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LETTER TO THE EDITOR

On the zero-field susceptibility in the $d=4$, $n=0$ limit: analysing for confluent logarithmic singularities

A J Guttmann†

Wheatstone Physics Laboratory, King's College, Strand, London WC2R 2LS, UK

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Abstract. A method is developed for the analysis of critical point singularities of the form $f(t) \sim A|t|^q|\ln|t||^p$ with q known. The $d=4$, $n=0$ susceptibility series is extended by two further terms, and is analysed under the assumption of a singularity of the above type with $q=-1$. It is found that $p=0{\cdot}23\pm0{\cdot}04$, in agreement with the calculation ($p=\frac{1}{4}$) of Larkin and Khmel'nitskii. The connective constant for the model is found to be $6{\cdot}7720\pm0{\cdot}0005$.

One of the earliest applications of field theoretical techniques to critical phenomena was the work of Larkin and Khmel'nitskii (1969) who calculated the leading term in the expansion of the zero-field susceptibility and specific heat of the n-vector model on a four-dimensional lattice. They obtained

$$\chi(t) \sim A|t|^{-1}|\ln|t||^{(n+2)/(n+8)} \tag{1}$$

and

$$C(t) \sim |\ln|t||^{(4-n)/(n+8)} \quad n<4; \qquad C(t) \sim \ln|\ln|t|| \quad n=4$$

where $t=(T-T_c)/T_c$ is the reduced temperature.

For the self-avoiding random walk model susceptibility ($n=0$) there is thus a term of the form $|\ln|t||^{1/4}$ modifying the simple pole at $t=0$. The first eleven terms of its series expansion were obtained by Fisher and Gaunt (1964). At that time it was generally believed that the dominant singularity was purely algebraic, that is of the form $t^{-\gamma}$, and that γ approached the mean field result ($\gamma=1$) smoothly as $d\to\infty$. Analysing their series by standard ratio and Padé techniques, they obtained $\gamma=1{\cdot}072\pm0{\cdot}002$.

However, neither the Padé nor the ratio method can successfully cope with confluent logarithmic terms of this type. The effect of the confluent term is to dramatically slow the rate of convergence. Equivalently, with a limited number of terms, these standard techniques result in extrapolations to slightly erroneous values of the critical parameters. In fact none of the contemporary methods of series analysis (Gaunt and Guttmann 1974) can handle singularities of this type. The recurrence relation method (Joyce and Guttmann 1973, Guttmann and Joyce 1972) can handle the following special cases:

$$\begin{aligned} f_1(t) &= \phi_1(t)+\phi_2(t)|t|^{\alpha}(\ln|t|)^n \\ f_2(t) &= (1+c(\ln|t|)^n)\phi_3(t)+|t|^{\alpha}\phi_4(t) \end{aligned} \tag{2}$$

† Permanent address: Department of Mathematics, University of Newcastle, Newcastle, NSW 2308, Australia.

where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are analytic in the neighbourhood of $t=0$, c is a constant, and n is an integer, but cannot handle the more general case

$$f(t) \sim A|t|^q |\ln|t||^p \tag{3}$$

for arbitrary p. In this Letter we develop a simple approach to this problem, which is appropriate when q is known, and discuss a generalisation to the case when q is not known.

We have extended the $n=0$, $d=4$ susceptibility series by two terms, by counting all the twelve and thirteen line theta graphs, dumbells, polygons and figure-eights and using Sykes' counting theorem (Sykes 1961). The first thirteen terms are found to be

$$\begin{aligned}\chi_0(v)_{n=0,d=4} = {} & 1+8v+56v^2+392v^3+2696v^4+18584v^5+127160v^6+871256v^7 \\ & +5946200v^8+40613816v^9+276750536v^{10}+1886784200v^{11} \\ & +12843449288v^{12}+87456597656v^{13}+\ldots. \end{aligned} \tag{4}$$

The first eleven terms reproduce those given by Fisher and Gaunt (1964), the last two are new.

