Generalized Hamiltonian Dynamics*

Yoichiro Nambu

The Enrico Fermi Institute and the Department of Physics, The University of Chicago, Chicago, Illinois 60637

(Received 26 December 1972)

Taking the Liouville theorem as a guiding principle, we propose a possible generalization of classical Hamiltonian dynamics to a three-dimensional phase space. The equation of motion involves two Hamiltonians and three canonical variables. The fact that the Euler equations for a rotator can be cast into this form suggests the potential usefulness of the formalism. In this article we study its general properties and the problem of quantization.

I. INTRODUCTION

A notable feature of the Hamiltonian description of classical dynamics is the Liouville theorem, which states that the volume of phase space occupied by an ensemble of systems is conserved. The theorem plays, among other things, a fundamental role in statistical mechanics. On the other hand, Hamiltonian dynamics is not the only formalism that makes a statistical mechanics possible. Any set of equations which lead to a Liouville theorem in a suitably defined phase space will do (provided of course that ergodicity may be assumed). With this in mind, let us consider the following scheme.

Let $(x, y, z) \equiv \vec{r}$ be a triplet of dynamical variables (canonical triplet) which span a three-dimensional phase space. This is a formal generalization of the conventional phase space spanned by a canonical pair (p, q). Next introduce two functions, H and G, of (x, y, z), which serve as a pair of "Hamiltonians" to determine the motion of points in phase space. More precisely, we postulate the following "Hamilton equations":

$$\frac{dx}{dt} = \frac{\partial(H, G)}{\partial(y, z)},\qquad \frac{dy}{dt} = \frac{\partial(H, G)}{\partial(z, x)},\qquad \frac{dz}{dt} = \frac{\partial(H, G)}{\partial(x, y)}, \tag{1}$$

or in a vector notation

$$\frac{d\vec{r}}{dt} = \vec{\nabla} H \times \vec{\nabla} G . \tag{1$'$}$$

For any function $F(x, y, z)$, then, we have

$$\frac{dF}{dt} = \frac{\partial(F,H, G)}{\partial(x, y, z)} = \vec{\nabla} F \cdot (\vec{\nabla} H \times \vec{\nabla} G). \tag{2}$$

We may call the right-hand side of (2) a generalized Poisson bracket (PB), to be denoted by $[F,H, G]$. Obviously a PB is antisymmetric under interchange of any pair of its components. As a result we have $H = F = 0$, i.e., both H and G are constants of motion. The orbit of a system in phase space is thus determined as the intersection of two surfaces, $H =$ const. and $G =$ const.

Equation (1) or (1$'$) also shows that the velocity field $d\vec{r}/dt$ is divergenceless,

$$\vec{\nabla} \cdot (\vec{\nabla} H \times \vec{\nabla} G) \equiv 0, \tag{3}$$

and this amounts to a Liouville theorem in our phase space.

The above properties immediately tempt us to construct a statistical mechanics where a canonical ensemble is characterized by a generalized Boltzmann distribution in phase space with a weight factor

$$e^{-\beta H - \gamma G}.$$

Two temperaturelike intensive parameters are thus required to specify the ensemble, much as in a grand canonical ensemble.

It is obvious that this kind of generalization can be extended to a phase space of any dimensionality, n. We would introduce an n-component vector x_i and $n-1$ Hamiltonians H_k, and postulate in lieu of Eqs. (1) and (2)

$$\frac{dx_i}{dt} = \sum_{jk\cdots l} \epsilon_{ijk\cdots l} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k} \cdots \frac{\partial H_{n-1}}{\partial x_l},\qquad \frac{dF}{dt} = \frac{\partial(F,H_1,H_2, \ldots ,H_{n-1})}{\partial(x_1, x_2, \ldots , x_n)}, \tag{4}$$

where $\epsilon_{ijk\cdots l}$ is the Levi-Civita tensor.

From the standpoint of physics, however, we must first examine the relevance and applicability of such generalizations. Are there real physical systems which may be described in this way? Or else can one think of these generalizations as a possible direction in which classical and quantum mechanics might develop?

In this paper we will limit ourselves to the three-dimensional case only. Then the first question can be answered affirmatively: Equation (1) is nothing but the Euler equation for a rigid rotator, if we identify $\vec{r}$ with the angular momentum $\vec{L}$ in the body-fixed frame, and G and H, respectively, with the total kinetic energy and the square of angular momentum in this frame:

$$G = \frac{1}{2}\left(\frac{L_x^2}{I_x} + \frac{L_y^2}{I_y} + \frac{L_z^2}{I_z}\right),$$
$$H = \tfrac{1}{2}(L_x^2 + L_y^2 + L_z^2). \tag{5}$$

We believe this to be justification enough to explore further the proposed ideas at least for the three-dimensional case. Needless to say, one may in general consider a number of canonical triplets $\vec{r}_n$, $n = 1, \ldots, N$ which form a $3N$-dimensional phase space, and write in place of Eq. (2)

$$\frac{dF}{dt} = \sum_n \frac{\partial(F, H, G)}{\partial(x_n, y_n, z_n)}. \tag{6}$$

For example, this will enable one to handle a model which simulates coupled spin systems.[1] For the moment, however, we are interested in the basic formalism only.

