

- Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$(\sigma_k)^2 = 1 \qquad (\sigma_k)^\dagger = \sigma_k \qquad (\sigma_k)^* = -\sigma_2\sigma_k\sigma_2$$

Therefore: $U^* = \sigma_2 U \sigma_2$

- $\widetilde{\Phi} = i\sigma_2\Phi^* \to i\sigma_2 U^*\Phi^* = i\sigma_2\sigma_2 U\sigma_2\Phi^* = Ui\sigma_2\Phi^* = U\widetilde{\Phi}$

- Lepton-Higgs Yukawa terms:

$$\overline{L_L}\,\Phi\,\ell_R \to \overline{L_L}\,U^\dagger\,U\,\Phi\,\ell_R = \overline{L_L}\,\Phi\,\ell_R$$
$$\overline{L_L}\,\widetilde{\Phi}\,\nu_R = \overline{L_L}\,U^\dagger\,U\,\widetilde{\Phi}\,\nu_R = \overline{L_L}\,\widetilde{\Phi}\,\nu_R$$

Dirac Mass Generation

$$L_L \equiv \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} \qquad \ell_R \qquad \nu_R$$

Lepton-Higgs Yukawa Lagrangian

$$\mathscr{L}_{H,\mathrm{L}} = -y^{\ell}\,\overline{L_L}\,\Phi\,\ell_R - y^{\nu}\,\overline{L_L}\,\widetilde{\Phi}\,\nu_R + \mathrm{H.c.}$$

Spontaneous Symmetry Breaking

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \qquad \widetilde{\Phi} = i\sigma_2\Phi^* = \frac{1}{\sqrt{2}}\begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mathscr{L}_{H,\mathrm{L}} = & -\frac{y^{\ell}}{\sqrt{2}}\left(\overline{\nu_L} \quad \overline{\ell_L}\right)\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}\ell_R \\ & -\frac{y^{\nu}}{\sqrt{2}}\left(\overline{\nu_L} \quad \overline{\ell_L}\right)\begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}\nu_R + \mathrm{H.c.} \end{aligned}$$

$$\mathscr{L}_{H,\mathrm{L}} = -y^{\ell}\,\frac{v}{\sqrt{2}}\,\overline{\ell_L}\,\ell_R - y^{\nu}\,\frac{v}{\sqrt{2}}\,\overline{\nu_L}\,\nu_R$$

$$- \frac{y^{\ell}}{\sqrt{2}}\,\overline{\ell_L}\,\ell_R\,H - \frac{y^{\nu}}{\sqrt{2}}\,\overline{\nu_L}\,\nu_R\,H + \text{H.c.}$$

$$m_{\ell} = y^{\ell}\,\frac{v}{\sqrt{2}} \qquad\qquad m_{\nu} = y^{\nu}\,\frac{v}{\sqrt{2}}$$

$$g_{\ell H} = \frac{y^{\ell}}{\sqrt{2}} = \frac{m_{\ell}}{v} \qquad\qquad g_{\nu H} = \frac{y^{\nu}}{\sqrt{2}} = \frac{m_{\nu}}{v}$$

$$v = \left(\sqrt{2}\,G_{\mathrm{F}}\right)^{-1/2} = 246\,\mathrm{GeV}$$

PROBLEM: $y^{\nu} \lesssim 10^{-11} \ll y^{e} \sim 10^{-6}$

Three-Generations Dirac Neutrino Masses

$L'_{eL} \equiv \begin{pmatrix} \nu'_{eL} \\ \ell'_{eL} \equiv e'_L \end{pmatrix}$	$L'_{\mu L} \equiv \begin{pmatrix} \nu'_{\mu L} \\ \ell'_{\mu L} \equiv \mu'_L \end{pmatrix}$	$L'_{\tau L} \equiv \begin{pmatrix} \nu'_{\tau L} \\ \ell'_{\tau L} \equiv \tau'_L \end{pmatrix}$
$\ell'_{eR} \equiv e'_R$	$\ell'_{\mu R} \equiv \mu'_R$	$\ell'_{\tau R} \equiv \tau'_R$
ν'_{eR}	$\nu'_{\mu R}$	$\nu'_{\tau R}$

Lepton-Higgs Yukawa Lagrangian

$$\mathscr{L}_{H,\mathrm{L}} = - \sum_{\alpha,\beta=e,\mu,\tau} \left[Y'^{\ell}_{\alpha\beta} \, \overline{L'_{\alpha L}} \, \Phi \, \ell'_{\beta R} + Y'^{\nu}_{\alpha\beta} \, \overline{L'_{\alpha L}} \, \widetilde{\Phi} \, \nu'_{\beta R} \right] + \text{H.c.}$$

Spontaneous Symmetry Breaking

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \qquad \widetilde{\Phi} = i\sigma_2 \, \Phi^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right) \sum_{\alpha,\beta=e,\mu,\tau} \left[Y'^{\ell}_{\alpha\beta}\, \overline{\ell'_{\alpha L}}\, \ell'_{\beta R} + Y'^{\nu}_{\alpha\beta}\, \overline{\nu'_{\alpha L}}\, \nu'_{\beta R} \right] + \mathrm{H.c.}$$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right) \left[\overline{\boldsymbol{\ell}'_L}\, Y'^{\ell}\, \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L}\, Y'^{\nu}\, \boldsymbol{\nu}'_R \right] + \mathrm{H.c.}$$

$$\boldsymbol{\ell}'_L \equiv \begin{pmatrix} e'_L \\ \mu'_L \\ \tau'_L \end{pmatrix} \qquad \boldsymbol{\ell}'_R \equiv \begin{pmatrix} e'_R \\ \mu'_R \\ \tau'_R \end{pmatrix} \qquad \boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \qquad \boldsymbol{\nu}'_R \equiv \begin{pmatrix} \nu'_{eR} \\ \nu'_{\mu R} \\ \nu'_{\tau R} \end{pmatrix}$$

$$Y'^{\ell} \equiv \begin{pmatrix} Y'^{\ell}_{ee} & Y'^{\ell}_{e\mu} & Y'^{\ell}_{e\tau} \\ Y'^{\ell}_{\mu e} & Y'^{\ell}_{\mu\mu} & Y'^{\ell}_{\mu\tau} \\ Y'^{\ell}_{\tau e} & Y'^{\ell}_{\tau\mu} & Y'^{\ell}_{\tau\tau} \end{pmatrix} \qquad Y'^{\nu} \equiv \begin{pmatrix} Y'^{\nu}_{ee} & Y'^{\nu}_{e\mu} & Y'^{\nu}_{e\tau} \\ Y'^{\nu}_{\mu e} & Y'^{\nu}_{\mu\mu} & Y'^{\nu}_{\mu\tau} \\ Y'^{\nu}_{\tau e} & Y'^{\nu}_{\tau\mu} & Y'^{\nu}_{\tau\tau} \end{pmatrix}$$

$$M'^{\ell} = \frac{v}{\sqrt{2}}\, Y'^{\ell} \qquad M'^{\nu} = \frac{v}{\sqrt{2}}\, Y'^{\nu}$$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\boldsymbol{\ell}'_L}\, Y'^{\ell}\, \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L}\, Y'^{\nu}\, \boldsymbol{\nu}'_R\right] + \mathrm{H.c.}$$

Diagonalization of Y'^{ℓ} and Y'^{ν} with unitary V^{ℓ}_L, V^{ℓ}_R, V^{ν}_L, V^{ν}_R

$$\boldsymbol{\ell}'_L = V^{\ell}_L\, \boldsymbol{\ell}_L \qquad \boldsymbol{\ell}'_R = V^{\ell}_R\, \boldsymbol{\ell}_R \qquad \boldsymbol{\nu}'_L = V^{\nu}_L\, \boldsymbol{n}_L \qquad \boldsymbol{\nu}'_R = V^{\nu}_R\, \boldsymbol{n}_R$$

Important general remark: unitary transformations are allowed because they leave invariant the kinetic terms in the Lagrangian

$$\begin{aligned}
\mathscr{L}_{\mathrm{kin}} &= \overline{\boldsymbol{\ell}'_L} i\partial\!\!\!/ \boldsymbol{\ell}'_L + \overline{\boldsymbol{\ell}'_R} i\partial\!\!\!/ \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L} i\partial\!\!\!/ \boldsymbol{\nu}'_L + \overline{\boldsymbol{\nu}'_R} i\partial\!\!\!/ \boldsymbol{\nu}'_R \\
&= \overline{\boldsymbol{\ell}_L} V^{\ell\dagger}_L i\partial\!\!\!/ V^{\ell}_L \boldsymbol{\ell}_L + \ldots \\
&= \overline{\boldsymbol{\ell}_L} i\partial\!\!\!/ \boldsymbol{\ell}_L + \overline{\boldsymbol{\ell}_R} i\partial\!\!\!/ \boldsymbol{\ell}_R + \overline{\boldsymbol{\nu}_L} i\partial\!\!\!/ \boldsymbol{\nu}_L + \overline{\boldsymbol{\nu}_R} i\partial\!\!\!/ \boldsymbol{\nu}_R
\end{aligned}$$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\boldsymbol{\ell}'_L}\, Y'^{\ell}\, \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L}\, Y'^{\nu}\, \boldsymbol{\nu}'_R\right] + \mathrm{H.c.}$$

Diagonalization of Y'^{ℓ} and Y'^{ν} with unitary V^{ℓ}_L, V^{ℓ}_R, V^{ν}_L, V^{ν}_R

$$\boldsymbol{\ell}'_L = V^{\ell}_L\, \boldsymbol{\ell}_L \qquad \boldsymbol{\ell}'_R = V^{\ell}_R\, \boldsymbol{\ell}_R \qquad \boldsymbol{\nu}'_L = V^{\nu}_L\, \boldsymbol{n}_L \qquad \boldsymbol{\nu}'_R = V^{\nu}_R\, \boldsymbol{n}_R$$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\boldsymbol{\ell}_L}\, V^{\ell\dagger}_L Y'^{\ell} V^{\ell}_R \boldsymbol{\ell}_R + \overline{\boldsymbol{n}_L}\, V^{\nu\dagger}_L Y'^{\nu} V^{\nu}_R \boldsymbol{n}_R\right] + \mathrm{H.c.}$$

$$V^{\ell\dagger}_L \; Y'^{\ell} \; V^{\ell}_R = Y^{\ell} \qquad Y^{\ell}_{\alpha\beta} = y^{\ell}_{\alpha}\, \delta_{\alpha\beta} \qquad (\alpha, \beta = e, \mu, \tau)$$

$$V^{\nu\dagger}_L \; Y'^{\nu} \; V^{\nu}_R = Y^{\nu} \qquad Y^{\nu}_{kj} = y^{\nu}_{k}\, \delta_{kj} \qquad (k, j = 1, 2, 3)$$

Real and Positive y^{ℓ}_{α}, y^{ν}_{k}

$$\begin{array}{ccccc} V^{\dagger}_L & Y' & V_R & = & Y \\ 9 & 18 & 9 & & 3 \end{array}$$

- Consider the Hermitian matrix $Y' Y'^{\dagger}$
- It has real eigenvalues and orthonormal eigenvectors:

$$Y' Y'^{\dagger} v_k = \lambda_k v_k \quad \Leftrightarrow \quad \sum_{\beta} (Y' Y'^{\dagger})_{\alpha\beta} (v_k)_{\beta} = \lambda_k (v_k)_{\alpha}$$

- Unitary diagonalizing matrix: $(V_L)_{\beta k} = (v_k)_{\beta}$

$$Y' Y'^{\dagger} V_L = \Lambda V_L \quad \Longrightarrow \quad V_L^{\dagger} Y' Y'^{\dagger} V_L = \Lambda \quad \text{with} \quad \Lambda_{kj} = \lambda_k \delta_{kj}$$

- The real eigenvalues λ_k are positive:

$$\begin{aligned} \lambda_k &= \sum_{\alpha} (V_L^{\dagger} Y')_{k\alpha} (Y'^{\dagger} V_L)_{\alpha k} = \sum_{\alpha} (V_L^{\dagger} Y')_{k\alpha} (V_L^{\dagger} Y')^{\dagger}_{\alpha k} \\ &= \sum_{\alpha} (V_L^{\dagger} Y')_{k\alpha} (V_L^{\dagger} Y')^{*}_{k\alpha} = \sum_{\alpha} |(V_L^{\dagger} Y')_{k\alpha}|^2 \geq 0 \end{aligned}$$

- Then, we can write $V_L^{\dagger} Y' Y'^{\dagger} V_L = Y^2 \quad \text{with} \quad (Y)_{kj} = y_k \delta_{kj}$

real and positive $y_k = \sqrt{\lambda_k}$

- Let us write Y' as $Y' = V_L Y V_R^\dagger$
- This is the diagonalizing equation if V_R is unitary.

$$V_R^\dagger = Y^{-1} V_L^\dagger Y' \qquad V_R = Y'^\dagger V_L Y^{-1} \qquad \text{with} \qquad Y^\dagger = Y$$

- $V_R^\dagger V_R = Y^{-1} V_L^\dagger Y' Y'^\dagger V_L Y^{-1} = Y^{-1} Y^2 Y^{-1} = \mathbb{1}$
- $V_R V_R^\dagger = Y'^\dagger V_L Y^{-1} Y^{-1} V_L^\dagger Y' = Y'^\dagger V_L Y^{-2} V_L^\dagger Y'$

$$Y^{-2} = V_L^\dagger (Y'^\dagger)^{-1} (Y')^{-1} V_L$$

$$V_R V_R^\dagger = Y'^\dagger V_L V_L^\dagger (Y'^\dagger)^{-1} (Y')^{-1} V_L V_L^\dagger Y' = Y'^\dagger (Y'^\dagger)^{-1} (Y')^{-1} Y' = \mathbb{1}$$

- In conclusion: $V_L^\dagger Y' V_R = Y$ with unitary V_L and V_R

$$(Y)_{kj} = y_k \delta_{kj} \qquad \text{with real and positive } y_k$$

Massive Chiral Lepton Fields

$V_L^{\ell\dagger}\,\boldsymbol{\ell}'_L = \boldsymbol{\ell}_L \equiv \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}$	$V_R^{\ell\dagger}\,\boldsymbol{\ell}'_R = \boldsymbol{\ell}_R \equiv \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}$
$V_L^{\nu\dagger}\,\boldsymbol{\nu}'_L = \boldsymbol{n}_L \equiv \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$	$V_R^{\nu\dagger}\,\boldsymbol{\nu}'_R = \boldsymbol{n}_R \equiv \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix}$

$$\begin{aligned}\mathscr{L}_{H,\mathrm{L}} &= -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\boldsymbol{\ell}_L}\,Y^{\ell}\,\boldsymbol{\ell}_R + \overline{\boldsymbol{n}_L}\,Y^{\nu}\,n_R\right] + \text{H.c.} \\ &= -\left(\frac{v+H}{\sqrt{2}}\right)\left[\sum_{\alpha=e,\mu,\tau} y_\alpha^{\ell}\,\overline{\ell_{\alpha L}}\,\ell_{\alpha R} + \sum_{k=1}^{3} y_k^{\nu}\,\overline{\nu_{kL}}\,\nu_{kR}\right] + \text{H.c.}\end{aligned}$$

Massive Dirac Lepton Fields

$$\ell_\alpha \equiv \ell_{\alpha L} + \ell_{\alpha R} \qquad (\alpha = e, \mu, \tau)$$

$$\nu_k = \nu_{kL} + \nu_{kR} \qquad (k = 1, 2, 3)$$

$$\mathscr{L}_{H,\mathrm{L}} = -\sum_{\alpha=e,\mu,\tau} \frac{y^\ell_\alpha\, v}{\sqrt{2}}\, \overline{\ell_\alpha}\, \ell_\alpha - \sum_{k=1}^{3} \frac{y^\nu_k\, v}{\sqrt{2}}\, \overline{\nu_k}\, \nu_k \qquad \text{Mass Terms}$$

$$- \sum_{\alpha=e,\mu,\tau} \frac{y^\ell_\alpha}{\sqrt{2}}\, \overline{\ell_\alpha}\, \ell_\alpha\, H - \sum_{k=1}^{3} \frac{y^\nu_k}{\sqrt{2}}\, \overline{\nu_k}\, \nu_k\, H \qquad \text{Lepton-Higgs Couplings}$$

Charged Lepton and Neutrino Masses

$$m_\alpha = \frac{y^\ell_\alpha v}{\sqrt{2}} \quad (\alpha = e, \mu, \tau) \qquad\qquad m_k = \frac{y^\nu_k\, v}{\sqrt{2}} \quad (k = 1, 2, 3)$$

