Chaotic Nature of the Spin-Glass Phase

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The description of the spin-glass phase in terms of a $T=0$ fixed point is shown to imply a "chaotic" phase in which the relative orientations of spins with large separations $L \gtrsim L^*$ are sensitive to small changes δT in the temperature or δJ in the bond strengths, where $L^* \propto 1/(\delta T)^{1/\zeta}$ or $1/(\delta J)^{1/\zeta}$, respectively, $\zeta = d_s/2 - y$, and $-y$ and d_S are the thermal eigenvalue at the $T=0$ fixed point and the "interfacial (fractal) dimension," respectively.

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Progress in the theory of spin glasses has until recently been largely confined to the Sherrington-Kirkpatrick (SK) model,[1-3] which serves as a mean-field model for the spin glass. Extension to short-range models by expansion around the solution[2] of the SK model has proved to be difficult,[4] and results from this avenue on the nature of ordering in finite-dimensional systems are still awaited.

Recently, however, a new approach has been proposed,[5-7] based on the properties of a $T=0$ fixed point. Motivated by numerical studies,[8,9] this approach describes an ordered phase very different from that of the SK model (no "replica-symmetry breaking,[2]" Almeida-Thouless line,[10] or "lack of self-averaging"[3,11]). The central role is played by an exponent y (called θ in Ref. 6) which characterizes the energetics of large-scale excitations from the ground state: The characteristic energy scale at length scale L varies as L^y. The system orders at low-temperature only for $y>0$, i.e., when the system scales to strong coupling. Numerical studies of interface (or "domain wall") energies indicate[8,9] that $y>0$ ($y \sim 0.2$) for Ising spin glasses in $d=3$, while $y<0$ ($y \sim -0.3$) for $d=2$. These results are supported by extensive Monte Carlo simulations.[12]

Here the $T=0$ scaling theory is used to investigate the microscopic structure of the ordered phase. It has been argued[6,7] that the smallness of L^y compared to the "naive" estimate L^{d-1} for the interface (free) energy of an overturned region of spins ("droplet") of size L implies large cancellations from different parts of the interface. Such cancellations should be sensitive to changes in T, implying that the relative orientations of spins sufficiently far apart can change with arbitrarily small changes in T.[6]

We first look at the related problem of small changes, not to T, but to the interaction strengths themselves. We work at $T=0$ and investigate the sensitivity of the ground state to such changes. Interesting results can be obtained even for $d=1$ and $d=2$ where the response to changes in T would be less interesting because of the absence of an ordered phase for $T>0$. We obtain the striking result that *at sufficiently large length scales the ground state is unstable against arbitrarily weak perturbations to the bonds.* More precisely, if δJ, J measure the strength of the bond perturbations and of the unperturbed bonds, respectively, the ground state is unstable on length scales larger than $L^* \sim (J/\delta J)^{1/\zeta}$, where $\zeta = d_S/2 - y$, and L^{d_S} is the typical area of an interface surrounding a droplet (i.e., low-energy excitation) of size L. The result is derived explicitly for $d=1$, while for $d=2$ it is inferred from the sensitivity of interface energies to bond perturbations. As a by-product we find $d_S \simeq 1.26$ for $d=2$. It is argued that for $d=3$, where $y>0$, the response to a temperature change for $T<T_c$ is given by a similar expression with δT replacing δJ.

A heuristic derivation of our results follows from an Imry-Ma–style domain argument.[13] Consider an Ising spin-glass with a continuous distribution, say Gaussian of width J, of exchange interactions. A low-energy excitation from the ground state, involving an overturned droplet of linear dimension L, costs energy of order JL^y. Now add a small random perturbation, say Gaussian of width J_0, to each bond. If the ground state remains unchanged, the contribution to the droplet energy from the perturbation is a sum of L^{d_S} independent random variables with random signs, i.e., a term of order $J_0 L^{d_S/2}$. Hence, *provided* $\zeta \equiv d_S/2 - y$ *is positive,* the ground state will be unstable to the perturbation on length scales $L \gtrsim L^* \approx (J/J_0)^{1/\zeta}$: The relative orientations of spins separated by more than L^* will be strongly affected. Such sensitivity to weak perturbations is a fundamental property of spin-glasses. Note that the nonnegativity of ζ for Ising spin-glasses is ensured by the inequalities $d_S \geq d-1$ and[6] $y \leq (d-1)/2$. By contrast, for an

Ising ferromagnet $d_S = d-1 = y$ giving $\zeta = -(d-1)/2$, i.e., the ferromagnetic ground state is stable against weak-bond perturbations for $d > 1$.