There is another direction in which Eq. (2) can be generalized. It is to assume

$$\frac{dF}{dt} = \sum_i \frac{\partial(F, H_i, G_i)}{\partial(x, y, z)}, \tag{7}$$

where (H_i, G_i) are a given set of functions. Since the Liouville theorem holds with respect to each term separately, it also holds for the sum. But the individual "Hamiltonians" H_i, G_i, are no longer constants of motion in general, though there may nevertheless be some constants of motion which are not directly related to the Hamiltonians. This may sound like an uninteresting and unnecessary digression, but we will see later that it becomes more natural when one tries quantization.

II. CANONICAL TRANSFORMATION

In this section we examine canonical transformations on the triplet (x, y, z), or more generally on the set of triplets (x_n, y_n, z_n).

We may call a mapping $(x, y, z) \to (x', y', z')$ a canonical transformation if

$$[x', y', z'] = \frac{\partial(x', y', z')}{\partial(x, y, z)} = 1. \tag{8}$$

Then

$$\frac{dF}{dt} = \frac{\partial(F, H, G)}{\partial(x, y, z)}$$
$$= \frac{\partial(F, H, G)}{\partial(x', y', z')}, \tag{9}$$

i.e., the Hamilton equations are form-invariant if we use the new set of variables. In particular, the Hamilton equations themselves generate an infinitesimal canonical transformation in view of the Liouville theorem (3). On the other hand, not all infinitesimal canonical transformations need be generated this way. Moreover, two sets (H, G) and (H', G') generate the same transformation if there is a functional relationship $H' = h(H, G)$, $G' = g(H, G)$ such that

$$\frac{\partial(H', G')}{\partial(H, G)} = 1, \tag{10}$$

because then

$$\frac{\partial(H, G)}{\partial(y, z)} = \frac{\partial(H', G')}{\partial(y, z)}, \quad \text{etc.}$$

Thus the Hamiltonians are defined up to the usual type of canonical transformations (10) where (H, G) are regarded as a canonical pair of variables. In order not to confuse this with the transformations on (x, y, z) we will refer to (10) as "gauge" transformations.

Linear canonical transformations form a special class. We may use the matrix notation

$$\vec{r}' = A\vec{r}. \tag{11}$$

Equation (8) implies that the matrix A must be unimodular: $\det A = 1$. In other words, linear canonical transformations form the group $SL(3, R)$. In order to generate them we can conveniently take H and G to be, respectively, linear and quadratic forms in $\vec{r}$:

$$H = \sum_i a_i r_i, \quad G = \sum_{ij} \tfrac{1}{2} r_i B_{ij} r_j, \tag{12}$$

where B_{ij} is a symmetric 3×3 matrix. Then

$$\dot{\vec{r}} = \vec{a} \times (B\vec{r})$$

or

$$\dot{r}_i = \sum_{jkl} \epsilon_{ijk} a_j B_{kl} r_l.$$

Hence

$$A_{ik} = \sum_{jl} \epsilon_{ijl} a_j B_{lk}. \tag{13}$$

The number of parameters is 3 for H and 6 for G but there is a redundancy under the scaling $H \to \lambda H$, $G \to G/\lambda$. Therefore the degrees of freedom are 8, the correct number for the group $SL(3, R)$.

The Euler equations for a rotator belongs to a slightly more complicated case where both H and G are quadratic forms. Let us assume in general

$$H = (\vec{r}, A\vec{r}), \quad G = (\vec{r}, B\vec{r}), \tag{14}$$

where A and B are symmetric matrices. It is possible, however, to bring them into a standard diagonal form

$$H=\lambda(\vec{r}\,',1\vec{r}\,'),\quad G=(\vec{r}\,',\Lambda\vec{r}\,') \tag{15}$$

by a linear canonical transformation, provided that A is positive (or negative) definite. For example, first diagonalize A by a rotation, then make it proportional to a unit matrix by a unimodular rescaling of the three coordinates. Finally, make G diagonal by a second rotation. Equation (5) for the Euler top corresponds precisely to such a form. One should note also that there is a freedom of linear gauge transformations between H and G, or equivalently between A and B:

$$\begin{aligned}&A\to\alpha A+\beta B,\\&B\to\gamma A+\delta B,\quad \alpha\delta-\beta\gamma=1\end{aligned} \tag{16}$$

which forms a group SL$(2,R)$.

