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Anomalous diffusion and anisotropic nonlinear Fokker–Planck equation

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Abstract

We analyse a N-dimensional anisotropic nonlinear Fokker–Planck equation by considering stationary and time-dependent solutions. The stationary solutions are obtained for very general situations, including those when the diffusion coefficients are spatial dependents. Time-dependent solutions are found in the absence of external force and with constant diffusion coefficients. When restricted to the bi-dimensional case, our investigation about time-dependent solutions focuses on situations where the diffusion coefficient are $D_x \propto |x|^{-\theta_x}$ and $D_y \propto |y|^{-\theta_y}$ with $\theta_x, \theta_y \in \mathscr{R}$. In general, we verify an anomalous behavior induced in a given direction due to the other directions.

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1. Introduction

The existence of the anomalous diffusion and its ubiquity has motivated the study of nonlinear Fokker–Planck equations, e.g., $\partial_t \rho = \mathscr{D}\nabla^2\rho^\nu$. In particular, this equation has been employed in many physical situations such as percolation of gases through porous media ($\nu \geqslant 2$) [1], thin saturated regions in porous media ($\nu = 2$) [2], a standard

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solid-on-solid model for surface growth ($\nu = 3$) and thin liquid films spreading under gravity (ν=4) [3]. In this direction, it is important to understand the properties concerning the nonlinear Fokker–Planck equation to characterize physical systems related to it. In recent works, it has been analyzed in several situations, for instance, by considering a linear external force [4–6], absorption term [7], spatial dependence in the diffusion coefficient (i.e., $\mathscr{D}(x) \propto |x|^{-\theta}$) [8,9] and the external force with a diffusion coefficient time-dependent [10]. In Ref. [11] the nonlinear Fokker–Planck equation was extended by using fractional derivatives. The relation between the solutions of the porous media equation and distribution of probability that emerges from the nonextensive statistics has been investigated in Refs. [4–6]. Other aspects are analyzed in Ref. [12–14].

These investigations essentially focused the nonlinear Fokker–Planck equation in one dimension or in N-dimension in an isotropic medium. However, there are situations that are characterized by an anomalous diffusion in anisotropic media such as a crystal with randomly distributed topological defects [15], in neutron scattering study of hydrated myoglobin [16], and a diffusion on bi-dimensional percolation of anisotropic clusters [17]. In this context, a careful analysis of the anisotropic case for the nonlinear Fokker–Planck equation has not been performed. Thus, it is an open question how to extend the above results for the nonlinear Fokker–Planck equation taking an anisotropic medium into account. In this direction, we focus our attention on the following anisotropic nonlinear Fokker–Planck equation:

$$\frac{\partial}{\partial t}\rho(\bar{x};t) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left\{ D_{ij}\frac{\partial}{\partial x_j}[\rho(\bar{x};t)]^{\nu}\right\} - \sum_{i=1}^{N}\frac{\partial}{\partial x_i}[F_i(\bar{x};t)\rho(\bar{x};t)] , \tag{1}$$

where $\bar{x} = (x_1, x_2, \ldots, x_N)$, $F_i(\bar{x};t)$ is an external force and D_{ij} are the diffusion coefficients, which can be time and spatial dependent in general. A particular but important case occurs when F_i is linear and $D_{ij} = \mathscr{D}_i|x|^{-\theta_i}\delta_{ij}$ with $\mathscr{D}_i$ and θ_i being constants. Note also that the cases mentioned above [4–9,11,12] can essentially be obtained from the unidimensional version of Eq. (1) by an adequate choice of F_i, $\mathscr{D}_i$, θ_i and ν.

In this work, we first investigate a stationary solution of Eq. (1) for the bi-dimensional case with $D_{ij} = D_i(x_i)\delta_{ij}$. Then, an N-dimensional stationary solution of Eq. (1) is obtained considering D_{ij} as a nonsingular spatial dependent matrix. After that, we start our analysis of time-dependent solutions for Eq. (1) by considering the bi-dimensional case without external force and $\theta_x = \theta_y = 0$. In this context, the N-dimensional case is also investigated. To study time-dependent solution with $\theta_x \neq 0$, $\theta_y \neq 0$, we return to the bi-dimensional case. In all cases, we emphasize that the solutions are analytically obtained and the presence of the nonlinearity produces an effective coupling among the directions, in contrast with the usual Fokker–Planck equation ($\nu = 1$).