Consider now the solvable case of an Ising chain. The Hamiltonian is

$$H = -\sum_{i=1}^{N-1} J_i S_i^{(1)} S_{i+1}^{(1)}.$$

If we fix the boundary spin $S_1^{(1)} = +1$ to break the time-reversal symmetry, the ground state is

$$S_L^{(1)} = \prod_{i=1}^{L-1} \operatorname{sgn}(J_i), \quad L = 2, \ldots, N.$$

For simplicity we add to each bond a constant, J_0, rather than a random variable (the Imry-Ma argument works equally well for this case). The new ground state is

$$S_L^{(2)} = \prod_{i=1}^{L-1} \operatorname{sgn}(J_i + J_0)$$

and the correlation between the new and old ground states, averaged over the disorder, is

$$[S_L^{(1)} S_L^{(2)}]_{\mathrm{av}} = \left[\int_{-\infty}^{\infty} dJ\, P(J) \operatorname{sgn}(J) \operatorname{sgn}(J+J_0)\right]^{L-1}$$
$$\simeq [1 - 2J_0 P(0)]^{L-1}$$

for $J_0 \ll 1$. In the "scaling limit" $L \to \infty$, $J_0 \to 0$, with $J_0 L$ fixed, we have $[S_L^{(1)} S_L^{(2)}]_{\mathrm{av}} \to \exp[-2LJ_0P(0)]$. Thus the two ground states decorrelate over a distance $L^* \sim 1/2J_0P(0)$, which agrees with the Imry-Ma argument since $d_S = 0$ and[7,8] $y = -1$ for $d = 1$. The "ground state overlap"

$$q^{(1,2)} \equiv N^{-1} \sum_{i=1}^{N} [S_i^{(1)} S_i^{(2)}]_{\mathrm{av}} \simeq \frac{L^*}{N} \to 0$$

in the thermodynamic limit. More generally if $P(J) \simeq A\nu |J|^{\nu-1}$ for $J \to 0$, one finds, in the scaling limit, $[S_L^{(1)} S_L^{(2)}]_{\mathrm{av}} \to \exp(-2ALJ_0^\nu)$, giving $L^* \propto J_0^{-\nu}$ in agreement with the Imry-Ma argument, since $d_S = 0$ and $y = -1/\nu$ for this case.[7]

For $d = 2$, computation of ground states is not straightforward for systems of reasonable size. Instead we study the "scale-dependent coupling," defined as the energy of an interface induced in a sample of $L \times (L+1)$ spins by a change of boundary conditions as described elsewhere.[7,8] Sensitivity of this coupling to perturbations in the bonds implies sensitivity of the ground state at scale L. To each sample $\{J_{i,j}^{(1)}\}$ is associated a second sample $\{J_{i,j}^{(2)}\}$, with $J_{ij}^{(2)} = J_{i,j}^{(1)} + J_0 K_{i,j}$. Both $J_{i,j}^{(1)}$ and $K_{i,j}$ are normally distributed with unit variance. If $J'(0)$ and $J'(J_0)$ are the corresponding "block couplings," defined as the boundary-condition–dependent part of the ground-state energy,[7,8] we expect $J'(0)$ and $J'(J_0)$ to become decorrelated for large L. This was measured through the correlation functions

$$C_1 = [J'(0) J'(J_0)]_{\mathrm{av}} / [J'(0)^2]_{\mathrm{av}}^{1/2} [J'(J_0)^2]_{\mathrm{av}}^{1/2}, \tag{1}$$

$$C_2 = [\operatorname{sgn}\{J'(0)\} \operatorname{sgn}\{J'(J_0)\}]_{\mathrm{av}}. \tag{2}$$