Let us now examine the case of many canonical triplets as represented by Eq. (6). Here one can define a PB to mean

$$[A,B,C]=\sum_{n=1}^{N}\frac{\partial(A,B,C)}{\partial(x_n,y_n,z_n)}\,. \tag{17}$$

The canonical variables then have the properties

$$\begin{aligned}[x_l,y_m,z_n]&=1\quad\text{if}\quad l=m=n\\&=0\quad\text{otherwise},\\ [x_l,x_m,z_n]&=[x_l,x_m,x_n]=0,\quad\text{etc.}\end{aligned} \tag{18}$$

A canonical transformation $\{r_n\}\to\{r_n'\}$ will be a transformation which leaves the PB form-invariant. In other words, Eq. (17) must hold true for $\{r_n\}$ when evaluated in terms of the new variables $\{r_n'\}$ and vice versa.

In the case of an infinitesimal canonical transformation $\vec{r}_n\to\vec{r}_n+\delta\vec{r}_n$, Eq. (18) leads to

$$\begin{aligned}\delta[x_l,y_m,z_n]&=[\delta x_l,y_m,z_n]\\&\quad+[x_l,\delta y_m,z_n]+[x_l,y_m,\delta z_n]\\&=0,\\ \delta[x_l,x_m,z_n]&=[\delta x_l,x_m,z_n]\\&\quad+[x_l,\delta x_m,z_n]+[x_l,x_m,\delta z_n]\\&=0,\\ \delta[x_l,x_m,x_n]&=[\delta x_l,x_m,x_n]\\&\quad+[x_l,\delta x_m,x_n]+[x_l,x_m,\delta x_n]\\&=0,\quad\text{etc.}\end{aligned} \tag{19}$$

Most of these conditions are satisfied trivially because the $\{r_n\}$ satisfy Eq. (18). The only nontrivial ones turn out to be

$$\begin{aligned}\delta[x_l,y_l,z_l]&=\frac{\partial}{\partial x_l}\delta x_l+\frac{\partial}{\partial y_l}\delta y_l+\frac{\partial}{\partial z_l}\delta z_l\\&=0,\end{aligned} \tag{20a}$$

$$\delta[x_l,x_m,z_m]=-\frac{\partial}{\partial y_m}\delta x_l=0\quad(m\neq l),\quad\text{etc.} \tag{20b}$$

The condition Eq. (20b) means that $\delta\vec{r}_n$ does not depend on any r_l, $l\neq n$. If so, then the first condition (20a) is a simple statement for canonical transformations of individual triplets. We are thus led to a rather unexciting result that the only continuous canonical transformations consist of independent transformations of the individual triplets. This contrasts with the case of usual canonical doublets where a general transformation involve all the variables simultaneously.

A corollary to the above result is that a Hamilton equation like Eq. (6) cannot be regarded as generating successive canonical transformations unless H and G are simple sums

$$H=\sum_n H_n(\vec{r}_n),\quad G=\sum_n G_n(\vec{r}_n)\,. \tag{21}$$

More general forms of H and G will satisfy Eq. (20a) but not (20b). On the other hand, Eq. (20a) alone is sufficient to guarantee that the Liouville theorem holds in the $3N$-dimensional phase space.

The problem of an infinitesimal canonical transformation can also be tackled from a different angle. The standard general solution to an equation like (20a) is

$$\delta\vec{r}=\vec{\nabla}\times\vec{A}, \tag{22}$$

where $\vec{A}$ is a vector field, determined up to the gradient of a scalar. Our Hamilton equation (2) corresponds to a special choice

$$\vec{A}=H\vec{\nabla}G\ (\text{or }-G\vec{\nabla}H). \tag{23}$$

As is well known, a transformation on (H,G) satisfying Eq. (10) has the property

$$H\delta G-H'\delta G'=\delta S \tag{24}$$

for some function $S(G,G')$. This induces a transformation on $\vec{A}$,

$$\vec{A}\to\vec{A}+\vec{\nabla}S, \tag{25}$$

which is indeed a gauge transformation in the conventional sense. On the other hand, Eq. (23) does not exhaust all possible fields $\vec{A}$, which have three independent components.

This brings us finally to the type of generalization represented by Eq. (7). Since more than two functions $\{H_n,G_n\}$ are now available, we should be able to represent an arbitrary $\vec{A}$ this way. The Hamiltonians are, however, again subject to gauge transformations $\{H_n,G_n\}\to\{H_n',G_n'\}$ such that

$$[H_i', G_j'] \equiv \sum_n \frac{\partial(H_i', G_j')}{\partial(H_n, G_n)} = \delta_{ij}\,, \tag{26}$$

$$[H_i', H_j'] = [G_i', G_j'] = 0\,.$$

These are precisely the ordinary canonical transformations with (H_n, G_n) being regarded as canonical pairs.