In order to analyse this series for a singularity of the form $\chi_0(v) \sim A(1-v/v_c)^{-1}|\ln(1-v/v_c)|^p$ we first form the ratios r_n of the suceptibility series

$$\chi_0(v) = \sum_{n\geqslant 0} a_n v^n; \qquad r_n = a_n/a_{n-1} \qquad n \geqslant 1. \tag{5}$$

Next, defining the function f by

$$v^{-p^*} f(v) \equiv v^{-p^*}(1-v)^{-1}\{\ln[1/(1-v)]\}^{p^*} = \sum_{n\geqslant 0} b_n v^n, \tag{6}$$

where the term v^{-p^*} is introduced to remove the branch point at the origin, we define the ratios $r_n^* = b_n/b_{n-1}$. Our analysis is then based on the observation that the sequence $R_n = r_n/r_n^*$ should approach v_c with zero slope as $n \to \infty$ when $p = p^*$.

Two caveats must be added however. Firstly, for loose-packed lattices, of which the $d=4$ hypercubic is one example, there is a singularity at $v=-v_c$ which introduces a characteristic oscillation in the ratios r_n, of periodicity two.

Secondly, and not independently, the assumed asymptotic form (3) is of course only dominant. There will be correction terms, undoubtedly o(1) but hopefully close to O(t).

The effect of the loose-packed lattice structure can be minimised by applying an Euler transformation to the series. We choose our favourite such transformation

$$x = 2v/(1+v/v_c^*). \tag{7}$$

The point $v=v_c^*$ is our estimate of the true critical point at $v=v_c$. The transformation maps the point $v=-v_c^*$ to infinity, while the point $v=v_c^*$ is a fixed point. Even if v_c is not known precisely, the point $v=-v_c$ is mapped sufficiently far from the origin of the x plane not to affect the extrapolations. Once the critical point in the x plane is determined, the transformation can be reverted to estimate the critical point v_c.

To allow for the effect of correction terms, whose precise form is not generally known, is more difficult. Linear and quadratic extrapolation can be tried, and if the correction terms are O(t^α), with α not much different from unity, or very slowly varying, such a procedure should ultimately be successful. In fact, field theoretical calculations (Brézin *et al* 1976) suggest that corrections to (3) are of the form $\ln|\ln|t||/\ln|t|$, which is an extremely slowly varying function. We have conducted numerical experiments on test functions of the form

$$g(t)=A|t|^{-1}|\ln|t||^{p}(1+c\ln|\ln|t||/\ln|t|) \tag{8}$$

for a range of values of c. We find that for $|c|<100$, the correction term had a negligible effect on the estimate of the exponent p. Rather, since the correction term is so slowly varying, it only has a significant effect on estimates of the amplitude A, which is of no interest here.