2. Stationary solutions

The stationary solution for the anisotropic nonlinear Fokker–Planck equation in the bi-dimensional case, when D_{ij}=$D_i\delta_{ij}$, can be written as $\nabla\cdot\bar{J}$=0 with $J_x = -D_x\partial\rho^{\nu}/\partial x + F_x\rho$ and $J_y = -D_y\partial\rho^{\nu}/\partial y + F_y\rho$. Note that a constant $\bar{J}$ is a solution for the stationary equation $\nabla \cdot \bar{J} = 0$. Then, we consider this solution by accomplishing the boundary conditions $\bar{J}(x \to \infty, y \to \infty) \to 0$, i.e., we investigate the case $\bar{J} = 0$. At this point, we suppose that $\bar{F} = -\nabla V$. Thus, $\partial \rho^{\nu}/\partial x = -(\rho/D_x)\partial V/\partial x$ and $\partial \rho^{\nu}/\partial y = -(\rho/D_y)\partial V/\partial y$. Therefore, the integrability condition for these two equations leads to

$$\frac{1}{D_x}\frac{\partial^2 V}{\partial x \partial y} - \frac{1}{D_x^2}\frac{\partial V}{\partial x}\frac{\partial D_x}{\partial y} = \frac{1}{D_y}\frac{\partial^2 V}{\partial x \partial y} - \frac{1}{D_y^2}\frac{\partial V}{\partial y}\frac{\partial D_y}{\partial x} \ . \tag{2}$$

If we restrict Eq. (2) for $D_x = D_x(x)$ and $D_y = D_y(y)$, it implies that $V(x, y) = V_x(x) + V_y(y)$ since $D_x \neq 0$, $D_y \neq 0$ and $D_x \neq D_y$ in general. Within this potential energy, we verify that the stationary solution under consideration is given by

$$\rho(x, y) = \rho_0 \exp_q \left[-\frac{\rho_0^{q-1}}{2-q} \left(\int_0^x \frac{1}{D_x} \frac{\mathrm{d}V_x}{\mathrm{d}x'} \, \mathrm{d}x' + \int_0^y \frac{1}{D_y} \frac{\mathrm{d}V_y}{\mathrm{d}y'} \, \mathrm{d}y' \right) \right] \ , \tag{3}$$

where $\rho_0 = \rho(0, 0)$ and $q = 2 - \nu$. In particular, if D_x and D_y are constants, we have $\rho(x, y) = \rho_0 \exp_q[-\beta_x V_x(x) - \beta_y V_y(y)]$, where $\beta_i = \rho_0^{q-1}/[(2-q)D_i]$ with $i = x, y$. The $\exp_q[x]$ ($\exp_q[x] \equiv [1 + (1-q)x]^{1/(1-q)}$ if $1 + (1-q)x \geqslant 0$ and $\exp_q[x] \equiv 0$ if $1 + (1-q)x < 0$) is the generalization of the exponential (q-exponential) related to the Tsallis formalism [18,19]. In fact, it is simple to verify that the q-exponential arises when the Tsallis entropy $S_q = \{1 - \int\int \mathrm{d}x\,\mathrm{d}y[\rho(x, y)]^q\}/(q-1)$ is maximized subject to adequate constraints, for instance, $\langle\langle V(x)\rangle\rangle = \langle\langle V(y)\rangle\rangle = cte$ with $\langle\langle \mathcal{O}\rangle\rangle \equiv [\int\int \mathrm{d}x\,\mathrm{d}y[\rho(x, y)]^q \mathcal{O}]/[\int\int \mathrm{d}x\,\mathrm{d}y[\rho(x, y)]^q]$ and $\int\int \mathrm{d}x\,\mathrm{d}y \rho(x, y) = 1$.

