Quasi-Nambu-Goldstone Modes in Bose-Einstein Condensates

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We show that quasi-Nambu-Goldstone (NG) modes, which play prominent roles in high energy physics but have been elusive experimentally, can be realized with atomic Bose-Einstein condensates. The quasi-NG modes emerge when the symmetry of a ground state is larger than that of the Hamiltonian. When they appear, the conventional vacuum manifold should be enlarged. Consequently, topological defects that are stable within the conventional vacuum manifold become unstable and decay by emitting the quasi-NG modes. Contrary to conventional wisdom, however, we show that the topological defects are stabilized by quantum fluctuations that make the quasi-NG modes massive, thereby suppressing their emission.

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Introduction.—Spontaneous symmetry breaking generally yields Nambu-Goldstone (NG) modes, which play a crucial role in determining low-energy behaviors of various systems from condensed matter to high energy physics. Although the NG theorem guarantees that NG modes do not acquire mass at any order of quantum corrections, we sometimes encounter situations in which soft modes appear which are massless in the zeroth order (tree approximation) but become massive due to quantum corrections. They are called quasi-NG modes [1] and were introduced in the context of gauge theories with symmetry breaking by Weinberg [2]. It was shown that such modes emerge if the symmetry of an effective potential of the zeroth order is higher than that of gauge symmetry, and the idea was invoked to account for the emergence of low-mass particles. On the other hand, Georgi and Pais demonstrated that quasi-NG modes also occur in cases in which the symmetry of the ground state is higher than that of the Hamiltonian [3].

Later on, quasi-NG modes turned out to play prominent roles in technicolor [4] and supersymmetry [5], both of which were proposed as candidates beyond the standard model for unification of fundamental forces. The quasi-NG modes of the Weinberg type [2] become an important ingredient in the physics of technicolor, which is a model to avoid a hierarchy problem because it does not assume the Higgs scalar field as an elementary particle. In this model, the quasi-NG mode is related to the vacuum that is energetically selected from among a large degenerate family of vacua (vacuum alignment problem). On the other hand, the quasi-NG modes of the Georgi-Pais type [3] are inevitable in theories with supersymmetry, which is among the most powerful guiding principles in contemporary elementary particle physics. This is because the vacuum condition in supersymmetric theories invariant under a group G is always invariant under its complex extension $G^{\mathbb{C}}$. This type of the quasi-NG mode is also believed to appear in the weak-coupling limit of A phases of ^{3}He [6] and spin-1 color superconductivity [7]. Despite their importance as described above, the direct experimental confirmation of the quasi-NG modes has yet to be made.

In this Letter, we point out that a spinor Bose-Einstein condensate is an ideal system to study the physics of quasi-NG modes. This system has recently been a subject of active research because of the great experimental manipulability and well-established microscopic Hamiltonians. We show that quasi-NG modes appear in a spin-2 nematic phase, which may be realized by using ^{87}Rb and a d-wave superfluid. In the nematic condensate, three phases, each of which has a different symmetry, are energetically degenerate to the zeroth order [8] as listed in Table I. We point out that this corresponds to the vacuum alignment problem in particle physics, and, to fully understand this, we must consider the degrees of freedom of quasi-NG modes due to the $\mathrm{U}(1)\times\mathrm{SO}(5)$ symmetry to the zeroth order. Considering this symmetry, we show that the vacuum (order parameter) manifold is enlarged to $\tilde{M}\cong[\mathrm{U}(1)\times S^4]/\mathbb{Z}_2$. Then, the vacuum alignment problem can be understood from the fact that each vacuum manifold of

TABLE I. Vacuum manifold M and the enlarged vacuum manifold $\tilde{M}$ of the uniaxial nematic ($\eta = n\pi/3$), biaxial nematic [$\eta = (n+1/2)\pi/3$], and dihedral-2 (other η) phases, where the parameter η characterizes the order parameter of the nematic phase [see Eq. (2)].