Lepton-Higgs coupling $\propto$ Lepton Mass

Quantization

$$\nu_k(x) = \int \frac{d^3p}{(2\pi)^3\, 2E_k} \sum_{h=\pm1} \left[a_k^{(h)}(p)\, u_k^{(h)}(p)\, e^{-ip\cdot x} + b_k^{(h)\dagger}(p)\, v_k^{(h)}(p)\, e^{ip\cdot x} \right]$$

$$p^0 = E_k = \sqrt{\vec{p}^2 + m_k^2}$$

$$(\not{p} - m_k)\, u_k^{(h)}(p) = 0$$
$$(\not{p} + m_k)\, v_k^{(h)}(p) = 0$$

$$\frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\, u_k^{(h)}(p) = h\, u_k^{(h)}(p)$$
$$\frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\, v_k^{(h)}(p) = -h\, v_k^{(h)}(p)$$

$$\{a_k^{(h)}(p), a_k^{(h')\dagger}(p')\} = \{b_k^{(h)}(p), b_k^{(h')\dagger}(p')\} = (2\pi)^3\, 2E_k\, \delta^3(\vec{p} - \vec{p}\,')\, \delta_{hh'}$$
$$\{a_k^{(h)}(p), a_k^{(h')}(p')\} = \{a_k^{(h)\dagger}(p), a_k^{(h')\dagger}(p')\} = 0$$
$$\{b_k^{(h)}(p), b_k^{(h')}(p')\} = \{b_k^{(h)\dagger}(p), b_k^{(h')\dagger}(p')\} = 0$$
$$\{a_k^{(h)}(p), b_k^{(h')}(p')\} = \{a_k^{(h)\dagger}(p), b_k^{(h')\dagger}(p')\} = 0$$
$$\{a_k^{(h)}(p), b_k^{(h')\dagger}(p')\} = \{a_k^{(h)\dagger}(p), b_k^{(h')}(p')\} = 0$$

Mixing

Charged-Current Weak Interaction Lagrangian

$$\mathscr{L}_{\mathrm{I}}^{(\mathrm{CC})} = -\frac{g}{2\sqrt{2}}\, j_W^\rho\, W_\rho + \mathrm{H.c.}$$

Weak Charged Current: $\quad j_W^\rho = j_{W,\mathrm{L}}^\rho + j_{W,\mathrm{Q}}^\rho$

Leptonic Weak Charged Current

$$j_{W,\mathrm{L}}^{\rho\dagger} = 2 \sum_{\alpha=e,\mu,\tau} \overline{\ell'_{\alpha L}}\, \gamma^\rho\, \nu'_{\alpha L} = 2\, \overline{\boldsymbol{\ell}'_L}\, \gamma^\rho\, \boldsymbol{\nu}'_L$$

$$\underline{\boldsymbol{\ell}'_L = V_L^\ell\, \boldsymbol{\ell}_L} \qquad \underline{\boldsymbol{\nu}'_L = V_L^\nu\, \boldsymbol{n}_L}$$

$$j_{W,\mathrm{L}}^{\rho\dagger} = 2\, \overline{\boldsymbol{\ell}_L}\, V_L^{\ell\dagger}\, \gamma^\rho\, V_L^\nu\, \boldsymbol{n}_L = 2\, \overline{\boldsymbol{\ell}_L}\, \gamma^\rho\, V_L^{\ell\dagger}\, V_L^\nu\, \boldsymbol{n}_L = 2\, \overline{\boldsymbol{\ell}_L}\, \gamma^\rho\, U\, \boldsymbol{n}_L$$

Mixing Matrix: $\quad \boxed{U = V_L^{\ell\dagger}\, V_L^\nu}$

- Definition: Left-Handed Flavor Neutrino Fields

$$\boldsymbol{\nu}_L = U\,\boldsymbol{n}_L = V_L^{\ell\dagger}\,V_L^{\nu}\,\boldsymbol{n}_L = V_L^{\ell\dagger}\,\boldsymbol{\nu}'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$$

- They allow us to write the Leptonic Weak Charged Current as in the SM:

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2\,\overline{\boldsymbol{\ell}_L}\,\gamma^\rho\,\boldsymbol{\nu}_L = 2 \sum_{\alpha=e,\mu,\tau} \overline{\ell_{\alpha L}}\,\gamma^\rho\,\nu_{\alpha L}$$

- Each left-handed flavor neutrino field is associated with the corresponding charged lepton field which describes a massive charged lepton:

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2\left(\overline{e_L}\,\gamma^\rho\,\nu_{eL} + \overline{\mu_L}\,\gamma^\rho\,\nu_{\mu L} + \overline{\tau_L}\,\gamma^\rho\,\nu_{\tau L}\right)$$

- In practice left-handed flavor neutrino fields are useful for calculations in the SM approximation of massless neutrinos (interactions).

- If neutrino masses must be taken into account, it is necessary to use

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2\,\overline{\boldsymbol{\ell}_L}\,\gamma^\rho\,U\,\boldsymbol{n}_L = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_{\alpha L}}\,\gamma^\rho\,U_{\alpha k}\,\nu_{kL}$$

Flavor Lepton Numbers

Flavor Neutrino Fields are useful for defining
Flavor Lepton Numbers
as in the SM

	L_e	L_μ	L_τ
$(\nu_e\,,\,e^-)$	$+1$	0	0
$(\nu_\mu\,,\,\mu^-)$	0	$+1$	0
$(\nu_\tau\,,\,\tau^-)$	0	0	$+1$

	L_e	L_μ	L_τ
$(\nu_e^c\,,\,e^+)$	-1	0	0
$(\nu_\mu^c\,,\,\mu^+)$	0	-1	0
$(\nu_\tau^c\,,\,\tau^+)$	0	0	-1

$$\boxed{L = L_e + L_\mu + L_\tau}$$

Standard Model: Lepton numbers are conserved

- L_e, L_μ, L_τ are conserved in the Standard Model with massless neutrinos

- Dirac mass term:

$$\mathscr{L}^{\mathrm{D}} = -\begin{pmatrix}\overline{\nu_{eL}} & \overline{\nu_{\mu L}} & \overline{\nu_{\tau L}}\end{pmatrix} \begin{pmatrix} m^{\mathrm{D}}_{ee} & m^{\mathrm{D}}_{e\mu} & m^{\mathrm{D}}_{e\tau} \\ m^{\mathrm{D}}_{\mu e} & m^{\mathrm{D}}_{\mu\mu} & m^{\mathrm{D}}_{\mu\tau} \\ m^{\mathrm{D}}_{\tau e} & m^{\mathrm{D}}_{\tau\mu} & m^{\mathrm{D}}_{\tau\tau} \end{pmatrix} \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix} + \mathrm{H.c.}$$

L_e, L_μ, L_τ are not conserved

- L is conserved: $\quad L(\nu_{\alpha R}) = L(\nu_{\beta L}) \implies |\Delta L| = 0$

- Leptonic Weak Charged Current is invariant under the global U(1) gauge transformations

$$\ell_{\alpha L} \to e^{i\varphi_\alpha}\,\ell_{\alpha L} \qquad \nu_{\alpha L} \to e^{i\varphi_\alpha}\,\nu_{\alpha L} \qquad (\alpha = e, \mu, \tau)$$

- If neutrinos are massless (SM), Noether's theorem implies that there is, for each flavor, a conserved current:

$$j_\alpha^\rho = \overline{\nu_{\alpha L}}\,\gamma^\rho\,\nu_{\alpha L} + \overline{\ell_\alpha}\,\gamma^\rho\,\ell_\alpha \qquad \partial_\rho j_\alpha^\rho = 0$$

and a conserved charge:

$$\mathrm{L}_\alpha = \int d^3x\, j_\alpha^0(x) \qquad \partial_0 \mathrm{L}_\alpha = 0$$

$$:\mathrm{L}_\alpha: \; = \int \frac{d^3p}{(2\pi)^3\,2E}\left[a_{\nu_\alpha}^{(-)\dagger}(p)\,a_{\nu_\alpha}^{(-)}(p) - b_{\nu_\alpha}^{(+)\dagger}(p)\,b_{\nu_\alpha}^{(+)}(p)\right] + \int \frac{d^3p}{(2\pi)^3\,2E}\sum_{h=\pm1}\left[a_{\ell_\alpha}^{(h)\dagger}(p)\,a_{\ell_\alpha}^{(h)}(p) - b_{\ell_\alpha}^{(h)\dagger}(p)\,b_{\ell_\alpha}^{(h)}(p)\right]$$

- Lepton-Higgs Yukawa Lagrangian:

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\sum_{\alpha=e,\mu,\tau} y^{\ell}_{\alpha}\,\overline{\ell_{\alpha L}}\,\ell_{\alpha R} + \sum_{k=1}^{3} y^{\nu}_{k}\,\overline{\nu_{kL}}\,\nu_{kR}\right] + \mathrm{H.c.}$$

- Mixing: $\nu_{\alpha L} = \sum_{k=1}^{3} U_{\alpha k}\,\nu_{kL} \qquad \Longleftrightarrow \qquad \nu_{kL} = \sum_{\alpha=e,\mu,\tau} U^{*}_{\alpha k}\,\nu_{\alpha L}$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right)\sum_{\alpha=e,\mu,\tau}\left[y^{\ell}_{\alpha}\,\overline{\ell_{\alpha L}}\,\ell_{\alpha R} + \overline{\nu_{\alpha L}}\sum_{k=1}^{3} U_{\alpha k}\,y^{\nu}_{k}\,\nu_{kR}\right] + \mathrm{H.c.}$$

- Invariant for

$$\ell_{\alpha L} \to e^{i\varphi_\alpha}\,\ell_{\alpha L}\,, \qquad \nu_{\alpha L} \to e^{i\varphi_\alpha}\,\nu_{\alpha L}$$

$$\ell_{\alpha R} \to e^{i\varphi_\alpha}\,\ell_{\alpha R}\,, \quad \sum_{k=1}^{3} U_{\alpha k}\,y^{\nu}_{k}\,\nu_{kR} \to e^{i\varphi_\alpha}\sum_{k=1}^{3} U_{\alpha k}\,y^{\nu}_{k}\,\nu_{kR}$$

- But kinetic part of neutrino Lagrangian is not invariant

$$\mathscr{L}^{(\nu)}_{\mathrm{kinetic}} = \sum_{\alpha=e,\mu,\tau}\overline{\nu_{\alpha L}}\,i\partial\!\!\!/\,\nu_{\alpha L} + \sum_{k=1}^{3}\overline{\nu_{kR}}\,i\partial\!\!\!/\,\nu_{kR}$$

because $\sum_{k=1}^{3} U_{\alpha k}\,y^{\nu}_{k}\,\nu_{kR}$ is not a unitary combination of the ν_{kR}'s

Total Lepton Number

- Dirac neutrino masses violate conservation of Flavor Lepton Numbers
- Total Lepton Number is conserved, because Lagrangian is invariant under the global U(1) gauge transformations

$$\nu_{kL} \to e^{i\varphi}\,\nu_{kL}\,, \qquad \nu_{kR} \to e^{i\varphi}\,\nu_{kR} \qquad (k = 1, 2, 3)$$
$$\ell_{\alpha L} \to e^{i\varphi}\,\ell_{\alpha L}\,, \qquad \ell_{\alpha R} \to e^{i\varphi}\,\ell_{\alpha R} \qquad (\alpha = e, \mu, \tau)$$

- From Noether's theorem:

$$j^\rho = \sum_{k=1}^{3} \overline{\nu_k}\,\gamma^\rho\,\nu_k + \sum_{\alpha=e,\mu,\tau} \overline{\ell_\alpha}\,\gamma^\rho\,\ell_\alpha \qquad \partial_\rho j^\rho = 0$$

Conserved charge: $\mathrm{L} = \int \mathrm{d}^3x\, j^0(x) \qquad \partial_0 \mathrm{L} = 0$

$$:\mathrm{L}: \;=\; \sum_{k=1}^{3} \int \frac{\mathrm{d}^3p}{(2\pi)^3\,2E} \sum_{h=\pm1} \left[a_{\nu_k}^{(h)\dagger}(p)\, a_{\nu_k}^{(h)}(p) - b_{\nu_k}^{(h)\dagger}(p)\, b_{\nu_k}^{(h)}(p) \right]$$
$$+ \sum_{\alpha=e,\mu,\tau} \int \frac{\mathrm{d}^3p}{(2\pi)^3\,2E} \sum_{h=\pm1} \left[a_{\ell_\alpha}^{(h)\dagger}(p)\, a_{\ell_\alpha}^{(h)}(p) - b_{\ell_\alpha}^{(h)\dagger}(p)\, b_{\ell_\alpha}^{(h)}(p) \right]$$

Mixing Matrix

- $U = V_L^{\ell\dagger}\, V_L^{\nu} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$
- A unitary $N \times N$ matrix depends on N^2 independent real parameters:

$$N = 3 \qquad \Longrightarrow \qquad \begin{array}{ll} \dfrac{N(N-1)}{2} = 3 & \quad \text{Mixing Angles} \\ \dfrac{N(N+1)}{2} = 6 & \quad \text{Phases} \end{array}$$

- Not all phases are physical observables!
- Neutrino Lagrangian:

$$\text{kinetic terms} + \text{mass terms} + \text{weak interactions}$$

- Mixing is due to the diagonalization of the mass terms.
- The kinetic terms are invariant under unitary transformations of the fermion fields.
- What is the effect of mixing in weak interactions?

- Weak Charged Current: $j^{\rho\dagger}_{W,\mathrm{L}} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_{\alpha L}} \, \gamma^{\rho} \, U_{\alpha k} \, \nu_{kL}$
- Apart from the Weak Charged Current, the Lagrangian is invariant under the global phase transformations (6 arbitrary phases)

$$\ell_{\alpha} \to e^{i\varphi_{\alpha}} \, \ell_{\alpha} \quad (\alpha = e, \mu, \tau)\,, \qquad \nu_{k} \to e^{i\varphi_{k}} \, \nu_{k} \quad (k = 1, 2, 3)$$

- Performing this transformation, the Weak Charged Current becomes

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_{\alpha L}} \, e^{-i\varphi_{\alpha}} \, \gamma^{\rho} \, U_{\alpha k} \, e^{i\varphi_{k}} \, \nu_{kL}$$

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2 \, \underbrace{e^{-i(\varphi_{e}-\varphi_{1})}}_{1} \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_{\alpha L}} \, \underbrace{e^{-i(\varphi_{\alpha}-\varphi_{e})}}_{2} \, \gamma^{\rho} \, U_{\alpha k} \, \underbrace{e^{i(\varphi_{k}-\varphi_{1})}}_{2} \, \nu_{kL}$$

- There are 5 independent combinations of the phases of the fields that can be chosen to eliminate 5 of the 6 phases of the mixing matrix
- 5 and not 6 phases of the mixing matrix can be eliminated because a common rephasing of all the lepton fields leaves the Weak Charged Current invariant $\Longleftrightarrow$ conservation of Total Lepton Number.

- The mixing matrix contains 1 Physical Phase.

- It is convenient to express the 3×3 unitary mixing matrix only in terms of the four physical parameters:

  3 Mixing Angles and 1 Phase

Standard Parameterization of Mixing Matrix

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13} \end{pmatrix}$$

$c_{ab} \equiv \cos\vartheta_{ab}$ $\quad$ $s_{ab} \equiv \sin\vartheta_{ab}$ $\quad$ $0 \leq \vartheta_{ab} \leq \dfrac{\pi}{2}$ $\quad$ $0 \leq \delta_{13} < 2\pi$

3 Mixing Angles ϑ_{12}, ϑ_{23}, ϑ_{13} and 1 Phase δ_{13}

Standard Parameterization

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of Different Phase Convention

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{i\delta_{23}} \\ 0 & -s_{23}e^{-i\delta_{13}} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of Different Parameterization

$$U = \begin{pmatrix} c'_{12} & s'_{12}e^{-i\delta'_{12}} & 0 \\ -s'_{12}e^{i\delta'_{12}} & c'_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c'_{23} & s'_{23} \\ 0 & -s'_{23} & c'_{23} \end{pmatrix} \begin{pmatrix} c'_{13} & 0 & s'_{13} \\ 0 & 1 & 0 \\ -s'_{13} & 0 & c'_{13} \end{pmatrix}$$

CP Violation

- $U \neq U^*$ $\Longrightarrow$ CP Violation (CPV)

- General conditions for CP violation (14 conditions):
  1. No charged leptons or neutrinos are degenerate in mass (6 conditions)
  2. No mixing angle is equal to 0 or $\pi/2$ (6 conditions)
  3. The physical phase is different from 0 or π (2 conditions)

- These 14 conditions are combined into the single condition

$$\det C \neq 0 \qquad \text{with} \qquad C = -i\,[M'^{\nu}\,M'^{\nu\dagger}\,,\,M'^{\ell}\,M'^{\ell\dagger}]$$

$$\det C = -2\,J\,(m_{\nu_2}^2 - m_{\nu_1}^2)\,(m_{\nu_3}^2 - m_{\nu_1}^2)\,(m_{\nu_3}^2 - m_{\nu_2}^2)$$
$$(m_{\mu}^2 - m_{e}^2)\,(m_{\tau}^2 - m_{e}^2)\,(m_{\tau}^2 - m_{\mu}^2) \neq 0$$

- Jarlskog invariant: $J = \mathrm{Im}\left[U_{e2}\,U_{e3}^*\,U_{\mu2}^*\,U_{\mu3}\right]$

[C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039, Z. Phys. C 29 (1985) 491]

[O. W. Greenberg, Phys. Rev. D 32 (1985) 1841]

[I. Dunietz, O. W. Greenberg, Dan-di Wu, Phys. Rev. Lett. 55 (1985) 2935]