III. QUANTIZATION

Can one "quantize" our system of equations just as the ordinary canonical formalism can be quantized? This is an intriguing question which we would like to investigate. The first problem is how to define a quantization. One supposes that quantization would be an algebraic mapping of the relationships which characterize the canonical formalism developed so far. As it turns out, this is not an easy task. The PB defined in (2) has two properties:

(a) Alternation law:

$$[A, B, C] = -[B, A, C] = [B, C, A] = \cdots\,.$$

In particular, $[A, A, C] = 0$, etc. (27a)

(b) Derivation law:

$$[A_1A_2, B, C] = [A_1, B, C]A_2 + A_1[A_2, B, C], \quad \text{etc.} \tag{27b}$$

The first property guarantees that the Hamiltonians are constants of motion. The second makes the PB appropriate for a differential equation. Therefore it should be natural to characterize a PB by Eqs. (27a) and (27b). But the main problem lies in satisfying both of them simultaneously. In fact we have not been able to find a solution beside the classical one (2).

In order to gain some more insight into the situation we recall that the correct Eulerian equations for a top are obtained in quantum theory by the Heisenberg equation (with $\hbar = 1$)

$$i\dot{F} = [F, G], \tag{28}$$

where G is the kinetic energy, Eq. (7), and the angular momenta L_i satisfy the commutation relations

$$[L_i, L_j] = -i\epsilon_{ijk} L_k\,. \tag{29}$$

The derivation law is duly satisfied by Eq. (28). [The anomalous sign on the right-hand side of Eq. (29) is well known,[2] but it is irrelevant to the discussion.] Comparing Eq. (28) with our classical form, Eqs. (2) and (5), we realize that the relations (29) were translated there as

$$[L_i, L_j] \to -i\epsilon_{ijk}\,\partial H/\partial L_k\,. \tag{30}$$

This suggests a commutator algebra different from (29):

$$[L_i, L_j] = i\epsilon_{ijk} K_k\,,\qquad [L_i, K_j] = i\delta_{ij}\,, \tag{31}$$

and

$$\partial H/\partial L_k = i[H, K_k]\,.$$

However Eq. (31) is incompatible with the Jacobi identity

$$[[L_1, L_2], L_3] + [[L_2, L_3], L_1] + [[L_3, L_1], L_2] = 0\,.$$

These considerations seem to indicate that we had better relax the constraints on a PB in order to find other solutions. If we want to keep only the physical consequences of Eqs. (27a) and (27b) the following conditions are also admissible for a PB:

(a′) $[H, H, G] = [G, H, G] = 0$, (32a)

(b′) $[A_1A_2, H, G] = A_1[A_2, H, G] + [A_1, H, G]A_2$, (32b)

where H and G are fixed. Still, we have three options,

(1) $(a)+(b')$, (2) $(a')+(b)$, and (3) $(a')+(b')$.

In the following we study in some detail the case (1) because the example of Pauli spin matrices falls in this category and suggests a way to define a PB. Case (2) retains the derivation law with respect to A, B, and C. This is the most difficult condition, and probably cannot be met. Case (3) will be discussed later. Let us consider an operator algebra $\mathfrak{A}$ generated by three elements, X, Y, and Z, and define a PB by

$$[A, B, C] \equiv ABC + BCA + CAB - BAC - ACB - CBA \tag{33a}$$

$$= A[B, C] + B[C, A] + C[A, B]$$
$$= [A, B]C + [B, C]A + [C, A]B$$
$$([A, B] \equiv AB - BA), \tag{33b}$$

which clearly satisfies Eq. (27a). The canonical triplet (X, Y, Z) will then have the property

$$[X, Y, Z] = i\,. \tag{34}$$

The Hamilton equation should read

$$i\frac{dF}{dt} = [F, H, G] \tag{35}$$

for arbitrary F. According to Eq. (33), a PB of Hermitian operators is anti-Hermitian, so that Eq. (35) is consistent if all physical quantities are translated into Hermitian operators.