Phase	$M \cong G/H$	$\tilde{M}$
Uniaxial nematic	$\mathrm{U}(1)\times S^2/\mathbb{Z}_2$	
Biaxial nematic	$[\mathrm{U}(1)\times \mathrm{SO}(3)]/D_4$	$[\mathrm{U}(1)\times S^4]/\mathbb{Z}_2$
Dihedral-2	$\mathrm{U}(1)\times \mathrm{SO}(3)/D_2$	

the NG modes $M \cong G/H$ is a submanifold of $\tilde{M}$. We show that the vacuum alignment problem is solved because of the quasi-NG modes becoming massive and that one of the vacuum manifolds is selected because the symmetry at the zeroth order is broken by quantum corrections. We find that a soliton that is unstable classically becomes stabilized by quantum corrections that suppress emissions of the quasi-NG modes as shown in Figs. 1(a) and 1(e). Notably, as soon as the quantum fluctuations are switched off at $t = 0$, the soliton begins to decay since the original vortex core structure disappears as shown in Figs. 1(b)–1(d); this is accompanied by emissions of quasi-NG modes as shown in Figs. 1(f)–1(h).

Symmetry of the ground state.—We start with a spin-2 condensate with mass M whose $G = \mathrm{U}(1) \times \mathrm{SO}(3)$ invariant Hamiltonian is given by

$$H = \int d^3r\Big[-\frac{\hbar^2}{2M}\psi_m^\dagger \Delta \psi_m + \frac{c_0}{2}(\psi_m^\dagger\psi_m)^2 + \frac{c_1}{2}(\psi_m^\dagger \mathbf{F}_{mn}\psi_n)^2 + \frac{c_2}{2}|\psi_m^\dagger T\psi_m|^2\Big], \tag{1}$$

where ψ_m ($m = 2, 1, \ldots, -2$) are field operators with magnetic quantum number m, F_i ($i = x, y, z$) are spin matrices, T denotes the time reversal operator $[T\psi_m = (-1)^m\psi_{-m}^\dagger]$, and repeated indices are assumed to be summed over $2, 1, \ldots, -2$. The coupling constants are related to the s-wave scattering lengths a_S in the total spin S channel by $c_0 = 4\pi\hbar^2(4a_2 + 3a_4)/7M$, $c_1 = 4\pi\hbar^2(a_4 - a_2)/7M$, and $c_2 = 4\pi\hbar^2(7a_0 - 10a_2 + 3a_4)/35M$. By applying the Gross-Pitaevskii theory to the Hamiltonian (1) under the assumption that the $\mathbf{k} = \mathbf{0}$ components are macroscopically occupied, we can show that the ground state of the nematic phase is realized for $c_2 < 0$ and $c_2 < 4c_1$ [9,10]. The order parameter of the nematic phase is given by

$$\phi_m = (\sin\eta/\sqrt{2}, 0, \cos\eta, 0, \sin\eta/\sqrt{2})^T, \tag{2}$$

where η is an additional parameter independent of G [8], and $\eta = n\pi/3$, $\eta = (n + 1/2)\pi/3$, and the other values correspond to the uniaxial nematic, biaxial nematic, and dihedral-2 phases, respectively, where $n \in \mathbb{Z}$. Since the state with η is equivalent to that with $\eta + \pi/3$ up to $\mathrm{U}(1) \times \mathrm{SO}(3)$, we restrict the domain of definition to $0 \le \eta < \pi/3$ unless otherwise stated.

Next, we show that the zeroth-order solution of the nematic phase has $\tilde{G} = \mathrm{U}(1) \times \mathrm{SO}(5)$ symmetry. As discussed in Ref. [11], the interactions proportional to c_0, c_1, and c_2 have SU(5), SO(3), and SO(5) symmetries, respectively, in addition to the U(1) symmetry. On the other hand, the zeroth-order solution for the nematic phase satisfies $|\phi_m^\dagger T\phi_m| = 1$ and $\phi_m^\dagger(\mathbf{F})_{mn}\phi_n = \mathbf{0}$. In fact, the first relation is sufficient to characterize the nematic phase, since it is proved that $|\langle\psi_m^\dagger T\psi_m\rangle| = 1$ implies $\langle\mathbf{F}\rangle = \mathbf{0}$ [10]. Within the zeroth order, the interaction term proportional to c_2 satisfies $|\phi_m^\dagger T\phi_m|^2 = 1$ and is invariant under a $\tilde{G}$ transformation, which implies that $|\phi_m^\dagger T\phi_m| = 1$ is preserved under the same transformation. Therefore, we conclude that the solution of the nematic phase has the $\tilde{G}$ symmetry even if $c_1 \neq 0$.