Example: $\vartheta_{12} = 0$

$$U = R_{23} R_{13} W_{12}$$

$$W_{12} = \begin{pmatrix} \cos\vartheta_{12} & \sin\vartheta_{12} e^{-i\delta_{12}} & 0 \\ -\sin\vartheta_{12} e^{-i\delta_{12}} & \cos\vartheta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vartheta_{12} = 0 \qquad \Longrightarrow \qquad W_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}$$

Real Mixing Matrix: $\qquad U = R_{23} R_{13}$

Example: $\vartheta_{13} = \pi/2$

$$U = R_{23} W_{13} R_{12}$$

$$W_{13} = \begin{pmatrix} \cos\vartheta_{13} & 0 & \sin\vartheta_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -\sin\vartheta_{13} e^{i\delta_{13}} & 0 & \cos\vartheta_{13} \end{pmatrix}$$

$$\vartheta_{13} = \pi/2 \qquad \Longrightarrow \qquad W_{13} = \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -e^{i\delta_{13}} & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}e^{i\delta_{13}} & 0 \\ s_{12}s_{23} - c_{12}c_{23}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}e^{i\delta_{13}} & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ |U_{\mu 1}| e^{i\lambda_{\mu 1}} & |U_{\mu 2}| e^{i\lambda_{\mu 2}} & 0 \\ |U_{\tau 1}| e^{i\lambda_{\tau 1}} & |U_{\tau 2}| e^{i\lambda_{\tau 2}} & 0 \end{pmatrix}$$

$$\lambda_{\mu 1} - \lambda_{\mu 2} = \lambda_{\tau 1} - \lambda_{\tau 2} \pm \pi \qquad \lambda_{\tau 1} - \lambda_{\mu 1} = \lambda_{\tau 2} - \lambda_{\mu 2} \pm \pi$$

$$\nu_k \to e^{i\varphi_k} \nu_k \quad (k = 1, 2, 3), \qquad \ell_\alpha \to e^{i\varphi_\alpha} \ell_\alpha \quad (\alpha = e, \mu, \tau)$$

$$U \to \begin{pmatrix} e^{-i\varphi_e} & 0 & 0 \\ 0 & e^{-i\varphi_\mu} & 0 \\ 0 & 0 & e^{-i\varphi_\tau} \end{pmatrix} \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ |U_{\mu 1}| e^{i\lambda_{\mu 1}} & |U_{\mu 2}| e^{i\lambda_{\mu 2}} & 0 \\ |U_{\tau 1}| e^{i\lambda_{\tau 1}} & |U_{\tau 2}| e^{i\lambda_{\tau 2}} & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{i\varphi_3} \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & e^{i(-\delta_{13} - \varphi_e + \varphi_3)} \\ |U_{\mu 1}| e^{i(\lambda_{\mu 1} - \varphi_\mu + \varphi_1)} & |U_{\mu 2}| e^{i(\lambda_{\mu 2} - \varphi_\mu + \varphi_2)} & 0 \\ |U_{\tau 1}| e^{i(\lambda_{\tau 1} - \varphi_\tau + \varphi_1)} & |U_{\tau 2}| e^{i(\lambda_{\tau 2} - \varphi_\tau + \varphi_2)} & 0 \end{pmatrix}$$

$$\varphi_1 = 0 \qquad \varphi_\mu = \lambda_{\mu 1} \qquad \varphi_\tau = \lambda_{\tau 1} \qquad \varphi_2 = \varphi_\mu - \lambda_{\mu 2} = \lambda_{\mu 1} - \lambda_{\mu 2}$$

$$\varphi_2 = \varphi_\tau - \lambda_{\tau 2} \pm \pi = \lambda_{\tau 1} - \lambda_{\tau 2} \pm \pi = \lambda_{\mu 1} - \lambda_{\mu 2}$$

Real Mixing Matrix: $U = \begin{pmatrix} 0 & 0 & \pm 1 \\ |U_{\mu 1}| & |U_{\mu 2}| & 0 \\ |U_{\tau 1}| & -|U_{\tau 2}| & 0 \end{pmatrix}$

Example: $m_{\nu_2} = m_{\nu_3}$

$$j_{W,\mathrm{L}}^{\rho\dagger} = 2\,\overline{\boldsymbol{\ell}_L}\,\gamma^\rho\,U\,\boldsymbol{n}_L$$

$$U = R_{12} R_{13} W_{23} \qquad \Longrightarrow \qquad j_{W,\mathrm{L}}^{\rho\dagger} = 2\,\overline{\boldsymbol{\ell}_L}\,\gamma^\rho\,R_{12} R_{13} W_{23}\,\boldsymbol{n}_L$$

$$W_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta_{23} & \sin\vartheta_{23} e^{-i\delta_{23}} \\ 0 & -\sin\vartheta_{23} e^{-i\delta_{23}} & \cos\vartheta_{23} \end{pmatrix}$$

$$W_{23}\boldsymbol{n}_L = \boldsymbol{n}'_L \qquad R_{12}R_{13} = U' \qquad \Longrightarrow \qquad j_{W,\mathrm{L}}^{\rho\dagger} = 2\,\overline{\boldsymbol{\ell}_L}\,\gamma^\rho\,U'\,\boldsymbol{n}'_L$$

ν_2 and ν_3 are indistinguishable if they have the same mass!

$\boldsymbol{n}'_L$ is equivalent to $\boldsymbol{n}_L$

Drop the prime $\Longrightarrow$ $j_{W,\mathrm{L}}^{\rho\dagger} = 2\,\overline{\boldsymbol{\ell}_L}\,\gamma^\rho\,U\,\boldsymbol{n}_L$

With the Real Mixing Matrix $U = R_{12} R_{13}$

Jarlskog Invariant

- Since physics in invariant under reparameterizations of the mixing matrix, all physical quantities can be expressed in terms of reparameterization-invariant quantities.
- Simplest invariants: $|U_{\alpha k}|^2 = U_{\alpha k} U_{\alpha k}^*$, $U_{\alpha k} U_{\alpha j}^* U_{\beta k}^* U_{\beta j}$
- Simplest CPV invariants: $\text{Im}\left[U_{\alpha k} U_{\alpha j}^* U_{\beta k}^* U_{\beta j}\right] = \pm J$

Jarlskog invariant: $J = \text{Im}\left[U_{e2} U_{e3}^* U_{\mu 2}^* U_{\mu 3}\right] = \text{Im} \begin{pmatrix} \cdot & \circ & \times \\ \cdot & \times & \circ \\ \cdot & \cdot & \cdot \end{pmatrix}$

- In standard parameterization:

$$
\begin{aligned}
J &= c_{12} s_{12} c_{23} s_{23} c_{13}^2 s_{13} \sin\delta_{13} \\
&= \frac{1}{8} \sin 2\vartheta_{12} \sin 2\vartheta_{23} \cos\vartheta_{13} \sin 2\vartheta_{13} \sin\delta_{13}
\end{aligned}
$$

- For CPV all mixing angles must be different from 0 and $\pi/2$!
- The Jarlskog invariant is useful for quantifying CPV in a parameterization-independent way.
- All measurable CPV effects depend on J.

Maximal CP Violation

- Maximal CP violation is defined as the case in which $|J|$ has its maximum possible value

$$|J|_{\max} = \text{Max}|\underbrace{c_{12}s_{12}}_{\frac{1}{2}}\,\underbrace{c_{23}s_{23}}_{\frac{1}{2}}\,\underbrace{c_{13}^2 s_{13}}_{\frac{2}{3\sqrt{3}}}\,\underbrace{\sin\delta_{13}}_{1}| = \frac{1}{6\sqrt{3}}$$

- In the standard parameterization it is obtained for

$$\vartheta_{12} = \vartheta_{23} = \pi/4\,, \quad s_{13} = 1/\sqrt{3}\,, \quad \sin\delta_{13} = \pm 1$$

- This case is called Trimaximal Mixing. All the absolute values of the elements of the mixing matrix are equal to $1/\sqrt{3}$:

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \mp\frac{i}{\sqrt{3}} \\ -\frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2} \mp \frac{i}{2\sqrt{3}} & -\frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & \mp i \\ -e^{\pm i\pi/6} & e^{\mp i\pi/6} & 1 \\ e^{\mp i\pi/6} & -e^{\pm i\pi/6} & 1 \end{pmatrix}$$

GIM Mechanism

[S.L. Glashow, J. Iliopoulos, L. Maiani, Phys. Rev. D 2 (1970) 1285]

- Neutral-Current Weak Interaction Lagrangian:

$$\mathscr{L}_{\mathrm{I}}^{(\mathrm{NC})} = -\frac{g}{2\cos\vartheta_{\mathrm{W}}} j_Z^\rho Z_\rho \qquad\qquad j_Z^\rho = j_{Z,\mathrm{L}}^\rho + j_{Z,\mathrm{Q}}^\rho$$

- Leptonic Weak Neutral Current: $\left(g_L^\nu = \frac{1}{2},\, g_L^\ell = -\frac{1}{2} + \sin^2\vartheta_{\mathrm{W}},\, g_R^\ell = \sin^2\vartheta_{\mathrm{W}}\right)$

$$j_{Z,\mathrm{L}}^\rho = 2g_L^\nu\, \overline{\boldsymbol{\nu}_L'}\, \gamma^\rho\, \boldsymbol{\nu}_L' + 2g_L^\ell\, \overline{\boldsymbol{\ell}_L'}\, \gamma^\rho \boldsymbol{\ell}_L' + 2g_R^\ell\, \overline{\boldsymbol{\ell}_R'}\, \gamma^\rho \boldsymbol{\ell}_R'$$

- Invariant under mixing transformations with unitary V_L^ℓ, V_R^ℓ, V_L^ν:

$$\begin{aligned} j_{Z,\mathrm{L}}^\rho &= 2g_L^\nu\, \overline{\boldsymbol{n}_L}\, V_L^{\nu\dagger}\, \gamma^\rho\, V_L^\nu\, \boldsymbol{n}_L + 2g_L^\ell\, \overline{\boldsymbol{\ell}_L}\, V_L^{\ell\dagger}\, \gamma^\rho\, V_L^\ell\, \boldsymbol{\ell}_L + 2g_R^\ell\, \overline{\boldsymbol{\ell}_R}\, V_R^{\ell\dagger}\, \gamma^\rho\, V_R^\ell\, \boldsymbol{\ell}_R \\ &= 2g_L^\nu\, \overline{\boldsymbol{n}_L}\, \gamma^\rho\, \boldsymbol{n}_L + 2g_L^\ell\, \overline{\boldsymbol{\ell}_L}\, \gamma^\rho \boldsymbol{\ell}_L + 2g_R^\ell\, \overline{\boldsymbol{\ell}_R}\, \gamma^\rho \boldsymbol{\ell}_R \end{aligned}$$

- Invariant also under the mixing transformation $\boldsymbol{\nu}_L = U\, \boldsymbol{n}_L$ which defines the flavor neutrino fields:

$$\begin{aligned} j_{Z,\mathrm{L}}^\rho &= 2g_L^\nu\, \overline{\boldsymbol{\nu}_L}\, U\, \gamma^\rho\, U^\dagger\, \boldsymbol{\nu}_L + 2g_L^\ell\, \overline{\boldsymbol{\ell}_L}\, \gamma^\rho \boldsymbol{\ell}_L + 2g_R^\ell\, \overline{\boldsymbol{\ell}_R}\, \gamma^\rho \boldsymbol{\ell}_R \\ &= 2g_L^\nu\, \overline{\boldsymbol{\nu}_L}\, \gamma^\rho\, \boldsymbol{\nu}_L + 2g_L^\ell\, \overline{\boldsymbol{\ell}_L}\, \gamma^\rho \boldsymbol{\ell}_L + 2g_R^\ell\, \overline{\boldsymbol{\ell}_R}\, \gamma^\rho \boldsymbol{\ell}_R \end{aligned}$$

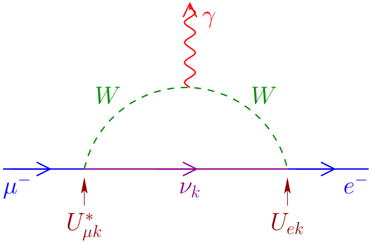
- Mixing has no effect in neutral-current weak interactions.

Lepton Numbers Violating Processes

Dirac mass term allows L_e, L_μ, L_τ violating processes

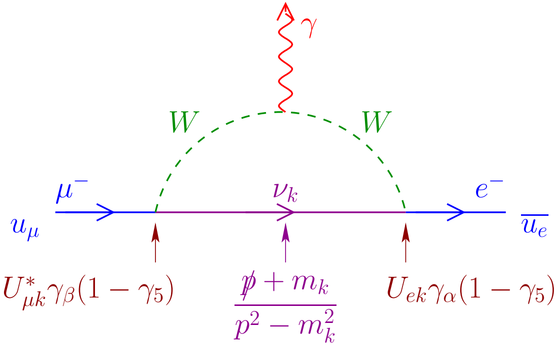
Example: $\mu^\pm \to e^\pm + \gamma\,, \qquad \mu^\pm \to e^\pm + e^+ + e^-$

$$\boxed{\mu^- \to e^- + \gamma}$$



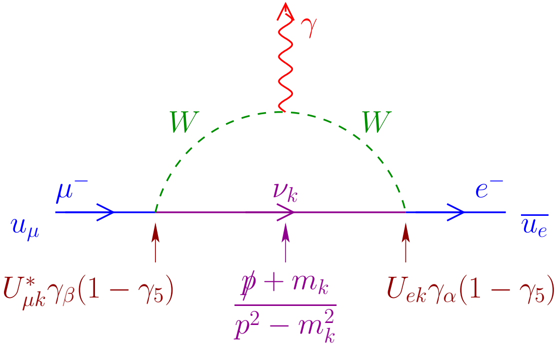
$$\sum_k U_{\mu k}^* U_{ek} = 0 \quad \Longrightarrow \quad \text{GIM suppression:} \quad A \propto \sum_k U_{\mu k}^* U_{ek} f(m_k)$$

$$
\begin{aligned}
\mathscr{L}_{\mathrm{I}}^{(\mathrm{CC})} &= -\frac{g}{2\sqrt{2}}\, W^{\alpha}\left[\overline{\nu_e}\gamma_{\alpha}\left(1-\gamma_5\right)e + \overline{\nu_{\mu}}\gamma_{\alpha}\left(1-\gamma_5\right)\mu + \ldots\right] \\
&= -\frac{g}{2\sqrt{2}}\, W^{\alpha}\sum_{k}\left[\overline{\nu_k}U_{ek}^{*}\gamma_{\alpha}\left(1-\gamma_5\right)e + \overline{\nu_k}U_{\mu k}^{*}\gamma_{\alpha}\left(1-\gamma_5\right)\mu + \ldots\right]
\end{aligned}
$$



$$
A \propto \sum_{k}\overline{u_e}U_{ek}\gamma_{\alpha}\left(1-\gamma_5\right)\frac{\not{p}+m_k}{p^2-m_k^2}U_{\mu k}^{*}\gamma_{\beta}\left(1-\gamma_5\right)u_{\mu}
$$

$$
\begin{aligned}
\mathscr{L}_{\mathrm{I}}^{(\mathrm{CC})} &= -\frac{g}{2\sqrt{2}}\, W^{\alpha} \left[\overline{\nu_e}\gamma_\alpha \left(1-\gamma_5\right) e + \overline{\nu_\mu}\gamma_\alpha \left(1-\gamma_5\right)\mu + \ldots\right] \\
&= -\frac{g}{2\sqrt{2}}\, W^{\alpha} \sum_k \left[\overline{\nu_k} U_{ek}^{*}\gamma_\alpha \left(1-\gamma_5\right) e + \overline{\nu_k} U_{\mu k}^{*}\gamma_\alpha \left(1-\gamma_5\right)\mu + \ldots\right]
\end{aligned}
$$



$$
A \propto \sum_k \overline{u_e}\, U_{ek}\gamma_\alpha \left(1-\gamma_5\right) \frac{\not{p}+\cancel{m_k}}{p^2-m_k^2} \left(1+\gamma_5\right)\gamma_\beta\, U_{\mu k}^{*}\, u_\mu
$$

$$\frac{1}{p^2 - m_k^2} = p^{-2}\left(1 - \frac{m_k^2}{p^2}\right)^{-1} \simeq p^{-2}\left(1 + \frac{m_k^2}{p^2}\right)$$

$$A \propto \sum_k U_{ek} U_{\mu k}^* \left(1 + \frac{m_k^2}{p^2}\right) = \sum_k U_{ek} U_{\mu k}^* \frac{m_k^2}{p^2} \to \sum_k U_{ek} U_{\mu k}^* \frac{m_k^2}{m_W^2}$$

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3} \underbrace{\frac{3\alpha}{32\pi} \left| \sum_k U_{ek} U_{\mu k}^* \frac{m_k^2}{m_W^2} \right|^2}_{\text{BR}}$$

[Petcov, SJNP 25 (1977) 340; Bilenky, Petcov, Pontecorvo, PLB 67 (1977) 309]

[Lee, Shrock, PRD 16 (1977) 1444]

Suppression factor: $\quad \frac{m_k}{m_W} \lesssim 10^{-11} \quad$ for $\quad m_k \lesssim 1\,\text{eV}$

$(\text{BR})_{\text{the}} \lesssim 10^{-47} \qquad (\text{BR})_{\text{exp}} \lesssim 10^{-11}$

Majorana Neutrino Masses and Mixing

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
  - Two-Component Theory of a Massless Neutrino
  - Majorana Equation
  - CP Symmetry
  - Effective Majorana Mass
  - Mixing of Three Majorana Neutrinos
- Dirac-Majorana Mass Term
- Sterile Neutrinos

Two-Component Theory of a Massless Neutrino

[Landau, NP 3 (1957) 127; Lee, Yang, PR 105 (1957) 1671; Salam, NC 5 (1957) 299]

- Dirac Equation: $(i\gamma^\mu \partial_\mu - m)\,\psi = 0$
- Chiral decomposition of a Fermion Field: $\psi = \psi_L + \psi_R$
- Equations for the Chiral components are coupled by mass:

$$i\gamma^\mu \partial_\mu \psi_L = m\,\psi_R$$
$$i\gamma^\mu \partial_\mu \psi_R = m\,\psi_L$$

- They are decoupled for a massless fermion: Weyl Equations (1929)

$$i\gamma^\mu \partial_\mu \psi_L = 0$$
$$i\gamma^\mu \partial_\mu \psi_R = 0$$

- A massless fermion can be described by a single chiral field ψ_L or ψ_R (Weyl Spinor), which has only two independent components (half the number of degrees of freedom of a Dirac field, which has four independent components).

