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Minireview: New results for old percolation

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Abstract

Percolation theory deals with the formation of clusters in disordered media. At the percolation threshold for the first time an infinite cluster appears. However, research of the last years showed that at this phase transition some old ideas were wrong: There can also be two or three clusters spanning from top to bottom, even in large two-dimensional lattices; and the probability of spanning is not given by simple real-space renormalization ideas.

1. Introduction

Percolation [1] is more than half a century old, and thousands of papers as well as several books were written about it. Thus, one should expect that progress would occur mostly in applications [2] while the basic foundations are no longer in flux. Here, we review recent work of others which changed widely held concepts on the number of percolating clusters and their description by simple renormalization group techniques.

In d-dimensional random site percolation, each of the L^d sites of a large lattice is either randomly occupied, with probability p, or empty, with probability $1 - p$. A cluster is a group of neighboring occupied sites; it is spanning or percolating if it connects the top and the bottom of the lattice. Usually, we are interested in the thermodynamic limit $L \to \infty$ when also the spanning clusters must become infinitely large. At low p there are mostly isolated sites and a few pairs and other small clusters; for p near unity most of the occupied sites form one infinite cluster; and thus some intermediate percolation threshold p_c is the phase transition between no and one infinite cluster. Some p_c values are known exactly, like $\frac{1}{2}$ in the triangular lattice, while others are known only numerically, like 0.592746... in the square lattice [3]. A mathematical theorem claimed that there can never be two or three infinite clusters coexisting with each other [4]; their number should be zero, one, or infinite.

This phase transition is of second order ("continuous") and should thus be described by renormalization group theories similar to thermal critical phenomena. Particularly

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successful was the cell-to-site transformation [5] where the $L \times L$ lattice is divided into many cells, and the whole $b \times b$ cell, which percolates with probability $R(p)$, is renormalized into a single site with occupation probability $p' = R(p)$. The fixed point $p' = p$ of this cell-to-site renormalization then is the percolation threshold,

$$R(p_c) = p_c$$

at least for sufficiently large b.

What was wrong with these ideas? Today it is believed [6] that with a low probability we may also have two or more spanning clusters at p_c (though only one above p_c) even on infinite square lattices, that $R(p_c)$ is not [4] equal to p_c and is only of limited universality. But Monte–Carlo estimates based on wrong ideas are still correct. And since for infinite lattices the problems occur only at $p = p_c$, they by definition do not affect series expansions in powers of p or $1 - p$.

2. Coexistence of "infinite" clusters

In two dimensions, an infinite system of east–west streets cannot coexist with one of north–south streets without some crossing points (bridges use already a third dimension.) Once streets cross each other, they are part of the same cluster in the definition of percolation theory. Thus, we cannot have one infinite cluster spanning from left to right coexisting with another one spanning from top to bottom, in two dimensions. In three and more dimensions, this is topologically possible, though, in general, the probability for it to happen goes to zero if the lattice size goes to infinity. With probability going to one, however, we have at intermediate concentrations p, like $0.31 < p < 0.69$ on the simple cubic lattice with nearest-neighbor interactions, two interpenetrating infinite networks of occupied and of empty sites. All this is well known and not controversial. Our question here is: Can there be two (or more) clusters of occupied sites, which both span the two- or three-dimensional lattice from top to bottom, even if the lattice size goes to infinity?

Because of the above theorem [4], and because computer pictures usually show at most one spanning cluster, many people believed the answer is no. This view was enforced when de Arcangelis [7] published "Multiplicity of infinite clusters in percolation above six dimensions", finding an infinite number of spanning clusters in seven dimensions, as predicted by Coniglio [8]. Thus, the possibility of infinitely many infinite clusters contained in the above theorem [4] was confirmed, but restricted to high dimensions; in two and three dimensions only one or no infinite cluster seemed possible.

However, independent of these numerical tests, a new theorem on the "Uniqueness of the infinite cluster ... for ... percolation" was published [9] around the same time; it was supported by later publications [10] ignoring Ref. [7]. Apparently, now the possibility of infinitely many infinite clusters was excluded mathematically. These new theorems were ignored by computer simulators. This separate development continued, at least for this author, up to an interdisciplinary workshop in June 1996 at the Technion in Israel; there I learned from the overview talk of J.E. Steif of the uniqueness claim. My protest on the basis of Refs. [7,8] found the reply that infinite clusters here were meant to cover a finite fraction of the whole lattice. Indeed, such a finite density is required in the uniqueness theorem of Ref. [9] while the infinitely many spanning clusters in seven dimensions were found right at the percolation threshold. At $p = p_c$, however, the largest clusters are "fractal" and have zero density, making the theorem of Ref. [9] not applicable. So we seemed to settle on zero or one spanning cluster, with only high dimensions right at p_c being an exception of little practical relevance.