Number of NG and quasi-NG modes.—We identify the enlarged vacuum manifold of the nematic phase based on $\tilde{G}$ and discuss the number of NG and quasi-NG modes in each phase. Since the vacuum manifold is independent of the location in the orbit, we choose the configuration $\bar{\phi}_m = (-i, 0, 0, 0, i)^T/\sqrt{2}$, which describes the five-dimensional representation of SO(3) in the spherical tensor basis and therefore describes the SO(5) in the same basis. To determine the isotropy group, let us next transform from this representation of SO(5) to the fundamental representation of SO(5). By using a unitary matrix

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 & 0 & 0 & -i \\ 0 & -i & 0 & -i & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3}$$

we can achieve it for the state and generators as

$$\varphi_k \equiv U\bar{\phi}_m = (1, 0, 0, 0, 0)^T \tag{4}$$

and $(\chi_{kl})_{ij} \equiv (UF_{kl}U^\dagger)_{ij} = -i(\delta_{ki}\delta_{lj} - \delta_{li}\delta_{kj})$, respectively, where F_{kl} is the corresponding SO(5) generator in the spherical tensor basis. In the fundamental representation, elements of the SO(5) group are expressed as $R_{kl}(t) = \exp(-i\chi_{kl}t)$ with real parameter t. Then, (4) is invariant

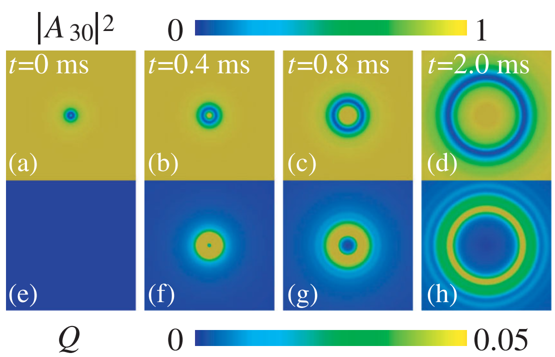

FIG. 1 (color online). Decay dynamics of a $\mathbb{Z}_2$ vortex that is stable under $M \cong \mathrm{U}(1) \times S^2/\mathbb{Z}_2$ in the uniaxial nematic phase. (a)–(d) Time evolution of the absolute square of the spin-singlet trio amplitude $|A_{30}|^2$ defined in Eq. (10). (e)–(h) Time evolution of the quasi-NG charge $Q \equiv \langle F_{35}\cos\theta + F_{13}\sin\theta\rangle$, where θ is the polar coordinate on a plane and F_{13} and F_{35} are introduced below Eq. (4). The quantum fluctuations that stabilize the vortex state are switched off at $t = 0$. Then the vortex decays in time as shown in (b)–(d). This decay is accompanied by the emission of quasi-NG modes as shown in (f)–(h). The numerical simulations are performed by using a modified Gross-Pitaevskii analysis based on a local density approximation [14]. In the simulations, we choose numerical parameters for typical experimental situations for spin-2 ^{87}Rb condensates with $n = 1.5 \times 10^{14}$ cm^{-3}.

under R_{23}, R_{24}, R_{25}, R_{34}, R_{35}, and R_{45}, which constitute an SO(4) as a subgroup of SO(5). Therefore, the enlarged vacuum manifold except for discrete groups is $U(1)\times SO(5)/SO(4)\cong U(1)\times S^4$. To consider the effect of discrete symmetry, let us utilize the degrees of freedom of $U(1)\times S^4$. Then, φ_k can be transformed as follows:

$$\varphi'_k = e^{i\phi}(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \sin\theta_1 \times \sin\theta_2\sin\theta_3\cos\theta_4, \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4)^T, \quad (5)$$

where $0\le\theta_1,\theta_2,\theta_3\le\pi$ and $0\le\theta_4\le 2\pi$. To ensure that φ'_k coincides with φ_k, it is necessary and sufficient that $\theta_1=\phi=0$, or $\theta_1=\phi=\pi$, which is isomorphic to $\mathbb{Z}_2$. The full enlarged vacuum manifold of the nematic phase is therefore given by

$$\tilde{M}\cong\frac{U(1)\times S^4}{\mathbb{Z}_2}\cong\frac{U(1)\times SO(5)}{\mathbb{Z}_2\ltimes SO(4)}. \quad (6)$$

Here, $\ltimes$ implies that the nontrivial element of $\mathbb{Z}_2$ does not commute with some elements of SO(4).