- Chiral representation of γ matrices:

$$\gamma^0 = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$P_L = \frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \qquad P_R = \frac{1+\gamma^5}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$

- Four-components Dirac spinor: $\psi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$

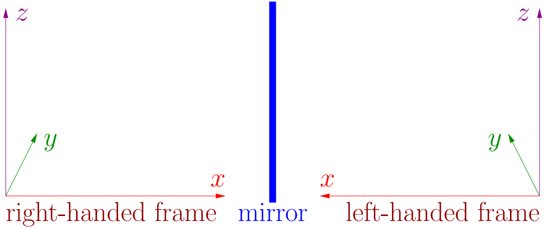
- The Weyl spinors ψ_L and ψ_R have only two components:

$$\psi_L = P_L \psi = \begin{pmatrix} 0 \\ \chi_L \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix} \qquad \psi_R = P_R \psi = \begin{pmatrix} \chi_R \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \\ 0 \\ 0 \end{pmatrix}$$

- The possibility to describe a physical particle with a Weyl spinor was rejected by Pauli in 1933 because it leads to parity violation:

$$\psi_L \stackrel{\mathsf{P}}{\rightleftharpoons} (\psi_L)^{\mathsf{P}} = (\psi^{\mathsf{P}})_R$$

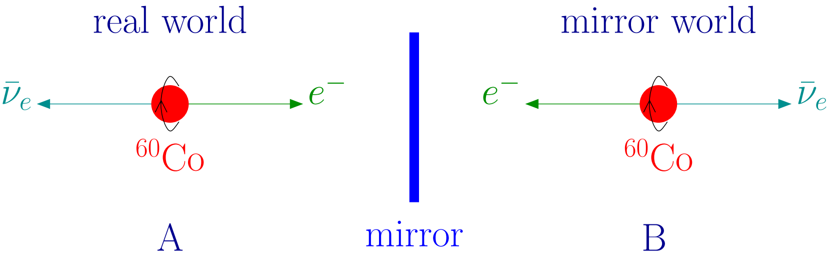
- Parity is the symmetry of space inversion (mirror transformation)



- Parity was considered to be an exact symmetry of nature
- 1956: Lee and Yang understand that Parity can be violated in Weak Interactions (1957 Physics Nobel Prize)
- 1957: Wu et al. discover Parity violation in β-decay of ^{60}Co

The experiment of Wu et al.

$$^{60}\text{Co} \to {}^{60}\text{Ni} + e^- + \bar{\nu}_e$$



- If parity were conserved the decays A and B schould be observed with the same frequency.

- Instead, Wu et al measured the asymmetry

$$\frac{A - B}{A + B} > 0.7$$

- Therefore, parity is violated in weak interactions.

- Parity: $x^\mu = (x^0, \vec{x}) \xrightarrow{\mathsf{P}} x_{\mathsf{P}}^\mu = (x^0, -\vec{x}) = x_\mu$
- The transformation of a fermion field $\psi(x)$ under parity is determined from the invariance of the theory under parity.
- Dirac Lagrangian:

$$\mathscr{L}_{\mathrm{D}}(x) = \overline{\psi}(x)\,(i\partial\!\!\!/ - m)\,\psi(x) = \overline{\psi}(x)\left(i\gamma^0\partial_0 + i\gamma^k\partial_k - m\right)\psi(x)$$

$$\downarrow \mathsf{P}$$

$$\overline{\psi}^{\mathsf{P}}(x_{\mathsf{P}})\left(i\gamma^0\partial_0 - i\gamma^k\partial_k - m\right)\psi^{\mathsf{P}}(x_{\mathsf{P}})$$

- It is equal to $\mathscr{L}_{\mathrm{D}}(x_{\mathsf{P}})$ if $\boxed{\psi^{\mathsf{P}}(x_{\mathsf{P}}) = \xi_{\mathsf{P}}\,\gamma^0\,\psi(x)}$
- Invariance is obtained from the action because $\left|\dfrac{\partial x_{\mathsf{P}}}{\partial x}\right| = 1$:

$$I_{\mathrm{D}} = \int \mathrm{d}^4x\,\mathscr{L}_{\mathrm{D}}(x) = \int \mathrm{d}^4x_{\mathsf{P}}\,\mathscr{L}_{\mathrm{D}}(x_{\mathsf{P}})$$

- $\psi(x) \xrightarrow{\mathsf{P}} \psi^{\mathsf{P}}(x_{\mathsf{P}}) = \xi_{\mathsf{P}}\,\gamma^0\,\psi(x)$

- $\psi_L(x) \xrightarrow{\mathsf{P}} (\psi_L)^{\mathsf{P}}(x_{\mathsf{P}}) = \xi_{\mathsf{P}}\,\gamma^0\,\psi_L(x)$

- $P_L(\psi_L)^{\mathsf{P}} = \xi_{\mathsf{P}}\,\dfrac{1-\gamma^5}{2}\,\gamma^0\,\psi_L = \xi_{\mathsf{P}}\,\gamma^0\,\dfrac{1+\gamma^5}{2}\,\psi_L = 0$

- $P_R(\psi_L)^{\mathsf{P}} = \xi_{\mathsf{P}}\,\dfrac{1+\gamma^5}{2}\,\gamma^0\,\psi_L = \xi_{\mathsf{P}}\,\gamma^0\,\dfrac{1-\gamma^5}{2}\,\psi_L = (\psi_L)^{\mathsf{P}}$

- Therefore $(\psi_L)^{\mathsf{P}}$ is right-handed: $\psi_L \overset{\mathsf{P}}{\rightleftharpoons} (\psi_L)^{\mathsf{P}} = (\psi^{\mathsf{P}})_R$

- Explicit swap of left and right components in the chiral representation:

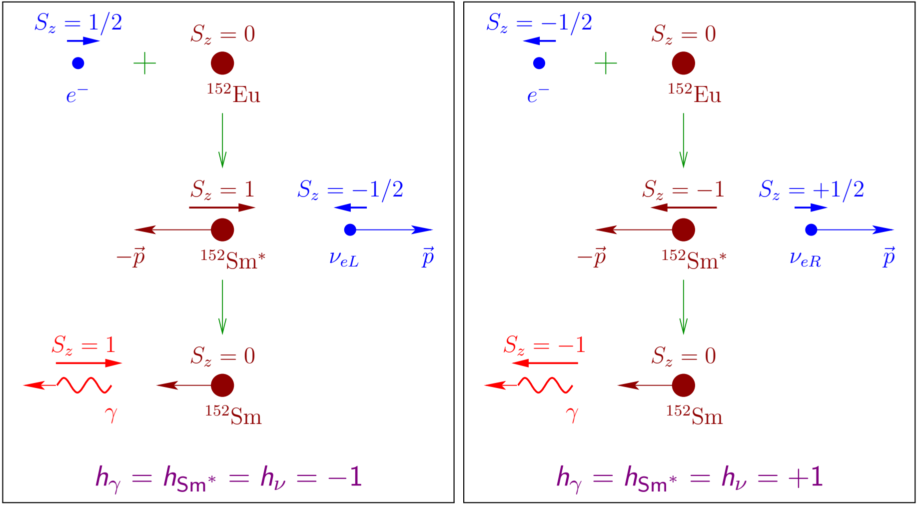
$$(\psi_L)^{\mathsf{P}} = \xi_{\mathsf{P}}\,\gamma^0\,\psi_L = \xi_{\mathsf{P}} \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \chi_L \end{pmatrix} = -\xi_{\mathsf{P}} \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}$$

- The discovery of parity violation in 1956-57 invalidated Pauli's reasoning, opening the possibility to describe massless particles with Weyl spinor fields $\Longrightarrow$ Two-component Theory of a Massless Neutrino (1957)

- 1958: Goldhaber, Grodzins and Sunyar measure the neutrino helicity with the electron capture process

$$
\begin{array}{l}
e^- + {}^{152}\mathrm{Eu} \to {}^{152}\mathrm{Sm}^* + \nu_e \\
\qquad\qquad\qquad\quad \downarrow \\
\qquad\qquad\qquad\quad {}^{152}\mathrm{Sm}^* \to {}^{152}\mathrm{Sm} + \gamma
\end{array}
$$

The neutrino helicity is the same as the measurable helicity of the photon when it is emitted in the same direction of the ${}^{152}\mathrm{Sm}^*$ recoil.



$h_\gamma = -0.91 \pm 0.19 \Longrightarrow$ NEUTRINOS ARE LEFT-HANDED: ν_L

[Goldhaber, Grodzins and Sunyar, PR 109 (1958) 1015]

$V-A$ Weak Interactions

[Feynman, Gell-Mann, PR 109 (1958) 193; Sudarshan, Marshak, PR 109 (1958) 1860; Sakurai, NC 7 (1958) 649]

- The Fermi Hamiltonian (1934) $H_\beta = g\,(\overline{p}\gamma^\alpha n)\,(\overline{e}\gamma^\alpha \nu) + \text{H.c.}$ explained only nuclear decays with $\Delta J = 0$.
- 1936: Gamow and Teller extension to describe observed nuclear decays with $|\Delta J| = 1$: [PR 49 (1936) 895]

$$H_\beta = \sum_{j=1}^{5} \left[g_j \left(\overline{p}\, \Omega^j\, n \right) \left(\overline{e}\, \Omega_j\, \nu_e \right) + g'_j \left(\overline{p}\, \Omega^j\, n \right) \left(\overline{e}\, \Omega_j\, \gamma_5\, \nu_e \right) \right] + \text{H.c.}$$

with $\Omega^1 = 1$, $\Omega^2 = \gamma^\alpha$, $\Omega^3 = \sigma^{\alpha\beta}$, $\Omega^4 = \gamma^\alpha\gamma^5$, $\Omega^5 = \gamma^5$

- 1958: Using simplicity arguments, Feynman and Gell-Mann, Sudarshan and Marshak, Sakurai propose the universal theory of parity-violating $V-A$ Weak Interactions:

$$H_W = \frac{G_F}{\sqrt{2}} \Big\{ \left[\overline{p}\gamma^\alpha \left(1-\gamma^5\right) n \right] \left[\overline{e}\gamma^\alpha \left(1-\gamma^5\right) \nu \right] + \left[\overline{\nu}\gamma^\alpha \left(1-\gamma^5\right) \mu \right] \left[\overline{e}\gamma^\alpha \left(1-\gamma^5\right) \nu \right] \Big\} + \text{H.c.}$$

in agreement with $\nu_L = \dfrac{1-\gamma^5}{2}\nu$

Quantization

$$\nu(x) = \int \frac{d^3p}{(2\pi)^3\, 2E} \sum_{h=\pm1} \left[a^{(h)}(p)\, u^{(h)}(p)\, e^{-ip\cdot x} + b^{(h)\dagger}(p)\, v^{(h)}(p)\, e^{ip\cdot x} \right]$$

$$p^0 = E = \sqrt{\vec{p}^2 + m^2}$$

$$(\not{p} - m)\, u^{(h)}(p) = 0$$
$$(\not{p} + m)\, v^{(h)}(p) = 0$$

$$\frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\, u^{(h)}(p) = h\, u^{(h)}(p)$$
$$\frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\, v^{(h)}(p) = -h\, v^{(h)}(p)$$

$$\{a^{(h)}(p), a^{(h')\dagger}(p')\} = \{b^{(h)}(p), b^{(h')\dagger}(p')\} = (2\pi)^3\, 2E\, \delta^3(\vec{p} - \vec{p}\,')\, \delta_{hh'}$$
$$\{a^{(h)}(p), a^{(h')}(p')\} = \{a^{(h)\dagger}(p), a^{(h')\dagger}(p')\} = 0$$
$$\{b^{(h)}(p), b^{(h')}(p')\} = \{b^{(h)\dagger}(p), b^{(h')\dagger}(p')\} = 0$$
$$\{a^{(h)}(p), b^{(h')}(p')\} = \{a^{(h)\dagger}(p), b^{(h')\dagger}(p')\} = 0$$
$$\{a^{(h)}(p), b^{(h')\dagger}(p')\} = \{a^{(h)\dagger}(p), b^{(h')}(p')\} = 0$$

Left-handed neutrino: $|\nu_L(p)\rangle = |\nu(p, h = -1)\rangle = a^{(-)\dagger}(p)|0\rangle$

Right-handed neutrino: $|\nu_R(p)\rangle = |\nu(p, h = +1)\rangle = a^{(+)\dagger}(p)|0\rangle$

Left-handed antineutrino: $|\bar{\nu}_L(p)\rangle = |\bar{\nu}(p, h = -1)\rangle = b^{(-)\dagger}(p)|0\rangle$

Right-handed antineutrino: $|\bar{\nu}_R(p)\rangle = |\bar{\nu}(p, h = +1)\rangle = b^{(+)\dagger}(p)|0\rangle$

Helicity and Chirality

$$\nu_L(x) = \int \frac{d^3p}{(2\pi)^3\, 2E} \sum_{h=\pm1} \left[a^{(h)}(p)\, u_L^{(h)}(p)\, e^{-ip\cdot x} + b^{(h)\dagger}(p)\, v_L^{(h)}(p)\, e^{ip\cdot x} \right]$$

$$u^{(h)\dagger}(p) u^{(h)}(p) = 2E \qquad u^{(h)\dagger}(p) \gamma^5 u^{(h)}(p) = 2h|\vec{p}|$$

$$v^{(h)\dagger}(p) v^{(h)}(p) = 2E \qquad v^{(h)\dagger}(p) \gamma^5 v^{(h)}(p) = -2h|\vec{p}|$$

$$u_L^{(h)\dagger}(p) u_L^{(h)}(p) = u^{(h)\dagger}(p) \left(\frac{1-\gamma^5}{2} \right) u^{(h)}(p) = E - h|\vec{p}|$$

$$u_L^{(-)\dagger}(p) u_L^{(-)}(p) = E + |\vec{p}| \simeq 2E - \frac{m^2}{2E}$$ left-handed neutrinos

$$u_L^{(+)\dagger}(p) u_L^{(+)}(p) = E - |\vec{p}| \simeq \frac{m^2}{2E}$$ suppressed right-handed neutrinos

$$v_L^{(h)\dagger}(p) v_L^{(h)}(p) = v^{(h)\dagger}(p) \left(\frac{1-\gamma^5}{2} \right) v^{(h)}(p) = E + h|\vec{p}|$$

$$v_L^{(-)\dagger}(p) v_L^{(-)}(p) = E - |\vec{p}| \simeq \frac{m^2}{2E}$$ right-handed antineutrinos

$$v_L^{(+)\dagger}(p) v_L^{(+)}(p) = E + |\vec{p}| \simeq 2E - \frac{m^2}{2E}$$ suppressed left-handed antineutrino

Massless Left-Handed Neutrino Field

$$\nu_L(x) = \int \frac{d^3p}{(2\pi)^3\, 2E} \left[a^{(-)}(p)\, u_L^{(-)}(p)\, e^{-ip\cdot x} + b^{(+)\dagger}(p)\, v_L^{(+)}(p)\, e^{ip\cdot x} \right]$$

Left-handed neutrino: $|\nu_L(p)\rangle = |\nu(p, h=-1)\rangle = a^{(-)\dagger}(p)|0\rangle$

Right-handed antineutrino: $|\bar{\nu}_R(p)\rangle = |\bar{\nu}(p, h=+1)\rangle = b^{(+)\dagger}(p)|0\rangle$

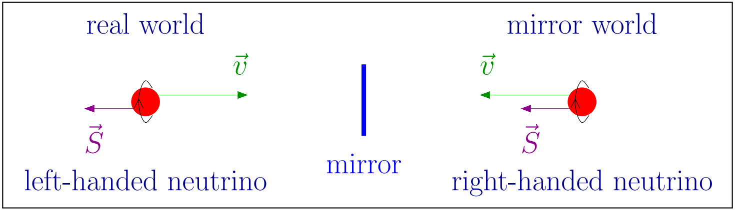
Standard Model

- Glashow (1961), Weinberg (1967) and Salam (1968) formulate the Standard Model of ElectroWeak Interactions (1979 Physics Nobel Prize) assuming that neutrinos are massless and left-handed

- Universal $V - A$ Weak Interactions

- Quantum Field Theory: $\nu_L \quad \Rightarrow \quad |\nu(h=-1)\rangle$ and $|\bar{\nu}(h=+1)\rangle$

- Parity is violated: $|\nu(h=-1)\rangle \xrightarrow{\mathsf{P}} \cancel{|\nu(h=+1)\rangle}$



- Particle-Antiparticle symmetry (Charge Conjugation) is violated:

$$|\nu(h=-1)\rangle \xrightarrow{\mathsf{C}} \cancel{|\bar{\nu}(h=-1)\rangle}$$

Majorana Equation

- Can a two-component spinor describe a massive fermion?