But then we learned from Aizenman [6] that also in two and three dimensions more than one spanning cluster should be possible. Indeed, computer simulations of Hu and Lin [11] found many spanning clusters at $p = p_c$ in long strips, and for quadratic or (hyper-) cubic shapes, Sen [12] found a small but finite probability in two–five dimensions to have two spanning clusters right at the percolation threshold; the one for three coexisting spanning clusters is even lower. Details depend on how the threshold is defined for a finite lattice, and the asymptotic decay law (Gaussian or other exponential) of the probability to have many spanning clusters is numerically not clear. Shchur and Kosyakov [12] confirmed this coexistence of spanning clusters in two dimensions. (Actually already de Arcangelis [7] published data showing more than one spanning cluster in five dimensions but explained that as a finite-size effect; her boss at that time and the referee of her paper must have been really stupid.)

So it seems one has to distinguish [6] between infinite clusters and spanning clusters such that uniqueness theorems are valid only for the first, while of the second we may have at $p = p_c$ several in two–five dimensions, and infinitely many in more than six dimensions. Sloppy definitions of infinity, while common in computational physics, are dangerous in theorems [4,13]. These problems, caused by the fractality of large critical clusters, appear only at the threshold:

For large enough lattices in the form of a square or (hyper-)cube, below p_c we have no spanning cluster and no cluster covering a positive fraction of the whole lattice, while above p_c the unique spanning cluster contains a finite fraction of the whole lattice. At p_c, one or more spanning clusters are possible, of which none has a positive density.

One difference at $p = p_c$ between the spanning cluster and the largest cluster is their identity. Imagine we want to evaluate some complicated property of the "infinite" cluster, like its electrical conductivity, and the lattice fills the whole universe; or computationally, it is stored on a huge and slow hard disk. There is a largest cluster of occupied sites (now called the infinite network) on this disk and we have marked it somehow. Now to evaluate the conductivity of a finite section of L^d sites in the center of the whole lattice, we get this section into the fast main memory of our workstation or PC. And then we increase L step by step to extrapolate to the usual $L \to \infty$ limit, keeping the same site at the center of the increasingly large sections. In this procedure, sites which belong to the infinite network in the smaller section also belong to it in the larger section; the infinite network keeps its identity. Spanning clusters at $p = p_c$, however, are volatile fractals [14]; sites which were not part of the spanning cluster for the smaller section may be part of it for the larger section, and the other way around. Thus, spanning clusters do not keep their identity as L increases. An infinite network can be thought to exist even before the L^d sites are looked at; a spanning cluster is directly linked to L.

(Of course, also the infinite network becomes volatile if we redefine it for every different L as the largest cluster of this finite section. And generally the definition of an infinite network as the largest cluster is problematic since also below p_c it leads to the existence of an infinite network with a mass increasing asymptotically as $\log(L)$. It is not easy to define [6] infinite clusters at p_c different from spanning clusters; perhaps [6] for infinite clusters we first let $L \to \infty$ above p_c and then $p \to p_c^+$, while for fractal spanning clusters we take first $p = p_c$ and then let $L \to \infty$. The first choice corresponds to $1 \ll \xi \ll L$ and the second to $1 \ll L \ll \xi$, where ξ is the bulk connectivity length and diverges at p_c. From now on we deal with spanning clusters only.)

3. Spanning probability

Now we ask: What is the probability $R(p)$ that a lattice with L^d sites percolates in the sense of having at least one cluster spanning from top to bottom? For large enough lattices, $R(p < p_c) = 0$ and $R(p > p_c) = 1$. Finite-size scaling theory predicts and computer simulation confirms since more than two decades [1] that in a small transition region $\Delta p \propto L^{-1/\nu}$ the spanning probability moves from close to zero towards close to one. Here, ν is the correlation length exponent for the typical cluster radius and is $\frac{4}{3}$ in two dimensions. One may define a size-dependent $p_c(L)$ by the condition that the spanning probability should be 50%: $R(p_c(L)) = \frac{1}{2}$. But small-cell renormalization theory [5,1] traditionally determines the critical point as a fixed point: $R(p_c) = p_c$. If we insert the bulk threshold value p_c, does this equation become valid for large enough cells?

It does not. Ziff [3] showed numerically that in a special square lattice case $R(p_c)$ is $\frac{1}{2}$ and different from 0.592746. More generally, it differs from p_c (and also from $\frac{1}{2}$) in three–six dimensions [15,16] and becomes unity in seven [6,17]. Actually, it was seen [18] before Ref. [3] that $R(p_c)$ on the square lattice differs from 0.593, but these authors did not ring the alarm bell with respect to simple renormalization group techniques [5]. Lack of communication across different fields of percolation theory again hampered progress.