In the linear case, $\nu = 1$, the previous $\rho(x, y)$ decouples in a product of a function of x times another function of y. More precisely, $\rho(x, y) = \rho_x(x)\rho_y(y)$, where $\rho_x(x)$ ($\rho_y(y)$) is proportional to the reduced distribution for x (y). Consequently, the mean value of any function of x (y) is not affected by the dynamics involving the y (x) variable. However, this scenario is drastically changed when $\nu \neq 1$, in contrast with the separability property of the potential energy that is not affected by the nonlinearity of the diffusion equation. To illustrate that the mean value for an arbitrary function of x contains a memory of the dynamics related to the y variable when $\nu \neq 1$, we consider the potential energy $V(x, y) = k_x|x|^n + k_y|y|^m$ with k_x, k_y, $\mathscr{D}_x$, and $\mathscr{D}_y$ being constants. In this case, when we integrate $\rho(x, y)$ over y, we obtain the reduced distribution $\rho(x) = \int \mathrm{d}y\ \rho(x, y; t) = \rho_0' \exp_{q'}[\beta_x'|x|^n]$ [20], where ρ_0' is a constant, $q' = (1 - q + mq)/(1 - q + m)$, $\beta_x' = [(m + 1 - q)/m]\ \beta_x$, and the integral is defined if $q < m + 1$. From this result we verify that the mean value of a generic function $\mathcal{O}(x)$, $\langle \mathcal{O}(x)\rangle = \int \mathrm{d}x\ \rho(x)\mathcal{O}(x)/\int \mathrm{d}x\ \rho(x)$, does not depend on the degree of anisotropy when $V_y(y) = k_y|y|^m$. However, q' and β_x' depend on the external force in the y direction. In addition, if we have $D_i = \mathscr{D}_i|x_i|^{\theta_i}$ with $\mathscr{D}_i = cte$, $\langle \mathcal{O}(x)\rangle$ will contain, in general, a dependence on θ_y, indicating the presence of a residual information of the degree of anisotropy. These facts can be interpreted as a memory induced by the nonlinearity of the diffusion equation (1).

For the N-dimensional case, if we suppose that $F_i = -\partial V/\partial x_i$, $D_{ij} = D_i(x_i)\delta_{ij}$ in Eq. (1) and employ the previous analysis, we can verify that $V(\bar{x}) = \sum_{i=1}^{N} V_i(x_i)$

and

$$\rho(\bar{x}) = \rho_0 \exp_q \left[-\frac{\rho_0^{q-1}}{2-q} \sum_{i=1}^{N} \int_0^x \frac{1}{D_i} \frac{\mathrm{d}V_{x_i'}}{\mathrm{d}x_i'} \,\mathrm{d}x_i' \right] . \tag{4}$$

On the other hand, when we have mixed derivatives, the generalization of this result for the N-dimensional case is not so immediate. In order to circumvent this difficulty, we employ another line of reasoning. First, we note that Eq. (1) can be written as the continuity equation $\partial\rho/\partial t + \sum_{i=1}^{N} \partial J_i/\partial x_i = 0$ with $J_i = -\sum_{j=1}^{N} D_{ij}\partial\rho^{\nu}/\partial x_j + F_i\rho$. Thus, a stationary solution obeys the equation $\sum_{i=1}^{N} \partial J_i/\partial x_i = 0$, where $J_i = 0$ is the solution to be investigated. Consequently, we have

$$F_i = \nu \sum_{j=1}^{N} D_{ij} \frac{\partial}{\partial x_j} \left(\frac{\rho^{\nu-1} - 1}{\nu - 1} \right) . \tag{5}$$

Note that the constant term $-1/(\nu-1)$ was introduced in order to recover $\ln\rho$ when $\nu \to 1$.

Now, supposing that the matrix D_{ij} is invertible and defining $\mathscr{F}_i \equiv \sum_{j=1}^{N} (D^{-1})_{ij} F_j$, Eq. (5) leads to $\mathscr{F}_i = [\nu/(\nu-1)]\partial(\rho^{\nu-1} - 1)/\partial x_i$. Therefore, the integrability condition related to $\mathscr{F}_i$, $\partial\mathscr{F}_i/\partial x_j = \partial\mathscr{F}_j/\partial x_i$, implies that $\mathscr{F}_i = -\partial\Phi/\partial x_i$, which is in general less restrictive than $F_i = -\partial V/\partial x_i$. From the last two expressions for $\mathscr{F}_i$ we verify that