  Yes! (E. Majorana, 1937)

- Trick: ν_R and ν_L are not independent: $\boxed{\nu_R = \nu_L^c = \mathcal{C}\,\overline{\nu_L}^T}$

  charge-conjugation matrix: $\mathcal{C}\,\gamma_\mu^T\,\mathcal{C}^{-1} = -\gamma_\mu$

- The relation between ν_R and ν_L must satisfy two requirements:
  - It must have the correct chirality.

    This is satisfied, because ν_L^c is right-handed: $P_R\nu_L^c = \nu_L^c \quad P_L\nu_L^c = 0$

  - It must be compatible with the chiral Dirac equations

$$
\begin{aligned}
i\gamma^\mu\partial_\mu\nu_L &= m\,\nu_R \\
i\gamma^\mu\partial_\mu\nu_R &= m\,\nu_L
\end{aligned}
$$

    Check:

$$
\begin{aligned}
i\gamma^\mu\partial_\mu\nu_R &= i\gamma^\mu\partial_\mu\mathcal{C}\overline{\nu_L}^T = i\mathcal{C}\mathcal{C}^{-1}\gamma^\mu\mathcal{C}\partial_\mu\overline{\nu_L}^T = -i\mathcal{C}(\gamma^\mu)^T\partial_\mu\overline{\nu_L}^T \\
&= -i\mathcal{C}(\partial_\mu\overline{\nu_L}\gamma^\mu)^T = m\,\mathcal{C}\overline{\nu_R}^T = m\,\nu_L \qquad \text{OK}
\end{aligned}
$$

- Other relations between ψ_R and ψ_L do not satisfy the two requirements.
- For example $\psi_R = \psi_L^{\mathsf{P}} = \gamma^0 \psi_L$ satisfies the chirality requirements, because $\psi_L \overset{\mathsf{P}}{\rightleftharpoons} \psi_R$, but

$$i\gamma^\mu \partial_\mu \psi_R = i\gamma^\mu \partial_\mu \gamma^0 \psi_L = i\gamma^0 (\gamma^\mu)^\dagger \partial_\mu \psi_L \neq i\gamma^0 \gamma^\mu \partial_\mu \psi_L = m\,\gamma^0 \psi_R = m\,\psi_L$$

- There are several relations which satisfy only the chirality requirements, for example $\quad \psi_R = \gamma^\mu \psi_L \quad$ for $\quad \mu = 0, 1, 2, 3$
- There is only one adequate relation $(\psi_R = \mathcal{C}\,\overline{\psi_L}^T)$ that can be derived from the chiral Dirac equations: consider $i\gamma^\mu \partial_\mu \psi_R = m\,\psi_L$

$$\begin{array}{rcl}
\text{Hermitian conj.} \times \gamma^0 & \Longrightarrow & -i\partial_\mu \underline{\psi_R^\dagger (\gamma^\mu)^\dagger \gamma^0} = m\,\overline{\psi_L} \\
\gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu & \Longrightarrow & -i\partial_\mu \overline{\psi_R} \gamma^\mu = m\,\overline{\psi_L} \\
\mathcal{C} \times \text{transpose} & \Longrightarrow & -i\mathcal{C}(\gamma^\mu)^T \partial_\mu \overline{\psi_R}^T = m\,\mathcal{C}\overline{\psi_L}^T \\
\mathcal{C}(\gamma^\mu)^T \mathcal{C}^{-1} = -\gamma^\mu & \Longrightarrow & i\gamma^\mu \partial_\mu \mathcal{C}\overline{\psi_R}^T = m\,\mathcal{C}\overline{\psi_L}^T
\end{array}$$

Identical to $i\gamma^\mu \partial_\mu \psi_L = m\,\psi_R$ for $\psi_R = \mathcal{C}\overline{\psi_L}^T \leftrightarrows \psi_L = \mathcal{C}\overline{\psi_R}^T$ (Majorana)

- $i\gamma^{\mu}\partial_{\mu}\nu_L = m\,\nu_R \quad \to \quad \boxed{i\gamma^{\mu}\partial_{\mu}\nu_L = m\,\nu_L^c}$ Majorana equation

- Majorana field: $\nu = \nu_L + \nu_R = \nu_L + \nu_L^c$

  $\boxed{\nu = \nu^c}$ Majorana condition

- $\nu = \nu^c$ implies the equality of particle and antiparticle

- Only neutral fermions can be Majorana particles

- For a Majorana field, the electromagnetic current vanishes identically:

$$\overline{\nu}\gamma^{\mu}\nu = \overline{\nu^c}\gamma^{\mu}\nu^c = -\nu^T\mathcal{C}^{\dagger}\gamma^{\mu}\mathcal{C}\overline{\nu}^T = \overline{\nu}\mathcal{C}\gamma^{\mu\,T}\mathcal{C}^{\dagger}\nu = -\overline{\nu}\gamma^{\mu}\nu = 0$$

- Only two independent components: in the chiral representation

$$\nu = \begin{pmatrix} i\sigma^2\chi_L^* \\ \chi_L \end{pmatrix} = \begin{pmatrix} \chi_{L2}^* \\ -\chi_{L1}^* \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$$

Majorana Lagrangian

Dirac Lagrangian

$$\begin{aligned} \mathscr{L}^{\mathrm{D}} &= \overline{\nu}\,(i\partial\!\!\!/ - m)\,\nu \\ &= \overline{\nu_L} i\partial\!\!\!/ \nu_L + \overline{\nu_R} i\partial\!\!\!/ \nu_R - m\,(\overline{\nu_R}\,\nu_L + \overline{\nu_L}\,\nu_R) \end{aligned}$$

$$\nu_R \quad \to \quad \nu_L^c = \mathcal{C}\,\overline{\nu_L}^T$$

$$\frac{1}{2}\,\mathscr{L}^{\mathrm{D}} \quad \to \quad \overline{\nu_L}\,i\partial\!\!\!/\,\nu_L - \frac{m}{2}\left(-\nu_L^T\,\mathcal{C}^\dagger\,\nu_L + \overline{\nu_L}\,\mathcal{C}\,\overline{\nu_L}^T\right)$$

Majorana Lagrangian

$$\begin{aligned} \mathscr{L}^{\mathrm{M}} &= \overline{\nu_L}\,i\partial\!\!\!/\,\nu_L - \frac{m}{2}\left(-\nu_L^T\,\mathcal{C}^\dagger\,\nu_L + \overline{\nu_L}\,\mathcal{C}\,\overline{\nu_L}^T\right) \\ &= \overline{\nu_L}\,i\partial\!\!\!/\,\nu_L - \frac{m}{2}\left(\overline{\nu_L^c}\,\nu_L + \overline{\nu_L}\,\nu_L^c\right) \end{aligned}$$

- Majorana Lagrangian: $\mathscr{L}^{\mathrm{M}} = \frac{1}{2}\,\overline{\nu}\,(i\partial\!\!\!/ - m)\,\nu\big|_{\nu=\nu^c}$

- Quantized Dirac Neutrino Field:

$$\nu(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\,2E} \sum_{h=\pm1} \left[a^{(h)}(p)\,u^{(h)}(p)\,e^{-ip\cdot x} + b^{(h)\dagger}(p)\,v^{(h)}(p)\,e^{ip\cdot x} \right]$$

- Quantized Majorana Neutrino Field: $\quad b^{(h)}(p) = a^{(h)}(p)$

$$\nu(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\,2E} \sum_{h=\pm1} \left[a^{(h)}(p)\,u^{(h)}(p)\,e^{-ip\cdot x} + a^{(h)\dagger}(p)\,v^{(h)}(p)\,e^{ip\cdot x} \right]$$

- A Majorana field has half the degrees of freedom of a Dirac field

Total Lepton Number

$$\cancel{L = +1} \quad \longleftarrow \quad \boxed{\nu = \nu^c} \quad \longrightarrow \quad \cancel{L = -1}$$

$$\nu_L \quad \Longrightarrow \quad L = +1 \qquad\qquad \nu_L^c \quad \Longrightarrow \quad L = -1$$

$$\mathscr{L}^{\text{M}} = \overline{\nu_L}\, i\partial\!\!\!/\, \nu_L - \frac{m}{2}\left(\overline{\nu_L^c}\,\nu_L + \overline{\nu_L}\,\nu_L^c\right)$$

Total Lepton Number is not conserved: $\boxed{\Delta L = \pm 2}$

Best process to find violation of Total Lepton Number:

Neutrinoless Double-β Decay

$$\mathcal{N}(A,Z) \to \mathcal{N}(A,Z+2) + 2e^- + \cancel{2\bar{\nu}_e} \qquad (\beta\beta_{0\nu}^-)$$

$$\mathcal{N}(A,Z) \to \mathcal{N}(A,Z-2) + 2e^+ + \cancel{2\nu_e} \qquad (\beta\beta_{0\nu}^+)$$

CP Symmetry

- Under a CP transformation

$$\nu_L(x) \xrightarrow{\text{CP}} \xi_\nu^{\text{CP}} \, \gamma^0 \, \nu_L^c(x_{\text{P}})$$

$$\nu_L^c(x) \xrightarrow{\text{CP}} -\xi_\nu^{\text{CP}*} \, \gamma^0 \, \nu_L(x_{\text{P}})$$

$$\overline{\nu_L}(x) \xrightarrow{\text{CP}} \xi_\nu^{\text{CP}*} \, \overline{\nu_L^c}(x_{\text{P}}) \, \gamma^0$$

$$\overline{\nu_L^c}(x) \xrightarrow{\text{CP}} -\xi_\nu^{\text{CP}} \, \overline{\nu_L}(x_{\text{P}}) \, \gamma^0$$

with $|\xi_\nu^{\text{CP}}|^2 = 1$, $x^\mu = (x^0, \vec{x})$, and $x_{\text{P}}^\mu = (x^0, -\vec{x})$

- The theory is CP-symmetric if there are values of the phase ξ_ν^{CP} such that the Lagrangian transforms as

$$\mathscr{L}(x) \xrightarrow{\text{CP}} \mathscr{L}(x_{\text{P}})$$

in order to keep invariant the action $I = \int \mathrm{d}^4x \, \mathscr{L}(x)$

- The Majorana Mass Term
$$\mathscr{L}^{\text{M}}_{\text{mass}}(x) = -\frac{1}{2}\, m \left[\overline{\nu^c_L}(x)\,\nu_L(x) + \overline{\nu_L}(x)\,\nu^c_L(x)\right]$$
transforms as
$$\mathscr{L}^{\text{M}}_{\text{mass}}(x) \xrightarrow{\text{CP}} -\frac{1}{2}\, m \Big[-(\xi^{\text{CP}}_\nu)^2\,\overline{\nu_L}(x_{\text{P}})\,\nu^c_L(x_{\text{P}}) - (\xi^{\text{CP}*}_\nu)^2\,\overline{\nu^c_L}(x_{\text{P}})\,\nu_L(x_{\text{P}}) \Big]$$

- $\mathscr{L}^{\text{M}}_{\text{mass}}(x) \xrightarrow{\text{CP}} \mathscr{L}^{\text{M}}_{\text{mass}}(x_{\text{P}})$ for $\boxed{\xi^{\text{CP}}_\nu = \pm i}$

- The one-generation Majorana theory is CP-symmetric

- The Majorana case is different from the Dirac case, in which the CP phase ξ^{CP}_ν is arbitrary

No Majorana Neutrino Mass in the SM

- Majorana Mass Term $\propto \left[\nu_L^T\,\mathcal{C}^\dagger\,\nu_L - \overline{\nu_L}\,\mathcal{C}\,\overline{\nu_L}^T\right]$ involves only the neutrino left-handed chiral field ν_L, which is present in the SM
- Eigenvalues of the weak isospin I, of its third component I_3, of the hypercharge Y and of the charge Q of the lepton and Higgs multiplets:

	I	I_3	Y	$Q = I_3 + \frac{Y}{2}$
lepton doublet $L_L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}$	$1/2$	$\begin{matrix} 1/2 \\ -1/2 \end{matrix}$	-1	$\begin{matrix} 0 \\ -1 \end{matrix}$
lepton singlet ℓ_R	0	0	-2	-1
Higgs doublet $\Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}$	$1/2$	$\begin{matrix} 1/2 \\ -1/2 \end{matrix}$	$+1$	$\begin{matrix} 1 \\ 0 \end{matrix}$

- $\nu_L^T\,\mathcal{C}^\dagger\,\nu_L$ has $I_3 = 1$ and $Y = -2 \Longrightarrow$ needed $Y = 2$ Higgs triplet ($I = 1$, $I_3 = -1$)
- Compare with Dirac Mass Term $\propto \overline{\nu_R}\nu_L$ with $I_3 = 1/2$ and $Y = -1$ balanced by $\phi_0 \to v$ with $I_3 = -1/2$ and $Y = +1$

Confusing Majorana Antineutrino Terminology

- A Majorana neutrino is the same as a Majorana antineutrino

- Neutrino interactions are described by the CC and NC Lagrangians

$$\mathscr{L}_{\mathrm{I,L}}^{\mathrm{CC}} = -\frac{g}{\sqrt{2}} \left(\overline{\nu_L} \, \gamma^{\mu} \, \ell_L \, W_{\mu} + \overline{\ell_L} \, \gamma^{\mu} \, \nu_L \, W_{\mu}^{\dagger} \right)$$

$$\mathscr{L}_{\mathrm{I},\nu}^{\mathrm{NC}} = -\frac{g}{2 \cos \vartheta_{\mathrm{W}}} \, \overline{\nu_L} \, \gamma^{\mu} \, \nu_L \, Z_{\mu}$$

- Dirac: ν_L $\begin{cases} \text{destroys left-handed neutrinos} \\ \text{creates right-handed antineutrinos} \end{cases}$

- Majorana: ν_L $\begin{cases} \text{destroys left-handed neutrinos} \\ \text{creates right-handed neutrinos} \end{cases}$

- Common implicit definitions:

  left-handed Majorana neutrino $\equiv$ neutrino

  right-handed Majorana neutrino $\equiv$ antineutrino

Effective Majorana Mass

- Dimensional analysis: Fermion Field $\sim [E]^{3/2}$ Boson Field $\sim [E]$
- Dimensionless action: $I = \int d^4x \, \mathscr{L}(x) \Longrightarrow \mathscr{L}(x) \sim [E]^4$
- Kinetic terms: $\overline{\psi} i \partial\!\!\!/ \psi \sim [E]^4$, $(\partial_\mu \phi)^\dagger \partial^\mu \phi \sim [E]^4$
- Mass terms: $m \overline{\psi} \psi \sim [E]^4$, $m^2 \phi^\dagger \phi \sim [E]^4$
- CC weak interaction: $g \, \overline{\boldsymbol{\nu}_L} \gamma^\rho \boldsymbol{\ell}_L W_\rho \sim [E]^4$
- Yukawa couplings: $y \, \overline{L_L} \Phi \ell_R \sim [E]^4$
- Product of fields $\mathscr{O}_d$ with energy dimension $d \equiv$ dim-d operator
- $\mathscr{L}_{(\mathscr{O}_d)} = C_{(\mathscr{O}_d)} \mathscr{O}_d \Longrightarrow C_{(\mathscr{O}_d)} \sim [E]^{4-d}$
- $\mathscr{O}_{d>4}$ are not renormalizable

- SM Lagrangian includes all $\mathscr{O}_{d\leq 4}$ invariant under $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$
- SM cannot be considered as the final theory of everything
- SM is an effective low-energy theory
- It is likely that SM is the low-energy product of the symmetry breaking of a high-energy unified theory
- It is plausible that at low-energy there are effective non-renormalizable $\mathscr{O}_{d>4}$ [S. Weinberg, Phys. Rev. Lett. 43 (1979) 1566]
- All $\mathscr{O}_d$ must respect $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, because they are generated by the high-energy theory which must include the gauge symmetries of the SM in order to be effectively reduced to the SM at low energies

- $\mathscr{O}_{d>4}$ is suppressed by a coefficient $\mathcal{M}^{4-d}$, where $\mathcal{M}$ is a heavy mass characteristic of the symmetry breaking scale of the high-energy unified theory:

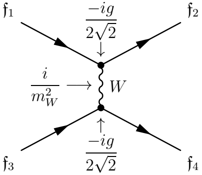
$$\mathscr{L} = \mathscr{L}_{\text{SM}} + \frac{g_5}{\mathcal{M}}\,\mathscr{O}_5 + \frac{g_6}{\mathcal{M}^2}\,\mathscr{O}_6 + \ldots$$

- Analogy with Fermi effective low-energy theory of weak interactions:

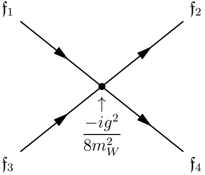
$$\mathscr{L}_{\text{eff}}^{(\text{CC})} \propto G_{\text{F}}\left(\overline{\nu_{eL}}\gamma^{\rho}e_L\right)\left(\overline{e_L}\gamma_{\rho}\nu_{eL}\right) + \ldots$$

$$\mathscr{O}_6 \to \left(\overline{\nu_{eL}}\gamma^{\rho}e_L\right)\left(\overline{e_L}\gamma_{\rho}\nu_{eL}\right) + \ldots \qquad\qquad \frac{g_6}{\mathcal{M}^2} \to \frac{G_{\text{F}}}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

$$G_{\mu\nu}^{(W)}(p) = i\,\frac{-g_{\mu\nu} + \frac{p_\mu\,p_\nu}{m_W^2}}{p^2 - m_W^2} \xrightarrow{p^2 \ll m_W^2} i\,\frac{g_{\mu\nu}}{m_W^2}$$



$\Longrightarrow$



- $\mathcal{M}^{4-d}$ is a strong suppression factor which limits the observability of the low-energy effects of the new physics beyond the SM
- The difficulty to observe the effects of the effective low-energy non-renormalizable operators increase rapidly with their dimensionality
- $\mathscr{O}_5 \quad \Longrightarrow \quad$ Majorana neutrino masses (Lepton number violation)
- $\mathscr{O}_6 \quad \Longrightarrow \quad$ Baryon number violation (proton decay)
- Majorana neutrino masses provide the most accessible low-energy window on new physics beyond the SM.
- Indeed, the existence of neutrino masses is the first and so far the only well established phenomenon beyond the SM.