We will next check the derivation law (27b). First of all it requires

$$[1, H, G] = 0 \tag{36}$$

as can be seen by putting $A_2 = 1$ in Eq. (27b). But this means, from Eq. (33b),

$$[H, G] = 0 . \tag{37}$$

For general A_1 and A_2, then, Eq. (27b) can be reduced to

$$\begin{aligned} &\dot{A}_1 A_2' - A_1' \dot{A}_2 = 0 , \\ &\dot{A} \equiv [A, H], \quad A' \equiv [A, G], \end{aligned} \tag{38}$$

or

$$[A_1']^{-1}\dot{A}_1 = \dot{A}_2[A_2']^{-1} = \dot{A}_1[A_1']^{-1} = [A_2']^{-1}\dot{A}_2$$

whenever the division is possible. Thus

$$\dot{A} = \alpha A' \quad (\text{or } A' = \alpha \dot{A}) \tag{39}$$

for any A such that A'^{-1} (or $\dot{A}^{-1}$) exists, which in turn implies

$$H = \alpha G + \beta \quad (\text{or } G = \alpha H + \beta). \tag{40}$$

Here α and β commute with all operators and therefore are c numbers. But then

$$[A, H, G] = -[A, \beta G] \quad (\text{or } [A, \beta H]) \tag{41}$$

and by redefining $\beta G \to -G$ (or $\beta H \to H$) we recover the Heisenberg equation.

We conclude thus that if Eq. (33) satisfies the conditions (27a) and (32b), one of the Hamiltonians is a c number, and Eq. (35) is equivalent to a Heisenberg equation. We may wonder at this point whether we can avoid the rather disappointing result if we go over to Eq. (7). Following the same arguments as before, we then arrive at the conditions

$$\begin{aligned} &\sum_i [H_i, G_i] = 0 , \\ &\sum_i [A_1, H_i][A_2, G_i] = \sum_i [A_1, G_i][A_2, H_i] \end{aligned} \tag{42}$$

in place of Eqs. (37) and (38). It is not clear, however, whether or not these more general relations admit nontrivial solutions.

Our next task is then to find realizations of the PB for the canonical triplet. For this purpose let us assume that the canonical variables $\vec{R} = (X, Y, Z)$ generate, under repeated commutator operations, a Lie subalgebra $\mathcal{a}'$ of $\mathcal{a}$ which is semisimple. Let the generators of $\mathcal{a}'$ be $\{R_n\}$, where the first three elements coincide with (X, Y, Z). Further let

$$\begin{aligned} &[R_2, R_3] \equiv i R_1', \\ &[R_3, R_1] \equiv i R_2', \\ &[R_1, R_2] \equiv i R_3'. \end{aligned} \tag{43}$$

Then (R_1', R_2', R_3') belong to the vector space of $\{R_n\}$. In view of Eq. (33b) we have

$$\begin{aligned} [X, Y, Z] &= i \sum_{i=1}^{3} R_i R_i' \\ &= i \sum_{i=1}^{3} R_i' R_i . \end{aligned} \tag{44}$$

Equation (34) demands that this be a c number (in an irreducible Hilbert space generated by $\vec{R}$). Therefore it must be a Casimir operator of the Lie algebra. Then $\vec{R}$ and $\vec{R}'$ must exhaust the generators, and hence $N \leq 6$. There are only two such possibilities, $N = 6$ and $N = 3$.

(a) If $N = 6$, the algebra is either $SO(4) \approx SU(2) \times SU(2)$, $SO(3,1) \approx SL(2, c)$, or $SO(2,2)$. For the first two cases, their generators $(\vec{L}, \vec{K})$ satisfy

$$\begin{aligned} &[L_1, L_2] = iL_3, \quad \text{etc.}, \\ &[L_1, K_2] = [K_1, L_2] = iK_3, \quad \text{etc.}, \\ &[K_1, K_2] = \pm i L_3, \quad \text{etc.}, \quad \begin{cases} SO(4) \\ SO(3,1) \end{cases} \end{aligned} \tag{45}$$

and the Casimir operators are

$$\begin{aligned} C_1 &= \vec{L}^2 \pm \vec{K}^2 , \\ C_2 &= \pm \vec{L} \cdot \vec{K} \\ &= -i(K_1[K_2, K_3] + K_2[K_3, K_1] + K_3[K_1, K_2]). \end{aligned} \tag{46}$$

Without loss of generality one may put $R_i = icK_i$. In this way the solution is found to be

$$\begin{aligned} &\vec{R} = C_2^{-1/3}\vec{K} , \\ &\vec{R}' = \pm C_2^{-2/3}\vec{L} . \end{aligned} \tag{47}$$

The results for the case of $SO(2,2)$ are obtained from that of $SO(4)$ by a similar change of metric.