$$\rho(\bar{x}) = \rho_0 \exp_q \left[-\frac{\rho_0^{q-1}}{2-q}\, \Phi(\bar{x}) \right] , \tag{6}$$

with $q = 2 - \nu$, $\Phi(0) = 0$, and $\rho_0 = \rho(0)$. To exemplify our last achievement we show that it contains the result (4) as a particular case if we suppose $D_{ij} = D_i(x_i)\delta_{ij}$ and $F_i(x_i) = -\mathrm{d}V_i(x_i)/\mathrm{d}x_i$. From the definition of $\mathscr{F}_i$ we have $\partial\Phi/\partial x_i = (1/D_i)\mathrm{d}V_i/\mathrm{d}x_i$ implying $\Phi(\bar{x}) = \sum_{i=1}^{N} \int_0^{x_i} (1/D_i)(\mathrm{d}V_i/\mathrm{d}x_i')\,\mathrm{d}x_i'$. Thus, $\rho(\bar{x})$ reduces to Eq. (4) that naturally recovers Eq. (3) for $N = 2$.

3. Time-dependent solutions

Now we start our discussion about the time-dependent solutions for Eq. (1) by considering the free case with $N = 2$ and $D_{ij} = \mathscr{D}_i\delta_{ij}$ ($\mathscr{D}_i = cte$). In this direction and considering an extension of the unidimensional case, we employ the ansatz

$$\rho(x, y; t) = \exp_q [-\beta_x(t)x^2 - \beta_y(t)y^2]/Z(t) , \tag{7}$$

with $q = 2 - \nu$, which recovers the usual Gaussian structure in the $q \to 1$ limit. Note that this ansatz is a kind of q-Zubarev-like nonequilibrium operator. By substituting Eq. (7) into Eq. (1), we obtain

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = 2\nu Z^{2-\nu}(\mathscr{D}_x\beta_x + \mathscr{D}_y\beta_y), \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta_x} = 4\nu\mathscr{D}_x Z^{1-\nu}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\beta_y} = 4\nu\mathscr{D}_y Z^{1-\nu} . \tag{8}$$

Since, in the unidimensional case, there is only one $\beta(t)$ with $\beta(t) \propto t^{\sigma}$ (or $\beta(t) \propto (t+t_0)^{\sigma}$) [4–6,8,9], we are motivated to try $\beta_x(t)=\mathscr{B}_x(t+t_0)^{-\alpha_x}$, $\beta_y(t)=\mathscr{B}_y(t+t_0)^{-\alpha_y}$ and $Z(t)=\mathscr{Z}(t+t_0)^{\alpha}$, where $\mathscr{B}_x$, $\mathscr{B}_y$, $\mathscr{Z}$, t_0, α_x, α_y, and α are the constants to be obtained. In this way, Eq. (8) leads to $\beta_x(t)=\mathscr{B}_x(t+t_0)^{-\alpha}$, $\beta_y(t)=\mathscr{B}_y(t+t_0)^{-\alpha}$, and $Z(t)=\mathscr{Z}(t+t_0)^{\alpha}$ with $\alpha=1/v$ and $\mathscr{D}_x\mathscr{B}_x=\mathscr{D}_y\mathscr{B}_y=\mathscr{Z}^{v-1}/4v^2$. Thus,

$$\rho(x,y;t)=\frac{\tilde{\rho}_0}{(t+t_0)^{\alpha}}\exp_q\left[-\frac{\tilde{\rho_0}^{q-1}}{(2-q)^2}\left(\frac{x^2}{4\mathscr{D}_x(t+t_0)^{\alpha}}+\frac{y^2}{4\mathscr{D}_y(t+t_0)^{\alpha}}\right)\right], \tag{9}$$

where $q=2-v$, $\tilde{\rho}_0=1/\mathscr{Z}$ and t_0 are positive constants. This result can be directly extended for the N-dimensional case when D_{ij} is diagonal and does not depend on t and $\bar{x}$. In fact, by considering the diagonal form for D_{ij} ($D_{ij}=\mathscr{D}_i\delta_{ij}$ with $\mathscr{D}_i>0$) and $F_i=0$, Eq. (1) reads $\partial\rho/\partial t=\sum_{i=1}^{N}\mathscr{D}_i\partial^2\rho^v/\partial x_i^2$, whose generalized Gaussian solution is $\rho(\bar{x};t)=[\tilde{\rho_0}/(t+t_0)^{\alpha N/2}]\exp_q\{-(\tilde{\rho_0}^{q-1}/(2-q)^2)\sum_{i=1}^{N}x_i^2/(\mathscr{D}_i(t+t_0)^{\alpha})\}$.