- There is only one dim-5 operator that can be constructed from the dim-2.5 $SU(2)_L \times U(1)_Y$ singlet with SM fields in the Lepton-Higgs Yukawa Lagrangian

$$\begin{array}{lcccccccccc} \mathscr{L}_{H,\mathrm{L}} = -y^{\ell} & \overline{L_L} & \Phi & \ell_R & -y^{\nu} & \overline{L_L} & \widetilde{\Phi} & \nu_R & +\mathrm{H.c.} \\ Y: & +1 & +1 & -2 & & +1 & -1 & 0 & \end{array}$$

- The dim-2.5 $SU(2)_L \times U(1)_Y$ singlet: $\overline{L_L}\,\widetilde{\Phi} = \overline{L_L}\, i\sigma_2\, \Phi^*$
- It must be multiplied by a spinor as ν_R to form a Lorentz scalar.
- As in the Majorana theory, ν_R can be replaced by

$$L_L^c = \mathcal{C}\,\overline{L_L}^T = \mathcal{C}\begin{pmatrix} \overline{\nu_L}^T \\ \overline{\ell_L}^T \end{pmatrix} \quad \text{with} \quad Y = +1$$

- To form a $SU(2)_L \times U(1)_Y$ singlet we need to multiply it by

$$\widetilde{\Phi}^T = -i\,\Phi^\dagger\,\sigma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} v + H(x) & 0 \end{pmatrix} \quad \text{with} \quad Y = -1$$

- The only dim-5 operator:

$$\begin{array}{lcccccc} \mathscr{O}_5 = (& \overline{L_L} & \widetilde{\Phi} &)(& \widetilde{\Phi}^T & L_L^c &) + \mathrm{H.c.} \\ Y: & +1 & -1 & & -1 & +1 & \end{array}$$

- $\mathrm{SU}(2)_L$ transformation:

$$L_L \to U L_L \qquad \Phi \to U\Phi \qquad \widetilde{\Phi} \to U\widetilde{\Phi} \qquad L_L^c \to U^* L_L^c$$

$$L_L^c = \mathcal{C}\overline{L_L}^T = \mathcal{C}\left(L_L^\dagger \gamma^0\right)^T \to \mathcal{C}\left(L_L^\dagger U^\dagger \gamma^0\right)^T = \mathcal{C}\left(L_L^\dagger \gamma^0 U^\dagger\right)^T$$
$$= \mathcal{C} U^* \left(L_L^\dagger \gamma^0\right)^T = U^* L_L^c$$

- $\mathrm{SU}(2)_L$ invariance:

$$\begin{aligned}\mathscr{O}_5 &= \left(\overline{L_L}\widetilde{\Phi}\right)\left(\widetilde{\Phi}^T L_L^c\right) + \text{H.c.} \\ &= \left(\overline{L_L} U^\dagger U \widetilde{\Phi}\right)\left(\widetilde{\Phi}^T U^T U^* L_L^c\right) + \text{H.c.} \\ &= \left(\overline{L_L}\widetilde{\Phi}\right)\left(\widetilde{\Phi}^T L_L^c\right) + \text{H.c.}\end{aligned}$$

because

$$U^T U^* = (U^\dagger U)^T = \mathbb{1}$$

- The only $SU(2)_L \times U(1)_Y$ invariant dim-5 Lagrangian term that can be constructed with SM fields:

$$\mathscr{L}_5 = -\frac{g_5}{\mathcal{M}} \left[\left(\overline{L_L}\, \widetilde{\Phi} \right) \left(\widetilde{\Phi}^T L_L^c \right) + \left(\overline{L_L^c}\, \widetilde{\Phi}^* \right) \left(\widetilde{\Phi}^\dagger L_L \right) \right]$$

- Electroweak Symmetry Breaking:

$$\widetilde{\Phi} = i\sigma_2 \Phi^* \quad \xrightarrow[\text{Breaking}]{\text{EW Symmetry}} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

- $$\mathscr{L}_5 \xrightarrow[\text{Breaking}]{\text{EW Symmetry}} \mathscr{L}^{\text{M}}_{\text{mass}} = -\frac{1}{2} \frac{g_5\, v^2}{\mathcal{M}} \left(\overline{\nu_L}\, \nu_L^c + \overline{\nu_L^c}\, \nu_L \right)$$

- Majorana neutrino mass: $$\boxed{m = \frac{g_5\, v^2}{\mathcal{M}}}$$

- General Seesaw Mechanism: $$m \propto \frac{v^2}{\mathcal{M}} = v\,\frac{v}{\mathcal{M}}$$

  natural explanation of the strong suppression of neutrino masses with respect to the electroweak scale

- Example: $\mathcal{M} \sim 10^{15}\,\text{GeV}$ (GUT scale)

$$v \sim 10^2\,\text{GeV} \implies \frac{v}{\mathcal{M}} \sim 10^{-13} \implies m \sim 10^{-2}\,\text{eV}$$

Mixing of Three Majorana Neutrinos

- $\boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix}$

$$\mathscr{L}^{\text{M}}_{\text{mass}} = \frac{1}{2} \boldsymbol{\nu}'^{T}_{L} \, \mathcal{C}^\dagger \, M^L \, \boldsymbol{\nu}'_L + \text{H.c.}$$

$$= \frac{1}{2} \sum_{\alpha,\beta=e,\mu,\tau} \nu'^{T}_{\alpha L} \, \mathcal{C}^\dagger \, M^L_{\alpha\beta} \, \nu'_{\beta L} + \text{H.c.}$$

- In general, the matrix M^L is a complex symmetric matrix

$$\sum_{\alpha,\beta} \nu'^{T}_{\alpha L} \, \mathcal{C}^\dagger \, M^L_{\alpha\beta} \, \nu'_{\beta L} = \sum_{\alpha,\beta} \left(\nu'^{T}_{\alpha L} \, \mathcal{C}^\dagger \, M^L_{\alpha\beta} \, \nu'_{\beta L} \right)^T$$

$$= - \sum_{\alpha,\beta} \nu'^{T}_{\beta L} \, M^L_{\alpha\beta} \, (\mathcal{C}^\dagger)^T \, \nu'_{\alpha L} = \sum_{\alpha,\beta} \nu'^{T}_{\beta L} \, \mathcal{C}^\dagger \, M^L_{\alpha\beta} \, \nu'_{\alpha L}$$

$$= \sum_{\alpha,\beta} \nu'^{T}_{\alpha L} \, \mathcal{C}^\dagger \, M^L_{\beta\alpha} \, \nu'_{\beta L}$$

$$M^L_{\alpha\beta} = M^L_{\beta\alpha} \quad \Longleftrightarrow \quad M^L = M^{L\,T}$$

- $\mathscr{L}^{\text{M}}_{\text{mass}} = \frac{1}{2} \boldsymbol{\nu}'^{T}_{L} \, \mathcal{C}^{\dagger} \, M^{L} \, \boldsymbol{\nu}'_{L} + \text{H.c.}$
- $\boldsymbol{\nu}'_{L} = V^{\nu}_{L} \, \boldsymbol{n}_{L} \quad \Longrightarrow \quad \mathscr{L}^{\text{M}}_{\text{mass}} = \frac{1}{2} \boldsymbol{\nu}'^{T}_{L} \, (V^{\nu}_{L})^{T} \, \mathcal{C}^{\dagger} \, M^{L} \, V^{\nu}_{L} \, \boldsymbol{\nu}'_{L} + \text{H.c.}$
- $(V^{\nu}_{L})^{T} \, M^{L} \, V^{\nu}_{L} = M \,, \qquad M_{kj} = m_{k} \, \delta_{kj} \qquad (k, j = 1, 2, 3)$
- Left-handed chiral fields with definite mass: $\boldsymbol{n}_{L} = V^{\nu\dagger}_{L} \, \boldsymbol{\nu}'_{L} = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$

$$\begin{aligned} \mathscr{L}^{\text{M}}_{\text{mass}} &= \frac{1}{2} \left(\boldsymbol{n}^{T}_{L} \, \mathcal{C}^{\dagger} \, M \, \boldsymbol{n}_{L} - \overline{\boldsymbol{n}_{L}} \, M \, \mathcal{C} \, \boldsymbol{n}^{T}_{L} \right) \\ &= \frac{1}{2} \sum_{k=1}^{3} m_{k} \left(\nu^{T}_{kL} \, \mathcal{C}^{\dagger} \, \nu_{kL} - \overline{\nu_{kL}} \, \mathcal{C} \, \nu^{T}_{kL} \right) \end{aligned}$$

- Majorana fields of massive neutrinos: $\nu_{k} = \nu_{kL} + \nu^{c}_{kL} \qquad \boxed{\nu^{c}_{k} = \nu_{k}}$
- $\boldsymbol{n} = \begin{pmatrix} \nu_{1} \\ \nu_{2} \\ \nu_{3} \end{pmatrix} \Longrightarrow \mathscr{L}^{\text{M}} = \frac{1}{2} \sum_{k=1}^{3} \overline{\nu_{k}} \, (i\partial\!\!\!/ - m_{k}) \, \nu_{k} \big|_{\nu_{k} = \nu^{c}_{k}}$

Mixing Matrix

- Leptonic Weak Charged Current:

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2\,\overline{\boldsymbol{\ell}_L}\,\gamma^\rho\,U\,\boldsymbol{n}_L \qquad \text{with} \qquad U = V_L^{\ell\dagger}\,V_L^{\nu}$$

- As in the Dirac case, we define the left-handed flavor neutrino fields as

$$\boldsymbol{\nu}_L = U\,\boldsymbol{n}_L = V_L^{\ell\dagger}\,\boldsymbol{\nu}'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$$

- In this way, as in the Dirac case, the Leptonic Weak Charged Current has the SM form

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2\,\overline{\boldsymbol{\ell}_L}\,\gamma^\rho\,\boldsymbol{\nu}_L = 2\sum_{\alpha=e,\mu,\tau}\overline{\ell_{\alpha L}}\,\gamma^\rho\,\nu_{\alpha L}$$

- Important difference with respect to Dirac case:
Two additional CP-violating phases: Majorana phases

- Majorana Mass Term $\mathscr{L}^{\mathrm{M}} = \frac{1}{2} \sum_{k=1}^{3} m_k \, \nu_{kL}^T \, \mathcal{C}^\dagger \, \nu_{kL} + \mathrm{H.c.}$ is not invariant under the global U(1) gauge transformations

$$\nu_{kL} \to e^{i\varphi_k} \, \nu_{kL} \quad (k = 1, 2, 3)$$

- For eliminating some of the 6 phases of the unitary mixing matrix we can use only the global phase transformations (3 arbitrary phases)

$$\ell_\alpha \to e^{i\varphi_\alpha} \, \ell_\alpha \quad (\alpha = e, \mu, \tau)$$

- Weak Charged Current: $j^{\rho\dagger}_{W,\mathrm{L}} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_{\alpha L}} \, \gamma^{\rho} \, U_{\alpha k} \, \nu_{kL}$

- Performing the transformation $\ell_{\alpha} \to e^{i\varphi_{\alpha}} \, \ell_{\alpha}$ we obtain

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_{\alpha L}} \, e^{-i\varphi_{\alpha}} \, \gamma^{\rho} \, U_{\alpha k} \, \nu_{kL}$$

$$j^{\rho\dagger}_{W,\mathrm{L}} = 2 \, \underbrace{e^{-i\varphi_{e}}}_{1} \sum_{\alpha=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_{\alpha L}} \, \underbrace{e^{-i(\varphi_{\alpha}-\varphi_{e})}}_{2} \, \gamma^{\rho} \, U_{\alpha k} \, \nu_{kL}$$

- We can eliminate 3 phases of the mixing matrix: one overall phase and two phases which can be factorized on the left.

- In the Dirac case we could eliminate also two phases which can be factorized on the right.

- In the Majorana case there are two additional physical Majorana phases which can be factorized on the right of the mixing matrix:

$$\boxed{U = U^{\mathrm{D}}\, D^{\mathrm{M}}} \qquad D^{\mathrm{M}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_2} & 0 \\ 0 & 0 & e^{i\lambda_3} \end{pmatrix}$$

- U^{D} is a Dirac mixing matrix, with one Dirac phase

- Standard parameterization:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_2} & 0 \\ 0 & 0 & e^{i\lambda_3} \end{pmatrix}$$

- $D^{\mathrm{M}} = \mathrm{diag}\left(e^{i\lambda_1}\,,\, e^{i\lambda_2}\,,\, e^{i\lambda_3}\right)$, but only two Majorana phases are physical

- All measurable quantities depend only on the differences of the Majorana phases because $e^{i(\lambda_k - \lambda_j)}$ remains constant under the allowed phase transformation

$$\ell_\alpha \to e^{i\varphi}\ell_\alpha \quad \Longrightarrow \quad e^{i\lambda_k} \to e^{i(\lambda_k - \varphi)}$$

- Our convention: $\lambda_1 = 0 \Longrightarrow D^{\mathrm{M}} = \mathrm{diag}\left(1\,,\, e^{i\lambda_2}\,,\, e^{i\lambda_3}\right)$

- CP is conserved if all the elements of each column of the mixing matrix are either real or purely imaginary:

$$\delta_{13} = 0 \text{ or } \pi \quad \text{and} \quad \lambda_k = 0 \text{ or } \pi/2 \text{ or } \pi \text{ or } 3\pi/2$$

Dirac-Majorana Mass Term

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
  - One Generation Dirac-Majorana Mass Term
  - Type-I Seesaw Mechanism
  - Three-Generation Mixing
- Sterile Neutrinos

One Generation Dirac-Majorana Mass Term

If ν_R exists, the most general mass term is the

Dirac-Majorana Mass Term

$$\mathscr{L}_{\text{mass}}^{\text{D+M}} = \mathscr{L}_{\text{mass}}^{\text{D}} + \mathscr{L}_{\text{mass}}^{L} + \mathscr{L}_{\text{mass}}^{R}$$

$$\mathscr{L}_{\text{mass}}^{\text{D}} = -m_{\text{D}} \, \overline{\nu_R} \, \nu_L + \text{H.c.}$$

Standard Dirac Mass Term

$$\mathscr{L}_{\text{mass}}^{L} = \frac{1}{2} \, m_L \, \nu_L^T \, \mathcal{C}^\dagger \, \nu_L + \text{H.c.}$$

ν_L Majorana Mass Term
Forbidden in the SM

$$\mathscr{L}_{\text{mass}}^{R} = \frac{1}{2} \, m_R \, \nu_R^T \, \mathcal{C}^\dagger \, \nu_R + \text{H.c.}$$

ν_R Majorana Mass Term
Allowed in the SM

- Column matrix of left-handed chiral fields: $N_L = \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} = \begin{pmatrix} \nu_L \\ \mathcal{C}\,\overline{\nu_R}^T \end{pmatrix}$

$$\mathscr{L}_{\text{mass}}^{\text{D+M}} = \frac{1}{2}\, N_L^T\, \mathcal{C}^\dagger\, M\, N_L + \text{H.c.} \qquad M = \begin{pmatrix} m_L & m_{\text{D}} \\ m_{\text{D}} & m_R \end{pmatrix}$$