Next, we discuss the number of NG and quasi-NG modes. As analyzed in Ref. [12], the number of the NG modes in the nematic phase is equal to the dimension of M, $\dim(M)$. Meanwhile, in Ref. [3], it has been shown that the number of quasi-NG modes, n, is given by

$$n=\dim(\tilde{M})-\dim(M), \quad (7)$$

where $\tilde{M}$ is the surface on which the effective potential assumes its minimum value to the zeroth order. This implies that M is a submanifold of $\tilde{M}$ and n is the dimension of the complementary space of M inside $\tilde{M}$.

In the nematic phase, we expect 5 soft modes because $\dim(\tilde{M})=5$, which is consistent with the result of the Bogoliubov theory since it predicts an equal number of massless modes [12,13]. While three and two of them are the NG and quasi-NG modes for the uniaxial nematic phase because of $\dim(M)=3$, four and one of them are the NG and quasi-NG modes for the dihedral-2 and biaxial nematic phases because of $\dim(M)=4$. (See Table II.) That is, the number of NG and quasi-NG modes changes in each phase, with the total number unchanged.

Quantum symmetry breaking.—Since the interactions proportional to c_0 and c_2 favor $\tilde{G}$ but the interaction proportional to c_1 breaks $\tilde{G}$, it is expected that the ground-state symmetry of the nematic phase at the zeroth order is broken by quantum corrections, thereby making quasi-NG modes massive. In fact, the ground-state energy per particle at the 1-loop level is evaluated as [12,13]

$$\Delta\epsilon = v(\eta)+\tilde{v}, \quad (8)$$

$$v(\eta)=\omega\sum_{j=0}^{2}[1+X\cos(2\eta+2\pi j/3)]^{5/2}, \quad (9)$$

where $\omega = 8\sqrt{M^3}[n(2c_1-c_2)]^{5/2}/15n\pi^2\hbar^3$, $X=-2c_1/(2c_1-c_2)$, $\tilde{v}$ describes an η-independent contribution, and $v(\eta)$ is the $\tilde{G}$ symmetry breaking contribution, which favors the uniaxial nematic phase for $c_1\ge 0$ and the biaxial nematic phase for $c_1\le 0$. In other words, the degeneracy at the zeroth order is lifted and one of the phases is selected, depending on the sign of c_1. This phenomenon corresponds to the vacuum alignment. Additionally, since $v(\eta)$ breaks the extra flat directions, masses of the quasi-NG modes that are of the order of ω arise. In this way, the quantum symmetry breaking of the $U(1)\times SO(5)$ symmetry indeed occurs.

Fate of topological defects.—Here, let us discuss the importance of the quasi-NG modes in terms of topological defects. If the symmetry breaking contribution (9) is negligible, some topological defects predicted from M become unstable. This is because, under such a situation, topological defects are characterized by $\pi_n(\tilde{M})$ instead of $\pi_n(M)$, where π_n represents the nth homotopy group. To illustrate this in detail, by using a modified Gross-Pitaevskii equation based on a local density approximation [14], we examine the (in)stability of the $\mathbb{Z}_2$ vortex in the uniaxial nematic phase, which is predicted from $\pi_1(S^2/\mathbb{Z}_2)=\mathbb{Z}_2$. In the $\mathbb{Z}_2$ vortex, the spin-singlet trio amplitude

$$\begin{aligned}A_{30} &\equiv 3\sqrt{6}(\phi_2\phi_{-1}^2+\phi_1^2\phi_{-2})/2 \\ &\quad +\phi_0(\phi_0^2-3\phi_1\phi_{-1}6\phi_2\phi_{-2}) \\ &=\cos 3\eta\end{aligned} \quad (10)$$

becomes zero at the vortex core and has a finite value in the other region. Initially, the $\mathbb{Z}_2$ vortex is prepared under the condition that the contribution (9) cannot be neglected, and, therefore, the vortex remains stable and the quasi-NG modes are not emitted spontaneously [Figs. 1(a) and 1(e)]. However, when the contribution (9) is negligible, the quasi-NG charge expands in a radial direction and the vortex core structure disappears with time, which indicates that the vortex decays due to the emission of the quasi-NG modes [Figs. 1(b)–1(d) and 1(f)–1(h)]. We emphasize that this drastic change in the vortex state is caused by quantum fluctuations.