Does this mean that all the numerical work based on the now wrong idea $R(p_c) = p_c$ were wrong? I have not yet found a single estimate which needs to be revised. The reason is that $R(p)$ for $L \to \infty$ jumps from zero to one. Thus, any constant C between zero and one can be used to define a size-dependent $p_c(L)$ through $R(p = p_c(L)) = C$; for large enough lattices this $p_c(L)$ converges to the proper bulk threshold [19].

Also, in ancient times [1] it was assumed that the derivative $dR(p)/dp$ approaches for large lattices a Gaussian; this derivative gives the probability that a lattice starts to percolate at a concentration p. For a fixed sequence of random numbers there is a well-defined onset of percolation when we increase p, and it is plausible that this onset follows a normal distribution for large systems. The width σ, defined as the standard deviation of this distribution of the L-dependent thresholds,

$$\sigma^2 = \langle p_c^2 \rangle_L - \langle p_c \rangle_L^2 ,$$

then varies as $L^{-1/\nu}$ and is proportional to the shift $\langle p_c \rangle_L - \langle p_c \rangle_\infty$. This is a convenient numerical tool to determine the bulk p_c by extrapolation without assuming any critical exponent ν. Again, the (unprecisely defined) convergence to a Gaussian turned out to be bad when tested numerically: The distribution has a finite skewness (third-order cumulant) $\langle (p_c - \langle p_c \rangle)^3 \rangle / \langle (p_c - \langle p_c \rangle)^2 \rangle^{3/2}$ which shows no intention to vanish for large lattices [20,21]. Also in this case, mathematical theorems were found independently of the simulations, but now the contact between the two methods was established much faster [20] and there is full agreement: Gaussians are out. And again, the determination of thresholds and exponents ν by the Gaussian assumption did not give wrong results, since $\sigma \propto L^{-1/\nu}$ remains valid and only the proportionality factor is influenced by the deviations from a Gaussian.

The reason why the distribution of thresholds in finite system does not follow the usual central limit theorem and its Gaussian distribution is [22] that the property of spanning is a collaborative effort of the whole lattice and not the sum of more or less independent contributions from smaller parts of the whole lattice. Therefore, the number of sites in the just-spanning cluster is not a self-averaging quantity with relative fluctuations vanishing for $L \to \infty$. That fact was known numerically since a long time but its indication against a Gaussian distribution was overlooked.

If neither $R(p_c)$ is equal to p_c nor its derivative dR/dp a good Gaussian, at least is $R(p_c)$ universal [3,21] i.e. independent of many details but dependent on dimensionality d? Yes, but only in a limited way [16,23]. As pointed out before for thermal critical phenomena [24], the universal finite-size scaling functions depend also on the boundary conditions and the shape of the sample, and this is true [16,23] also for $R(p_c)$. Even the choice of the algorithm used may play a role in such studies right at the critical point [25]. On the other hand, for square- and triangular-site percolation, $R(p_c)$ could not be distinguished numerically; and also site and bond percolation on the simple cubic lattice lead to the same $R(p_c)$ within error bars. Some universality seems to persist [26] at the threshold, but not too much.

By the way, if you prefer the simple but wrong renormalization picture of Ref. [5] over the more formal one of Ref. [21], its validity can be restored: Just replace cell-to-site renormalization (from L to 1) by cell-to-cell renormalization (from L to $L/2$). According to Hu et al. [27], $R(p_c)$ for lattice size L agrees with $R(p_c)$ for lattice size $L/2$, provided L is large enough.

4. Lessons learned?

From the cited literature, in particular Refs. [7,12,16,20] the reader gets a more detailed impression of past errors and present progress, and from Ref. [6], we learn the difficulties of defining infinite clusters properly.

It pays off to read "enemy" literature: if simulators would have read more theorems of mathematical physicists, and vice versa, progress on the number of spanning clusters would have been faster. With Gaussian versus non-Gaussian threshold distributions [20], contact between mathematicians and simulators was (by luck?) established much faster, and the problem was (hopefully) clarified in a much shorter time. These percolation problems are not the only cases where one side's research was triggered by the other side's work.

It also pays off to check widespread assumptions and supposedly exact theorems if one cannot find numerical evidence for them in the literature. While in most cases one confirms the expectations, one may also find surprising discrepancies, if for example the theorems were not applicable, misunderstood, unclear, or based on very slow convergence. Of course, computer simulations can also be wrong for similar reasons, and important or controversial results should be rechecked.

Finally, one should be cautious in declaring a field nearly finished. We should have learned from the Hitchcock movie "The Trouble With Harry" that dead bodies may turn up again and again. To my surprise, percolation theory is not dead.

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