- The Dirac-Majorana Mass Term has the structure of a Majorana Mass Term for two chiral neutrino fields coupled by the Dirac mass

- Diagonalization: $n_L = U^\dagger\, N_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix}$

$$U^T\, M\, U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad \text{Real } m_k \geq 0$$

- $\mathscr{L}_{\text{mass}}^{\text{D+M}} = \frac{1}{2} \sum_{k=1,2} m_k\, \nu_{kL}^T\, \mathcal{C}^\dagger\, \nu_{kL} + \text{H.c.} = -\frac{1}{2} \sum_{k=1,2} m_k\, \overline{\nu_k}\, \nu_k$

$$\nu_k = \nu_{kL} + \nu_{kL}^c$$

- Massive neutrinos are Majorana! $\qquad \nu_k = \nu_k^c$

Real Mass Matrix

- CP is conserved if the mass matrix is real: $M = M^*$
- $M = \begin{pmatrix} m_L & m_{\text{D}} \\ m_{\text{D}} & m_R \end{pmatrix}$ we consider real and positive m_R and m_{D} and real m_L
- A real symmetric mass matrix can be diagonalized with $U = \mathcal{O}\,\rho$

$$\mathcal{O} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \qquad \rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \qquad \rho_k^2 = \pm 1$$

- $\mathcal{O}^T\,M\,\mathcal{O} = \begin{pmatrix} m'_1 & 0 \\ 0 & m'_2 \end{pmatrix}$ $\qquad \tan 2\vartheta = \dfrac{2 m_{\text{D}}}{m_R - m_L}$

$$m'_{2,1} = \frac{1}{2}\left[m_L + m_R \pm \sqrt{(m_L - m_R)^2 + 4\,m_{\text{D}}^2} \right]$$

- m'_1 is negative if $m_L m_R < m_{\text{D}}^2$

$$U^T M U = \rho^T \mathcal{O}^T M \mathcal{O} \rho = \begin{pmatrix} \rho_1^2 m'_1 & 0 \\ 0 & \rho_2^2 m'_2 \end{pmatrix} \implies \boxed{m_k = \rho_k^2\,m'_k}$$

- m_2' is always positive:

$$m_2 = m_2' = \frac{1}{2}\left[m_L + m_R + \sqrt{(m_L - m_R)^2 + 4\,m_{\mathrm{D}}^2}\right]$$

- If $m_L m_R \geq m_{\mathrm{D}}^2$, then $m_1' \geq 0$ and $\rho_1^2 = 1$

$$m_1 = \frac{1}{2}\left[m_L + m_R - \sqrt{(m_L - m_R)^2 + 4\,m_{\mathrm{D}}^2}\right]$$

$$\rho_1 = 1 \text{ and } \rho_2 = 1 \quad \Longrightarrow \quad U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

- If $m_L m_R < m_{\mathrm{D}}^2$, then $m_1' < 0$ and $\rho_1^2 = -1$

$$m_1 = \frac{1}{2}\left[\sqrt{(m_L - m_R)^2 + 4\,m_{\mathrm{D}}^2} - (m_L + m_R)\right]$$

$$\rho_1 = i \text{ and } \rho_2 = 1 \quad \Longrightarrow \quad U = \begin{pmatrix} i\cos\vartheta & \sin\vartheta \\ -i\sin\vartheta & \cos\vartheta \end{pmatrix}$$

- If Δm^2 is small, there are oscillations between active ν_a generated by ν_L and sterile ν_s generated by ν_R^c:

$$P_{\nu_a \to \nu_s}(L, E) = \sin^2 2\vartheta \, \sin^2 \left(\frac{\Delta m^2 L}{4 E} \right)$$

$$\Delta m^2 = m_2^2 - m_1^2 = (m_L + m_R) \sqrt{(m_L - m_R)^2 + 4 \, m_\mathrm{D}^2}$$

- It can be shown that the CP parity of ν_k is $\xi_k^{\mathrm{CP}} = i \, \rho_k^2$:

$$\nu_k(x) \xrightarrow{\mathrm{CP}} \xi_k^{\mathrm{CP}} \gamma^0 \overline{\nu_k}^T (x_\mathrm{P}) \qquad \xi_1^{\mathrm{CP}} = i \, \rho_1^2 \qquad \xi_2^{\mathrm{CP}} = i$$

- Special cases:
  - $m_L = m_R \implies$ Maximal Mixing
  - $m_L = m_R = 0 \implies$ Dirac Limit
  - $|m_L|, m_R \ll m_\mathrm{D} \implies$ Pseudo-Dirac Neutrinos
  - $m_L = 0 \quad m_\mathrm{D} \ll m_R \implies$ Seesaw Mechanism

Maximal Mixing

$$\boxed{m_L = m_R}$$

$$\tan 2\vartheta = \frac{2m_{\text{D}}}{m_R - m_L} \quad \Longrightarrow \quad \vartheta = \pi/4$$

$$m'_{2,1} = m_L \pm m_{\text{D}}$$

$$\begin{cases} \rho_1^2 = +1\,, \quad m_1 = m_L - m_{\text{D}} \quad \text{if} \quad m_L \geq m_{\text{D}} \\ \rho_1^2 = -1\,, \quad m_1 = m_{\text{D}} - m_L \quad \text{if} \quad m_L < m_{\text{D}} \end{cases}$$

$$m_2 = m_L + m_{\text{D}}$$

$$\underline{m_L < m_{\text{D}}}$$

$$\begin{cases} \nu_{1L} = \dfrac{-i}{\sqrt{2}} (\nu_L - \nu_R^c) \\ \nu_{2L} = \dfrac{1}{\sqrt{2}} (\nu_L + \nu_R^c) \end{cases}$$

$$\begin{cases} \nu_1 = \nu_{1L} + \nu_{1L}^c = \dfrac{-i}{\sqrt{2}} \left[(\nu_L + \nu_R) - (\nu_L^c + \nu_R^c) \right] \\ \nu_2 = \nu_{2L} + \nu_{2L}^c = \dfrac{1}{\sqrt{2}} \left[(\nu_L + \nu_R) + (\nu_L^c + \nu_R^c) \right] \end{cases}$$

Dirac Limit

$$\boxed{m_L = m_R = 0}$$

- $m'_{2,1} = \pm m_{\rm D} \quad \Longrightarrow \quad \begin{cases} \rho_1^2 = -1 & m_1 = m_{\rm D} & \xi_1^{\rm CP} = -i \\ \rho_2^2 = +1 & m_2 = m_{\rm D} & \xi_2^{\rm CP} = i \end{cases}$
- The two Majorana fields ν_1 and ν_2 can be combined to give one Dirac field:

$$\nu = \frac{1}{\sqrt{2}} (i\nu_1 + \nu_2) = \nu_L + \nu_R$$

- A Dirac field ν can always be split in two Majorana fields:

$$\begin{aligned} \nu &= \frac{1}{2} \left[(\nu - \nu^c) + (\nu + \nu^c) \right] \\ &= \frac{i}{\sqrt{2}} \left(-i \, \frac{\nu - \nu^c}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} \left(\frac{\nu + \nu^c}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} (i\nu_1 + \nu_2) \end{aligned}$$

- A Dirac field is equivalent to two Majorana fields with the same mass and opposite CP parities

Pseudo-Dirac Neutrinos

$$\boxed{|m_L|\,,\; m_R \ll m_{\text{D}}}$$

- $m'_{2,1} \simeq \dfrac{m_L + m_R}{2} \pm m_{\text{D}}$
- $m'_1 < 0 \quad \Longrightarrow \quad \rho_1^2 = -1 \quad \Longrightarrow \quad m_{2,1} \simeq m_{\text{D}} \pm \dfrac{m_L + m_R}{2}$
- The two massive Majorana neutrinos are almost degenerate in mass and have opposite CP parities $(\xi_1^{\text{CP}} = -i, \quad \xi_2^{\text{CP}} = i)$
- The best way to reveal pseudo-Dirac neutrinos are active-sterile neutrino oscillations due to the small squared-mass difference

$$\Delta m^2 \simeq m_{\text{D}} \, (m_L + m_R)$$

- The oscillations occur with practically maximal mixing:

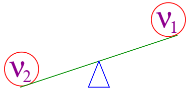
$$\tan 2\vartheta = \frac{2 m_{\text{D}}}{m_R - m_L} \gg 1 \quad \Longrightarrow \quad \vartheta \simeq \pi/4$$

Type-I Seesaw Mechanism

[Minkowski, PLB 67 (1977) 42; Yanagida (1979); Gell-Mann, Ramond, Slansky (1979); Mohapatra, Senjanovic, PRL 44 (1980) 912]

$$\boxed{m_L = 0 \quad m_{\text{D}} \ll m_R}$$

- $\mathscr{L}^L_{\text{mass}}$ is forbidden in the SM $\Longrightarrow$ $m_L = 0$
- $m_{\text{D}} \lesssim v \sim 100\,\text{GeV}$ is generated by SM Higgs Mechanism (protected by SM symmetries)
- m_R is not protected by SM symmetries $\Longrightarrow$ $m_R \sim \mathcal{M}_{\text{GUT}} \gg v$
- $$\left.\begin{array}{l} m'_1 \simeq -\dfrac{m_{\text{D}}^2}{m_R} \\ m'_2 \simeq m_R \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} \rho_1^2 = -1\,, & m_1 \simeq \dfrac{m_{\text{D}}^2}{m_R} \\ \rho_2^2 = +1\,, & m_2 \simeq m_R \end{array}\right.$$



- Natural explanation of smallness of neutrino masses
- Mixing angle is very small: $\tan 2\vartheta = 2\,\dfrac{m_{\text{D}}}{m_R} \ll 1$
- ν_1 is composed mainly of active ν_L: $\nu_{1L} \simeq -i\,\nu_L$
- ν_2 is composed mainly of sterile ν_R: $\nu_{2L} \simeq \nu_R^c$

Connection with Effective Lagrangian Approach

- Dirac–Majorana neutrino mass term with $m_L = 0$:

$$\mathscr{L}^{\text{D+M}} = -m_{\text{D}} \left(\overline{\nu_R}\, \nu_L + \overline{\nu_L}\, \nu_R \right) + \frac{1}{2}\, m_R \left(\nu_R^T\, \mathcal{C}^\dagger\, \nu_R + \nu_R^\dagger\, \mathcal{C}\, \nu_R^* \right)$$

- Above the electroweak symmetry-breaking scale:

$$\mathscr{L}^{\text{D+M}} = -y^\nu \left(\overline{\nu_R}\, \widetilde{\Phi}^\dagger\, L_L + \overline{L_L}\, \widetilde{\Phi}\, \nu_R \right) + \frac{1}{2}\, m_R \left(\nu_R^T\, \mathcal{C}^\dagger\, \nu_R + \nu_R^\dagger\, \mathcal{C}\, \nu_R^* \right)$$

- If $m_R \gg v \implies \nu_R$ is static $\implies$ kinetic term in equation of motion can be neglected:

$$0 \simeq \frac{\partial \mathscr{L}^{\text{D+M}}}{\partial \nu_R} = m_R\, \nu_R^T\, \mathcal{C}^\dagger - y^\nu\, \overline{L_L}\, \widetilde{\Phi}$$

$$\nu_R \simeq -\frac{y^\nu}{m_R}\, \widetilde{\Phi}^T\, \mathcal{C}\, \overline{L_L}^T$$

$$\mathscr{L}^{\text{D+M}} \quad \to \quad \mathscr{L}_5^{\text{D+M}} \simeq -\frac{1}{2}\, \frac{(y^\nu)^2}{m_R} \left(L_L^T\, \sigma_2\, \Phi \right) \mathcal{C}^\dagger \left(\Phi^T\, \sigma_2\, L_L \right) + \text{H.c.}$$

$$\mathscr{L}_5 = \frac{g}{\mathcal{M}} \left(L_L^T \, \sigma_2 \, \Phi \right) \mathcal{C}^\dagger \left(\Phi^T \, \sigma_2 \, L_L \right) + \text{H.c.}$$

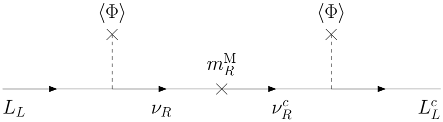
$$\mathscr{L}_5^{\text{D+M}} \simeq -\frac{1}{2} \, \frac{(y^\nu)^2}{m_R} \left(L_L^T \, \sigma_2 \, \Phi \right) \mathcal{C}^\dagger \left(\Phi^T \, \sigma_2 \, L_L \right) + \text{H.c.}$$

$$g = -\frac{(y^\nu)^2}{2} \qquad\qquad \mathcal{M} = m_R$$

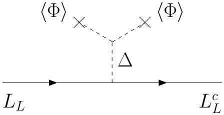
- See-saw mechanism is a particular case of the effective Lagrangian approach.
- See-saw mechanism is obtained when dimension-five operator is generated only by the presence of ν_R with $m_R \sim \mathcal{M}$.
- In general, other terms can contribute to $\mathscr{L}_5$.

Three Seesaw Types

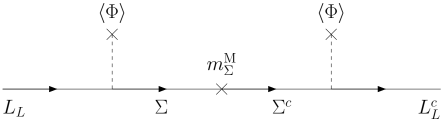
- Since combining the two doublets L_L and Φ one can form singlets and triplets, there are three types of Seesaw types that can be generated at the tree level.
- Type-I Seesaw: intermediate fermion singlets ν_R



- Type-II Seesaw: coupling with boson triplets Δ



- Type-III Seesaw: intermediate fermion triplets Σ



Generalized Seesaw Mechanism

- General effective Dirac-Majorana mass matrix:

$$M = \begin{pmatrix} m_L & m_\mathrm{D} \\ m_\mathrm{D} & m_R \end{pmatrix}$$

- m_L generated by dim-5 operator:

$$m_L \ll m_\mathrm{D} \ll m_R$$

- Eigenvalues:

$$\begin{vmatrix} m_L - \mu & m_\mathrm{D} \\ m_\mathrm{D} & m_R - \mu \end{vmatrix} = 0$$

$$\mu^2 - (\cancel{m_L} + m_R)\, \mu + m_L m_R - m_\mathrm{D}^2 = 0$$

$$\mu = \frac{1}{2}\left[m_R \pm \sqrt{m_R^2 - 4\left(m_L m_R - m_\mathrm{D}^2\right)}\right]$$

$$\mu = \frac{1}{2}\left[m_R \pm \sqrt{m_R^2 - 4\left(m_L m_R - m_{\text{D}}^2\right)}\right]$$
$$= \frac{1}{2}\left[m_R \pm m_R\left(1 - 4\frac{m_L m_R - m_{\text{D}}^2}{m_R^2}\right)^{1/2}\right]$$
$$\simeq \frac{1}{2}\left[m_R \pm m_R\left(1 - 2\frac{m_L m_R - m_{\text{D}}^2}{m_R^2}\right)\right]$$

$$+ \quad \to \quad m_{\text{heavy}} \simeq m_R$$
$$- \quad \to \quad m_{\text{light}} \simeq \left| m_L - \frac{m_{\text{D}}^2}{m_R} \right|$$

Dominant type I seesaw: $m_L \ll \frac{m_{\text{D}}^2}{m_R} \implies m_{\text{light}} \simeq \frac{m_{\text{D}}^2}{m_R}$

Dominant type II or III seesaw: $m_L \gg \frac{m_{\text{D}}^2}{m_R} \implies m_{\text{light}} \simeq m_L$

Right-Handed Neutrino Mass Term

$$\mathcal{L}_R^{\mathrm{M}} = -\frac{1}{2}\, m \left(\overline{\nu_R^c}\, \nu_R + \overline{\nu_R}\, \nu_R^c \right)$$

- $\mathcal{L}_R^{\mathrm{M}}$ respects the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ SM symmetry
- $\mathcal{L}_R^{\mathrm{M}}$ breaks Lepton number conservation

Three possibilities:

- Lepton number can be explicitly broken
- Lepton number is spontaneously broken locally, with a massive vector boson coupled to the lepton number current
- Lepton number is spontaneously broken globally and a massless Goldstone boson appears in the theory (Majoron)