Geometrical interpretation.—The mathematical structure of the nematic phase can be interpreted from the viewpoint of differential geometry [15]. The target space $\tilde{M}$ is parametrized by both NG and quasi-NG modes. To identify the quasi-NG modes, we eliminate the NG modes

TABLE II. Broken generators within (2) and $0\le\eta<\pi/3$, the number of ordinary NG modes [U(1), F_x, F_y, and F_z], and that of quasi-NG modes (F_{13} and F_{35}) in each phase, where $F_x=-F_{14}-F_{25}+\sqrt{3}F_{23}$, $F_y=-F_{12}+F_{45}-\sqrt{3}F_{34}$, and $F_z=2F_{15}+F_{24}$.

Phase	Broken generator	N_{NG}	$N_{quasi\text{-}NG}$
Uniaxial nematic	U(1), F_x, F_y, F_{13}, F_{35}	3	2
Biaxial nematic	U(1), F_x, F_y, F_z, F_{35}	4	1
Dihedral-2	U(1), F_x, F_y, F_z, F_{35}	4	1

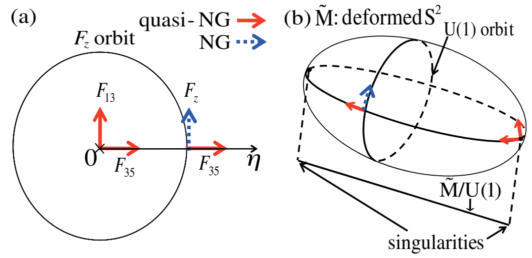

FIG. 2 (color online). (a) Graphical interpretation of the NG–quasi-NG change within $0 \leq \eta < \pi/3$. Note that this is a vertical slice about the directions along which the NG modes on U(1), F_x, and F_y fluctuate. (b) A simple example of the NG–quasi-NG change in which the target space is the rugby ball (homeomorphic to S^2), the base space $S^2/\mathrm{U}(1)$, and the structure group U(1). Since U(1) symmetry is broken at the generic points, we have one NG mode along the U(1) orbit and one quasi-NG mode along the base space $S^2/\mathrm{U}(1)$ orthogonal to the U(1) orbit. However, two quasi-NG modes appear at the singularities of the base space where the U(1) symmetry is restored. This is because neither of the two directions is generated by U(1) at the U(1) fixed point.

by taking a quotient of $\tilde{M}$ by G. Then, the quasi-NG mode corresponds to the orbit space $\tilde{M}/G$, and it is parametrized by quantities made of ψ_m, which are G-invariant but not $\tilde{G}$-invariant. There should exist n such invariants with n defined in (7) at generic points, where the dimension of unbroken symmetry dim(H) is the lowest. For the nematic phase, where the generic points are the dihedral-2 and biaxial nematic phases with $n = 1$, there should be one polynomial to describe $\tilde{M}/G$. In fact, we can show that this is uniquely determined to be the absolute value of the spin-singlet trio amplitude $|A_{30}|$. From this result, we find that the orbit space $\tilde{M}/G$ can be characterized by η. Physically, the direction of η is that of the quasi-NG mode, which is generated by F_{35}. The target space $\tilde{M}$ can be regarded as a (singular) fiber bundle over the base space $\tilde{M}/G$, with fiber M and structure group G. The fiber shrinks at certain points, where H is enhanced and which correspond to the singular points in $\tilde{M}/G$. Since the number of NG modes is reduced at such points, some of the NG modes need to change to quasi-NG modes to keep the total number of soft modes unchanged. In fact, the NG mode generated by F_z changes to the quasi-NG mode generated by F_{13} in the uniaxial nematic phase, which corresponds to a singularity in $\tilde{M}/G$, as shown in Fig. 2.

In conclusion, we have shown that quasi-NG modes can be realized with the spin-2 nematic phase because the vacuum manifold at the zeroth order is enlarged to $\tilde{M}$ and each M is a submanifold of $\tilde{M}$. This corresponds to the vacuum alignment problem, where a unique vacuum is selected by considering quantum fluctuations. We expect that the phenomena illustrated in Fig. 1 open up a new possibility in quantum vortices that can be tested experimentally by controlling quantum fluctuations.

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[1] Although this was originally called the pseudo-NG mode, this term is currently used to represent a soft mode which has a small mass attributed to explicitly symmetry breaking terms at the zeroth order. Hence, we refer to it as the quasi-NG mode to avoid confusion with the pseudo-NG mode.
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