(b) If $N = 3$, the algebra is either $SO(3) \approx SU(2)$ or $SO(2,1) \approx SU(1,1)$. Their generators $\vec{L}$ satisfy

$$\begin{aligned} &[L_1, L_2] = \pm i L_3 , \quad \begin{cases} SO(3) \\ SO(2,1) \end{cases} \\ &[L_2, L_3] = iL_1, \quad [L_3, L_1] = iL_2, \end{aligned} \tag{48}$$

with the Casimir operator

$$\begin{aligned} C &= L_1^2 + L_2^2 \pm L_3^2 \\ &= -i(L_1[L_2, L_3] + L_2[L_3, L_1] + L_3[L_1, L_2]). \end{aligned} \tag{49}$$

Thus the solution is

$$\begin{aligned} &\vec{R} = (X, Y, Z) = C^{-1/3}(L_1, L_2, L_3), \\ &\vec{R}' = C^{-2/3}(L_1, L_2, \pm L_3), \quad \begin{cases} SO(3) \\ SO(2,1). \end{cases} \end{aligned} \tag{50}$$

We are left with the cases where $\mathfrak{A}'$ is not semisimple. We will discuss some representative cases without claiming completeness.

(c) Euclidean algebra E(3) and its noncompact version E(2, 1). These are derived from SO(4) and SO(3, 1) by group contraction. The six generators $(\vec{L}, \vec{P})$ obey the commutation relations

$$\begin{aligned} &[L_1, L_2] = \pm i L_3, \quad [L_2, L_3] = i L_1, \quad [L_3, L_1] = i L_2, \\ &[L_1, P_2] = -[L_2, P_1] = i P_3, \\ &[L_2, P_3] = \pm [L_3, P_2] = \pm i P_1, \\ &[L_3, P_1] = \pm [L_1, P_3] = i P_2, \\ &[P_i, P_j] = 0, \qquad [P_i, L_i] = 0. \end{aligned} \tag{51}$$

The Casimir operators are

$$\begin{aligned} &C_1 = P_1^2 + P_2^2 + P_3^2, \\ &C_2 = L_1 P_1 + L_2 P_2 + L_3 P_3. \end{aligned} \tag{52}$$

The solutions for R and R' are given by

$$\begin{aligned} &\vec{R} = (\pm C_2)^{-1/3}(L_1, L_2, P_3), \\ &\vec{R}' = \pm C_2^{-2/3}(P_1, P_2, L_3) \end{aligned} \tag{53}$$

or

$$\begin{aligned} &\vec{R} = C_2^{-1/3}(P_1, L_2, L_3), \\ &\vec{R}' = C_2^{-2/3}(L_1, P_2, P_3). \end{aligned}$$

[These are equivalent for E(3).]

(d) E(2) and E(1, 1) derived from SO(3) and SO(2, 1) by contraction. The generators are (L, P_1, P_2) with the commutation relations

$$[L, P_1] = i P_2, \quad [L, P_2] = \mp i P_1, \quad [P_1, P_2] = 0, \tag{54}$$

and the Casimir operator is

$$C = \pm P_1^2 + P_2^2. \tag{55}$$

The solutions are

$$\begin{aligned} &\vec{R} = C^{-1/3}(L, P_1, P_2), \\ &\vec{R}' = C^{-2/3}(0, \pm P_1, P_2). \end{aligned} \tag{56}$$

(e) This case is characterized by the commutation relations of the Galilean group in one dimension,

$$[X, Y] = i Z^{-1}, \quad [X, Z] = [Y, Z] = 0, \tag{57}$$

for which we have a representation

$$\vec{R} = \left(i \frac{d}{dq}, \frac{q}{x}, x\right), \quad \vec{R}' = \left(0, 0, \frac{1}{x}\right). \tag{58}$$

(f) By putting $x = \text{const.} = 1$ in Eq. (58) we recover the canonical formalism of quantum mechanics:

$$\vec{R} = (-p, q, 1), \quad [p, q] = -i. \tag{59}$$

IV. USE OF NONASSOCIATIVE ALGEBRAS

In this section we pose the question whether it is possible to realize the PB relations by means of nonassociative algebras.[3] As a matter of fact, we immediately see that there is some hope, because it takes three elements to characterize the nonassociative nature of an algebra, just as we have a PB involving three observables. The analog of a PB in a nonassociative algebra is the associator

$$(a, b, c) \equiv (ab)c - a(bc) \tag{60}$$

which is zero if the multiplication table is actually associative.

Let us now see whether or not one can identify the associator with the PB under some combination of conditions (27) and (32). First take the alternative (1) stated there. Equation (27a) then demands that the associator is alternative:

$$(a, b, c) = -(b, a, c) = (b, c, a) = \cdots. \tag{61}$$

That means that we are dealing with the alternative algebra of Cayley and Dickson.[3] The Cayley-Dickson algebra C is an algebra over real numbers (or any suitable field) with eight basic elements u_i, $i = 0, \ldots, 7$, where u_0 serves as the unit element; it is the only possible algebra of this kind.