Let us now focus on the free case $N=2$ with $D_x=\mathscr{D}_x|x|^{-\theta_x}$ and $D_y=\mathscr{D}_y|y|^{-\theta_y}$ taking $\theta_x\neq 0$ and $\theta_y\neq 0$ into account. By following the approach employed above, we can verify that

$$\rho(x,y;t)=\exp_q[-\beta_x(t)|x|^{2+\theta_x}-\beta_y(t)|y|^{2+\theta_y}]/Z(t) \tag{10}$$

is a solution for Eq. (1) with $q=2-v$ and $\beta_x(t)$, $\beta_y(t)$, and $Z(t)$ obeying a system of equations like (8). From these equations we can deduce the relations $Z(t)\,[\beta_x(t)]^{1/(2+\theta_x)}[\beta_y(t)]^{1/(2+\theta_y)}=\mathscr{C}_0$ and $1/[(2+\theta_x)^2\mathscr{D}_x\beta_x(t)]=1/[(2+\theta_y)^2\mathscr{D}_y\beta_y(t)]+\mathscr{C}_1$, where $\mathscr{C}_0$ and $\mathscr{C}_1$ are constants. By using the last relations and considering $\mathscr{C}_1=0$, we verify that

$$\beta_x(t)=\{v(2+\theta_x)^2\mathscr{D}_x\mathscr{C}^{1-v}t/\alpha'\}^{-\alpha'}, \tag{11}$$

where $\mathscr{C}=\mathscr{C}_0\{(2+\theta_x)^2\mathscr{D}_x/[(2+\theta_y)^2\mathscr{D}_y]\}^{-1/(2+\theta_x)}$ and

$$\alpha'=\frac{(2+\theta_x)(2+\theta_y)}{(2+\theta_x)(2+\theta_y)+(4+\theta_x+\theta_y)(v-1)}. \tag{12}$$

Note that this $\beta_x(t)$ recovers the first one when $\theta_x=\theta_y=0$ and $t_0=0$. From these results, a complete information is achieved for $\langle x^2\rangle$ and $\langle y^2\rangle$. In particular, we have that $\langle x^2\rangle \propto \beta_x^{-2/(2+\theta_x)}(t)$ and $\langle y^2\rangle \propto \beta_y^{-2/(2+\theta_y)}(t)$. These expressions for $\langle x^2\rangle$ and $\langle y^2\rangle$ indicate that there is a coupling between x and y directions since β_x (β_y) depends on $\mathscr{D}_x$, $\mathscr{D}_y$, θ_x, and θ_y.

4. Summary and conclusions

In summary, we have worked out the nonlinear Fokker–Planck equation (1) in several situations by incorporating an anisotropic dependence in the diffusion coefficients and also in the external force. We first obtained stationary solutions in a very general context, including situations where the diffusion coefficients are spatial dependent. After that, we considered the time-dependent solutions in the absence of external force. From our solutions we verified that the nonlinearity ($\nu \neq 1$) produces a kind of memory in the system since the reduced distribution for a given direction contains parameters related to the others. This memory effect, for $\langle x^2 \rangle$, is not enough to identify the degree of anisotropy when the diffusion coefficients are constant, but is only for the presence of other dimensions. In contrast, when the diffusion coefficients are spatial dependent, the solutions contain more information about the degree of anisotropy. In this context, we remark that the solutions found here can not be written as $\rho(x, y; t) = \rho_x(x, t)\rho_y(y, t)$, as in the usual case, due to the nonlinearity of equation under investigation. In addition, it must be stressed that this conclusion refers to the cases whose diffusion coefficients D_{ij} do not exhibit mixed terms, i.e., $D_{ij} = 0$ if $i \neq j$. Thus, the memory effect reported in this work is only due to the nonlinearity and not from the mixed derivative terms that occur when $D_{ij} \neq 0$ if $i \neq j$.

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