Our subsequent analysis is of the series in the x variable (7), with v_c initially chosen to be $v_c=1/6{\cdot}769$. We first form the ratios $R_n=r_n/r_n^*$ for $p^*=0$ and $p^*=1$. For $p^*=0$ (no confluent logarithmic term) we find for R_8-R_{13} the decreasing sequence of values 6·826, 6·820, 6·815, 6·811, 6·808 and 6·805 respectively. For $p^*=1$, corresponding to a singularity of the form $t^{-1}|\ln|t||$, we find the corresponding quantities to be an increasing sequence, namely 6·558, 6·587, 6·610, 6·628, 6·643, 6·655. Clearly then $0<p^*<1$. Next, with $p^*=\frac{1}{3}$ we obtain 6·7400, 6·7457, 6·7499, 6·7530, 6·7555 and 6·7574 for R_8-R_{13}, while with $p^*=0{\cdot}15$ we find the sequence: 6·7880, 6·7872, 6·7864, 6·7855, 6·7844, 6·7839. We can therefore immediately write $0{\cdot}150<p^*<0{\cdot}333$. To confine p^* more closely within this interval a more detailed analysis is required, and this is given in table 1. For $p^*=0{\cdot}2$ we see that the ratios R_n are slowly increasing, though R_{13} reverses this trend. For $p^*=0{\cdot}25$ the ratios are also slowly increasing, though less slowly than for $p^*=0{\cdot}2$. In order to account for corrections to our assumed asymptotic form, we also show the linear and quadratic extrapolants of the sequence $\{R_n\}$, corresponding to the first and second columns of a Neville table. For both $p^*=0{\cdot}2$ and $p^*=0{\cdot}25$ the linear extrapolants are slowly decreasing, at a similar rate. For the quadratic extrapolants, the last three estimates for $p^*=0{\cdot}25$ are identical to five significant figures, while for $p^*=0{\cdot}20$ the quadratic extrapolants are still increasing. Thus we marginally favour $p^*=0{\cdot}25$ over $p^*=0{\cdot}20$. Performing a similar analysis for a number of values of p^* around $p^*=0{\cdot}20$ and $p^*=0{\cdot}25$ allows us to make the estimate $p^*=0{\cdot}23\pm0{\cdot}04$. Clearly, this is in excellent agreement with the result $p=\frac{1}{4}$ given by the calculation of Larkin and Khmel'nitskii (1969). We also estimate $1/x_c=6{\cdot}7705\pm0{\cdot}0005$ from this data which gives (by reverting (7)) $1/v_c=6{\cdot}7720\pm0{\cdot}0005$ compared to the value $6{\cdot}7680\pm0{\cdot}0015$

Table 1. Ratios and Neville table analysis of the self-avoiding walk generating function on the $d=4$ hypercubic lattice, assuming $\chi(t)\sim|t|^{-1}|\ln|t||^{p^*}$.

	$p^*=0{\cdot}2$			$p^*=0{\cdot}25$		
n	Ratios $R_n=r_n/r_n^*$	Linear extrapolants	Quadratic extrapolants	Ratios $R_n=r_n/r_n^*$	Linear extrapolants	Quadratic extrapolants
6	6·76958	6·80407	6·77025	6·75103	6·80855	6·77271
7	6·77311	6·79431	6·76990	6·75775	6·78803	6·77173
8	6·77500	6·78816	6·76974	6·76194	6·79131	6·77117
9	6·77600	6·78405	6·76966	6·76470	6·78676	6·77083
10	6·77652	6·78117	6·76964	6·76659	6·78354	6·77062
11	6·77675	6·77908	6·76967	6·76791	6·78117	6·77051
12	6·77682	6·77753	6·76976	6·76887	6·77939	6·77049
13	6·77678	6·77635	6·76989	6·76957	6·77803	6·77053

obtained by Fisher and Gaunt (1964). This slight discrepancy clearly illustrates the effect of weak confluent terms in slowing down the rate of convergence in standard methods of series analysis.

With the apparent small discrepancies in the predictions of critical exponents by renormalisation group methods and by series analysis, it is of great interest to resolve these differences. A possible mechanism for drastically slowing down the rate of convergence of series is the presence of weak confluent singularities. Renormalisation group methods suggest the presence of such terms in certain cases, such as the one considered here. It is therefore both timely and important to develop methods of analysis that can handle such confluences. A first step in this direction has been taken in this Letter.

An obvious extension when q (see equation (3)) is not known is to investigate the series for a range of values of p^* and q^*. It is by no means certain that unique values will be obtained. Indeed, since we have seen (Fisher and Gaunt 1964) that under the assumption $p^*=0$, a sensible estimate of q^* can be obtained, it seems likely that a small region in the p^*–q^* plane will be obtained. If either p or q is known, the other can then be obtained quite accurately. Also, if the critical point is known, this fact can be used to decrease the region of possible values in the p^*–q^* plane.

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