Unfortunately, we run into trouble with the derivation law (32b). The associator does not in general satisfy the derivation law, and conversely any derivation algebra over the Cayley numbers is known to be generated instead by operations of the form $\sum_{a,b} D_{a,b}$ on C, where

$$\begin{aligned} D_{a,b}\, x &= a(bx) - b(ax) + (xb)a - (xa)b + (bx)a - b(xa), \\ &\equiv D(a, b; x). \end{aligned} \tag{62}$$

$D(a, b; x)$ is antisymmetric only with respect to a, b, and for associative algebras reduces to

$$D(a, b; x) = [x, [a, b]]. \tag{63}$$

Suppose we now identify the PB $[A, B, C]$ with $D(B, C; A)$, and $B = H$, $C = G$ in the Hamilton equation. We have then switched to the alternative (3), which requires Eq. (32a):

$$D(H, G; H) = D(H, G; G) = 0. \tag{64}$$

This amounts to

$$[H, (HG) - (GH)] = [G, (HG) - (GH)] = 0, \tag{65}$$

but there is no nontrivial solution to it.

On the other hand, we can give up Eq. (32a) if we are considering Eq. (9) as the classical basis. Then the form

$$\frac{dF}{dt} = \sum_i D(H_i G_i ; F) \tag{66}$$

will certainly serve as a possible Hamilton equation. Since the multiplication table is preserved under Eq. (66), it induces an automorphism on C. It is known that automorphisms on C form a Lie group of type G_2. An element of C which is left invariant under (66) will be a constant of motion.

Let us leave the Cayley algebra and next shift our attention to the Jordan algebra J, which is commutative (but not associative).[3,4] The associator has the property

$$(a, b, a) = 0, \quad (a, b, a^2) = 0. \tag{67}$$

(The first is an identity which follows from the definition.) A derivation operator on J is given by

$$D_{a,b}\, x = (a, b, x) - (b, a, x). \tag{68}$$

This allows one to define the PB as

$$\begin{aligned} -i[A, B, C] &= D_{B,C} A \\ &= (B, C, A) - (C, B, A) \end{aligned} \tag{69}$$

and the Hamilton equation as

$$i\frac{dF}{dt} = [F, H, G], \tag{70a}$$

or more generally

$$= \sum_i [F, H_i, G_i]. \tag{70b}$$

As before, we may add the condition (32a) in the case of Eq. (70a).

Now it is known that a Jordan algebra can be derived from an associative algebra $\mathfrak{A}$ if we define the multiplication to mean

$$axb = \tfrac{1}{2}(ab + ba), \tag{71}$$

where ab stands for a multiplication in the associative algebra $\mathfrak{A}$. Going back to this realization, we then find the PB (69) is given by

$$[A, B, C] = i[A, [B, C]]. \tag{72}$$

This is the same form as Eq. (63), and Eqs. (70a) and (70b) reduce to

$$i\frac{dF}{dt} = [F, \mathcal{H}], \quad \mathcal{H} = i[H, G] \text{ or } i\sum_i [H_i, G_i]. \tag{73}$$

In other words, it is equivalent to a Heisenberg equation with Hamiltonian $\mathcal{H}$. In addition, an example of canonical triplet $\vec{R}$ satisfying

$$-i[X, Y, Z] = [X, [Y, Z]] = 1 \tag{74}$$

can readily be found. In fact

$$\vec{R} = \left(i\frac{\partial}{\partial x}, xy, i\frac{\partial}{\partial y} \right). \tag{75}$$

Finally there exists one Jordan algebra (the exceptional Jordan algebra) which is not isomorphic to an algebra generated by Eq. (71).[3,4] It is an algebra of 3×3 matrices M with elements in a Cayley algebra. M has the typical form

$$M = \begin{pmatrix} \alpha & c & \bar{b} \\ \bar{c} & \beta & a \\ b & \bar{a} & \gamma \end{pmatrix}, \tag{76}$$

where α, β, γ are c numbers; a, b, c are Cayley numbers, where $\bar{a}$ is obtained from a by the involution $u_0 \to u_0$, $u_i \to -u_i$ $(i \neq 0)$. This should again offer another possibility of realizing the PB relations.

V. SUMMARY

Taking the Liouville theorem as a guiding principle, we have proposed a possible generalization of classical Hamiltonian dynamics to a three-dimensional phase space. The equation of motion involves two Hamiltonians and three canonical variables. A more general form may have many triplets and many Hamiltonian pairs. Such a formalism does not seem totally irrelevant to physics because the Eulerian top problem can be cast into this form, and it offers a new possibility in statistical mechanics. An attempt to find a "quantized" version of the formalism, however, has been only partially successful. In the process, the correspondence between classical and quantized versions is largely lost. One is repeatedly led to discover that the quantized version is essentially equivalent to the ordinary quantum theory. This may be an indication that quantum theory is pretty much unique, although its classical analog may not be.