Singlet Majoron Model

[Chikashige, Mohapatra, Peccei, Phys. Lett. B98 (1981) 265, Phys. Rev. Lett. 45 (1980) 1926]

$$\mathcal{L}_\Phi = -y_d \left(\overline{L_L}\, \Phi\, \nu_R + \overline{\nu_R}\, \Phi^\dagger\, L_L \right) \xrightarrow[\langle\Phi\rangle \neq 0]{} -m_{\mathrm{D}} \left(\overline{\nu_L}\, \nu_R + \overline{\nu_R}\, \nu_L \right)$$

$$\mathcal{L}_\eta = -y_s \left(\eta\, \overline{\nu_R^c}\, \nu_R + \eta^\dagger\, \overline{\nu_R}\, \nu_R^c \right) \xrightarrow[\langle\eta\rangle \neq 0]{} -\frac{1}{2}\, m_R \left(\overline{\nu_R^c}\, \nu_R + \overline{\nu_R}\, \nu_R^c \right)$$

$$\eta = 2^{-1/2} \left(\langle\eta\rangle + \rho + i\,\chi \right) \qquad \mathcal{L}_{\mathrm{mass}} = -\frac{1}{2} \left(\overline{\nu_L^c} \;\; \overline{\nu_R} \right) \begin{pmatrix} 0 & m_{\mathrm{D}} \\ m_{\mathrm{D}} & m_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \mathrm{H.c.}$$

$$\underset{\text{scale of } L \text{ violation}}{m_R} \gg \underset{\text{EW scale}}{m_{\mathrm{D}}} \Longrightarrow \text{Type-I Seesaw:} \quad \boxed{m_1 \simeq \frac{m_{\mathrm{D}}^2}{m_R}}$$

ρ = massive scalar, χ = Majoron (massless pseudoscalar Goldstone boson)

The Majoron is weakly coupled to the light neutrino

$$\mathcal{L}_{\chi-\nu} = \frac{i y_s}{\sqrt{2}}\, \chi \left[\overline{\nu_2} \gamma^5 \nu_2 - \frac{m_{\mathrm{D}}}{m_R} \left(\overline{\nu_2} \gamma^5 \nu_1 + \overline{\nu_1} \gamma^5 \nu_2 \right) + \left(\frac{m_{\mathrm{D}}}{m_R} \right)^2 \overline{\nu_1} \gamma^5 \nu_1 \right]$$

Three-Generation Mixing

$$\mathscr{L}_{\text{mass}}^{\text{D+M}} = \mathscr{L}_{\text{mass}}^{\text{D}} + \mathscr{L}_{\text{mass}}^{L} + \mathscr{L}_{\text{mass}}^{R}$$

$$\mathscr{L}_{\text{mass}}^{\text{D}} = -\sum_{s=1}^{N_S} \sum_{\alpha=e,\mu,\tau} \overline{\nu'_{sR}}\, M^{\text{D}}_{s\alpha}\, \nu'_{\alpha L} + \text{H.c.}$$

$$\mathscr{L}_{\text{mass}}^{L} = \frac{1}{2} \sum_{\alpha,\beta=e,\mu,\tau} \nu'^{T}_{\alpha L}\, \mathcal{C}^{\dagger}\, M^{L}_{\alpha\beta}\, \nu'_{\beta L} + \text{H.c.}$$

$$\mathscr{L}_{\text{mass}}^{R} = \frac{1}{2} \sum_{s,s'=1}^{N_S} \nu'^{T}_{sR}\, \mathcal{C}^{\dagger}\, M^{R}_{ss'}\, \nu'_{s'R} + \text{H.c.}$$

$$\boldsymbol{N}'_L \equiv \begin{pmatrix} \boldsymbol{\nu}'_L \\ \boldsymbol{\nu}'^{C}_R \end{pmatrix} \qquad \boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \qquad \boldsymbol{\nu}'^{C}_R \equiv \begin{pmatrix} \nu'^{C}_{1R} \\ \vdots \\ \nu'^{C}_{N_S R} \end{pmatrix}$$

$$\mathscr{L}_{\text{mass}}^{\text{D+M}} = \frac{1}{2}\, \boldsymbol{N}'^{T}_L\, \mathcal{C}^{\dagger}\, M^{\text{D+M}}\, \boldsymbol{N}'_L + \text{H.c.} \qquad M^{\text{D+M}} = \begin{pmatrix} M^L & M^{\text{D}\,T} \\ M^{\text{D}} & M^R \end{pmatrix}$$

- Diagonalization of the Dirac-Majorana Mass Term $\Longrightarrow$ massive Majorana neutrinos

- See-Saw Mechanism $\Longrightarrow$ right-handed neutrinos have large Majorana masses and are decoupled from the low-energy phenomenology.

- If all right-handed neutrinos have large Majorana masses, at low energy we have an effective mixing of three Majorana neutrinos.

- It is possible that not all right-handed neutrinos have large Majorana masses: some right-handed neutrinos may correspond to low-energy Majorana particles which belong to new physics beyond the Standard Model.

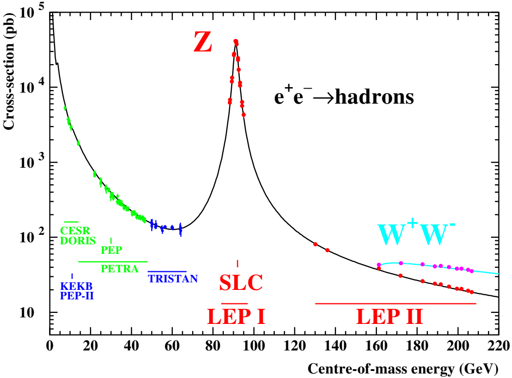
- Light anti-ν_R are called sterile neutrinos

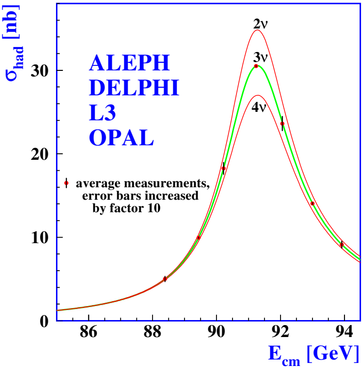
$$\nu_R^c \to \nu_{sL} \qquad \text{(left-handed)}$$

Sterile Neutrinos

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
- Sterile Neutrinos
  - Number of Flavor and Massive Neutrinos?
  - Sterile Neutrinos
  - Fundamental Fields in QFT

Number of Flavor and Massive Neutrinos?





[LEP, Phys. Rept. 427 (2006) 257, arXiv:hep-ex/0509008]

$$\Gamma_Z = \sum_{\ell=e,\mu,\tau} \Gamma_{Z\to\ell\bar{\ell}} + \sum_{q\neq t} \Gamma_{Z\to q\bar{q}} + \Gamma_{\text{inv}} \qquad \Gamma_{\text{inv}} = N_\nu \, \Gamma_{Z\to\nu\bar{\nu}}$$

$$\boxed{N_\nu = 2.9840 \pm 0.0082}$$

Improved cross section: $N_\nu = 2.9975 \pm 0.0074$ [Janot, Jadach, arXiv:1912.02067]

$$e^+ e^- \to Z \xrightarrow{\text{invisible}} \sum_{a=\text{active}} \nu_a \bar{\nu}_a \quad \Longrightarrow \quad \nu_e \; \nu_\mu \; \nu_\tau$$

3 light active flavor neutrinos

mixing $\Rightarrow$ $$\nu_{\alpha L} = \sum_{k=1}^{N} U_{\alpha k} \nu_{kL} \qquad \alpha = e, \mu, \tau$$

$N \geq 3$
no upper limit!

Mass Basis:	ν_1	ν_2	ν_3	ν_4	ν_5	$\cdots$
Flavor Basis:	ν_e	ν_μ	ν_τ	ν_{s_1}	ν_{s_2}	$\cdots$
	ACTIVE			STERILE		

$$\boxed{\nu_{\alpha L} = \sum_{k=1}^{N} U_{\alpha k} \nu_{kL} \qquad \alpha = e, \mu, \tau, s_1, s_2, \ldots}$$

Sterile Neutrinos

- Sterile means no standard model interactions

  [Pontecorvo, Sov. Phys. JETP 26 (1968) 984]

- Obviously no electromagnetic interactions as normal active neutrinos
- Thus sterile means no standard weak interactions
- But sterile neutrinos are not absolutely sterile:
  - Gravitational Interactions
  - New non-standard interactions of the physics beyond the Standard Model which generates the masses of sterile neutrinos
- Active neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ can oscillate into sterile neutrinos (ν_s)
- Observables:
  - Disappearance of active neutrinos
  - Indirect evidence through combined fit of data
- Powerful window on new physics beyond the Standard Model

No GIM with Sterile Neutrinos

[Lee, Shrock, PRD 16 (1977) 1444; Schechter, Valle PRD 22 (1980) 2227]

- Neutrino Neutral-Current Weak Interaction Lagrangian:

$$\mathscr{L}_{\mathrm{I}}^{(\mathrm{NC})} = -\frac{g}{2\cos\vartheta_{\mathrm{W}}} Z_\rho \overline{\boldsymbol{\nu}_L'} \gamma^\rho \boldsymbol{\nu}_L'$$

- The transformation to active flavor neutrino fields is independent of the existence of sterile neutrinos: $\boldsymbol{\nu}_L' = V_L^\ell \boldsymbol{\nu}_L$

$$\mathscr{L}_{\mathrm{I}}^{(\mathrm{NC})} = -\frac{g}{2\cos\vartheta_{\mathrm{W}}} Z_\rho \overline{\boldsymbol{\nu}_L} \gamma^\rho \boldsymbol{\nu}_L = -\frac{g}{2\cos\vartheta_{\mathrm{W}}} Z_\rho \sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}} \gamma^\rho \nu_{\alpha L}$$

- Mixing with sterile neutrinos: $\displaystyle \nu_{\alpha L} = \sum_{k=1}^{3+N_s} U_{\alpha k} \nu_{kL}$

- No GIM: $\displaystyle \mathscr{L}_{\mathrm{I}}^{(\mathrm{NC})} = -\frac{g}{2\cos\vartheta_{\mathrm{W}}} Z_\rho \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} \overline{\nu_{jL}} \gamma^\rho \nu_{kL} \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k}$

- $\displaystyle \sum_{\alpha=e,\mu,\tau,s_1,\ldots} U_{\alpha j}^* U_{\alpha k} = \delta_{jk}$ but $\displaystyle \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k} \neq \delta_{jk}$

Effect on Invisible Width of Z Boson?

- Amplitude of $Z \to \nu_j \bar{\nu}_k$ decay:

$$A(Z \to \nu_j \bar{\nu}_k) = \langle \nu_j \bar{\nu}_k | - \int d^4x \, \mathscr{L}_{\mathrm{I}}^{(\mathrm{NC})}(x) | Z \rangle$$
$$= \frac{g}{2\cos\vartheta_{\mathrm{W}}} \langle \nu_j \bar{\nu}_k | \int d^4x \, \overline{\nu_{jL}}(x) \gamma^\rho \nu_{kL}(x) Z_\rho(x) | Z \rangle \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* U_{\alpha k}$$

- If $m_k \ll m_Z/2$ for all k's, the neutrino masses are negligible in all the matrix elements and we can approximate

$$\frac{g}{2\cos\vartheta_{\mathrm{W}}} \langle \nu_j \bar{\nu}_k | \int d^4x \, \overline{\nu_{jL}}(x) \, \gamma^\rho \, \nu_{kL}(x) \, Z_\rho(x) | Z \rangle \simeq A_{\mathrm{SM}}(Z \to \nu_\ell \bar{\nu}_\ell)$$

- $A_{\mathrm{SM}}(Z \to \nu_\ell \bar{\nu}_\ell)$ is the Standard Model amplitude of Z decay into a massless neutrino-antineutrino pair of any flavor $\ell = e, \mu, \tau$

- $A(Z \to \nu_j \bar{\nu}_k) \simeq A_{\mathrm{SM}}(Z \to \nu_\ell \bar{\nu}_\ell) \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* \, U_{\alpha k}$

- $P(Z \to \nu \bar{\nu}) = \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} |A(Z \to \nu_j \bar{\nu}_k)|^2$

- $P(Z \to \nu\bar{\nu}) \simeq P_{\mathrm{SM}}(Z \to \nu_\ell\bar{\nu}_\ell) \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} \left| \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^{*} \, U_{\alpha k} \right|^2$
- Effective number of neutrinos in Z decay:

$$N_\nu^{(Z)} = \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} \left| \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^{*} \, U_{\alpha k} \right|^2$$

- Using the unitarity relation $\displaystyle\sum_{k=1}^{3+N_s} U_{\alpha k} \, U_{\beta k}^{*} = \delta_{\alpha\beta}$ we obtain

$$\begin{aligned} N_\nu^{(Z)} &= \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^{*} \, U_{\alpha k} \sum_{\beta=e,\mu,\tau} U_{\beta j} \, U_{\beta k}^{*} \\ &= \sum_{\alpha=e,\mu,\tau} \sum_{\beta=e,\mu,\tau} \underbrace{\sum_{j=1}^{3+N_s} U_{\alpha j}^{*} \, U_{\beta j}}_{\delta_{\alpha\beta}} \underbrace{\sum_{k=1}^{3+N_s} U_{\alpha k} \, U_{\beta k}^{*}}_{\delta_{\alpha\beta}} = \sum_{\alpha=e,\mu,\tau} 1 = 3 \end{aligned}$$

- $\boxed{N_\nu^{(Z)} = 3}$ independently of the number of light sterile neutrinos!

Effect of Heavy Sterile Neutrinos

[Jarlskog, PLB 241 (1990) 579; Bilenky, Grimus, Neufeld, PLB 252 (1990) 119]

- $N_\nu^{(Z)} = \sum_{j=1}^{3+N_s} \sum_{k=1}^{3+N_s} \left| \sum_{\alpha=e,\mu,\tau} U_{\alpha j}^* \, U_{\alpha k} \right|^2 R_{jk}$ with

$$R_{jk} = \left(1 - \frac{m_j^2 + m_k^2}{2m_Z^2} - \frac{(m_j^2 - m_k^2)^2}{2m_Z^4} \right) \frac{\lambda(m_Z^2, m_j^2, m_k^2)}{m_Z^2} \theta(m_Z - m_j - m_k)$$

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$

- $R_{jk} \leq 1 \quad \Longrightarrow \quad \boxed{N_\nu^{(Z)} \leq 3}$

Fundamental Fields in QFT

- Each elementary particle is described by a field which is an irreducible representation of the Poincaré group (Lorentz group + space-time translations).
- In this way
  - Under Poincaré transformation an elementary particle remains itself.
  - Lagrangian is constructed with invariant products of elementary fields.
- Spinorial structure of a particle is determined by its representation under the restricted Lorentz group of proper and orthochronous Lorentz transformation (no space or time inversions).

- Restricted Lorentz group is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$.
- Classification of fundamental representations:

$(0,0)$	scalar φ
$(1/2,0)$	left-handed Weyl spinor χ_L (Majorana if massive)
$(0,1/2)$	right-handed Weyl spinor χ_R (Majorana if massive)

- All representations are constructed combining the two fundamental Weyl spinor representations.

$(1/2,1/2)$	four-vector v^μ (irreducible)
$(1/2,0)+(0,1/2)$	four-component Dirac spinor ψ (reducible)

- Two-component Weyl (Majorana if massive) spinor is more fundamental than four-component Dirac spinor.

- Two-component left-handed Weyl (Majorana if massive) spinor:

$$\chi_L = \begin{pmatrix} \chi_{L1} \\ \chi_{L2} \end{pmatrix}$$

- Two-component right-handed Weyl (Majorana if massive) spinor:

$$\chi_R = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix}$$

- Four-component Dirac spinor: $\psi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$

- Lorentz transformation: $v^\mu \to v'^\mu = \Lambda^\mu{}_\nu \, v^\nu$

$$g_{\mu\nu} \, \Lambda^\mu{}_\rho \, \Lambda^\nu{}_\sigma = g_{\rho\sigma} \qquad g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

- Restricted Lorentz transformation: $\Lambda^\mu{}_\nu = [e^\omega]^\mu{}_\nu \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & v_1 & v_2 & v_3 \\ -v_1 & 0 & \theta_3 & -\theta_2 \\ -v_2 & -\theta_3 & 0 & \theta_1 \\ -v_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}$$

- 6 parameters:
  - 3 for rotations: $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$
  - 3 for boosts: $\vec{v} = (v_1, v_2, v_3)$

$$\chi_L \to \chi'_L = \Lambda_L \chi_L \qquad \Lambda_L = e^{i\left(\vec{\theta} - i\vec{v}\right)\cdot\vec{\sigma}/2}$$

$$\chi_R \to \chi'_R = \Lambda_R \chi_R \qquad \Lambda_R = e^{i\left(\vec{\theta} + i\vec{v}\right)\cdot\vec{\sigma}/2}$$

- Four-component form of two-component left-handed Weyl (Majorana if massive) spinor:

$$\psi_L = \begin{pmatrix} 0 \\ \chi_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$$

- Majorana mass term:

$$\mathscr{L}^L_{\text{mass}} = \underbrace{\frac{1}{2} m_L \psi_L^T \mathcal{C}^\dagger \psi_L + \text{H.c.}}_{\text{four-component form}} = \underbrace{-\frac{1}{2} m_L \chi_L^T i\sigma^2 \chi_L + \text{H.c.}}_{\text{two-component form}}$$

$$(1/2, 0) \times (1/2, 0) = \underbrace{(1, 0)}_{\text{symmetric}} + \underbrace{(0, 0)}_{\text{antisymmetric}}$$

σ^2 is antisymmetric!

- Anticommutativity of spinors is necessary, otherwise

$$\chi_L^T i\sigma^2 \chi_L = \left(\chi_L^T i\sigma^2 \chi_L\right)^T = -\chi_L^T i\sigma^2 \chi_L = 0$$