On the other hand, there remains some possibility that nonassociative algebras may also be incorporated into the new formalism. Jordan was first led to what is now known as Jordan algebra in an attempt to reformulate and generalize quantum mechanics.[4] Although our starting point and motivation were different from Jordan's we have also found the potential significance of nonassociative algebras.

I would like to express my appreciation to Professor K. Husimi who kindly took an interest in my ideas when the contents of the first section were conceived more than twenty years ago. I would also like to acknowledge the inspirations I derived from recent communications with Professor F. Gürsey[5] and Dr. Pierre Ramond regarding nonassociative algebra.

*Work supported in part by the National Science Foundation (Contract No. NSF GP 32904 X) and the John Simon Guggenheim Foundation.

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PHYSICAL REVIEW D VOLUME 7, NUMBER 8 15 APRIL 1973

Degeneracy of Relativistic Cyclotron Motion

Sadaaki Yanagawa
Department of Applied Physics, Faculty of Engineering, University of Tokyo, Tokyo, Japan
(Received 5 July 1972)

The quantum-mechanical problem of the relativistic cyclotron motion of a charged particle in a uniform magnetic field is solved by consideration of the symmetry which the system obeys. It is shown that its symmetry is isomorphic to the Lie group called G(0,1) or G(1,0), and doubly degenerate infinite series of wave functions with a constant energy eigenvalue are labeled by the eigenvalues of the operators $\mathcal{H}^2$, L_z+S_z, and S_z. Here $\mathcal{H}$ is the relativistic Hamiltonian referred to in the present problem, and L_z and S_z are the usual orbital and spin angular momentum operators, respectively.

I. INTRODUCTORY REMARK

In a previous paper[1] it was shown that the nonrelativistic Hamiltonian H^{nr},

$$H^{\mathrm{nr}}=\frac{1}{2m}(\Pi_x{}^2+\Pi_y{}^2)$$
$$\equiv\frac{1}{2m}\left[\left(P_x-\frac{eH}{2c}y\right)^2+\left(P_y+\frac{eH}{2c}x\right)^2\right], \tag{1}$$

which expresses the motion of a free electron in a uniform magnetic field H directed in the z direction, apart from the z component of the space coordinates, has a symmetry of the Lie group G(0, b) generated by the infinitesimal operators $A_\pm$, A_3^{nr}, and E (identity) defined as

$$A_\pm=A_x\pm iA_y ,$$
$$A_x=-\frac{\partial}{\partial x}-i\frac{eH}{2c\hbar}y ,$$
$$A_y=-\frac{\partial}{\partial y}+i\frac{eH}{2c\hbar}x , \tag{2}$$

and

$$A_3^{\mathrm{nr}}=\frac{1}{i}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\equiv\frac{L_z}{\hbar} .$$

They satisfy the following commutation relations:

$$[A_+,A_-]=-\frac{2eH}{c\hbar}E\equiv-bE ,$$
$$[A_3^{\mathrm{nr}},A_+]=A_+ , \tag{3}$$

and

$$[A_3^{\mathrm{nr}},A_-]=-A_- .$$

Each operator commutes with H^{nr}, and all degenerate eigenfunctions $\psi_n, \psi_n', \ldots$ of semi-infinite numbers with a constant eigenvalue $(n+\frac{1}{2})\hbar\omega_c$ can be obtained by operating with the $A_\pm$ operators on any eigenfunction with the same eigenvalue; namely,

$$A_+\psi_n=\psi_n'$$
or
$$A_-\psi_n=\psi_n'' . \tag{4}$$

These functions are given explicitly in Ref. 1, and each function is labeled by the eigenvalue of A_3^{nr}. Here, A_+ or A_- is nothing but the raising or lowering operator for angular momentum (L_+ or L_-), respectively. Further, when we define operators $B_\pm$ and B_3 as

$$B_+=\left(\frac{\hbar}{2m\omega_c}\right)^{1/2}A_++\left(\frac{m\omega_c}{2\hbar}\right)^{1/2}(x+iy)$$
$$\equiv-i\left(\frac{c}{2e\hbar H}\right)^{1/2}(\Pi_x+i\Pi_y) ,$$
$$B_-=-\left(\frac{\hbar}{2m\omega_c}\right)^{1/2}A_-+\left(\frac{m\omega_c}{2\hbar}\right)^{1/2}(x-iy) \tag{5}$$
$$\equiv i\left(\frac{c}{2e\hbar H}\right)^{1/2}(\Pi_x-i\Pi_y) ,$$