This investigation also allows a computation of the mean interface length $[L_{\mathrm{int}}]_{\mathrm{av}}$. If J_0 is chosen sufficiently small that the ground states are unaffected by the perturbation [$J_0 = 10^{-7}$ was found to be adequate for the range of sizes ($L \le 12$) explored; for large L, of course, $J_0 \lesssim \mathrm{const} L^{-1/\zeta}$ would be required] then for a particular sample $J'(0) - J'(J_0) = (L_{\mathrm{int}})^{1/2} J_0 z$, where z is a normally distributed random variable with $[z^2]_{\mathrm{av}} = 1$. Thus

$$[L_{\mathrm{int}}]_{\mathrm{av}} = \lim_{J_0 \to 0} \{[J'(0) - J'(J_0)]^2\}_{\mathrm{av}} / J_0^2.$$

The data presented in Fig. 1, where the error on each point is no larger than its size, leads to the estimate $d_S = 1.26 \pm 0.03$ for the "fractal dimension of the interface."[6] Combined with[7,8] $y = -0.29 \pm 0.01$ our estimate for d_S gives $\zeta = 0.92 \pm 0.02$ for $d = 2$. The quoted errors are the statistical errors associated with finite sampling. Possible systematic errors, associated with the failure to reach asymptotically large L, are much more difficult to estimate. For $d = 1$ we have $\zeta = 1$, suggesting that the exponent ζ may depend rather weakly on dimension.

To test for sensitivity to the shape of the sample some data were collected for samples of $L \times (2L+1)$ spins, the long direction being that in which the changing boundary conditions were applied.[7,8] These data lie on a parallel straight line within the statistical error.

The above considerations suggest that the functions

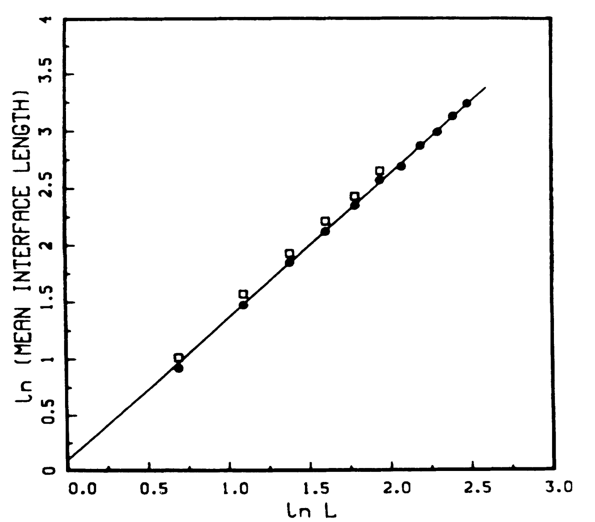

FIG. 1. Scale dependence of the mean interface length for two-dimensional Ising spin-glasses at $T = 0$. Filled circles, $L \times (L+1)$ systems; open squares, $L \times (2L+1)$ systems. The straight line has slope 1.26.

C_1, C_2 for different J_0 and L can be collapsed onto a single curve by a suitable choice of abscissa. For sufficiently small J_0, $J'(J_0)=J'(0)+(L_{\rm int})^{1/2}J_0 z$ as discussed above and $[J'(0)J'(J_0)]_{\rm av}=[J'(0)^2]_{\rm av}$ since the cross term averages to zero. Normalizing as in (1) yields $C_1 \simeq (1+x^2)^{-1/2}$, $x \lesssim 1$, where $x=J_0[L_{\rm int}]_{\rm av}^{1/2}/[J'(0)^2]_{\rm av}^{1/2} \sim J_0 L^{1/\zeta}$ is the appropriate scaling variable, as suggested by the Imry-Ma argument. A similar small-x approximation for C_2 requires knowledge of the probability distribution for $J'(0)$ which, unfortunately, is known only numerically.[7,8] To leading order in x, however, one obtains

$$C_2 \simeq 1-(2/\pi)^{1/2}P_L(0)[J'(0)^2]_{\rm av}^{1/2}x+\dots,$$

where $P_L(J')$ is the probability distribution normalized on the interval $(0,\infty)$. From Ref. 8 we estimate $[J'(0)^2]_{\rm av}^{1/2} \simeq 1.03L^y$ while Fig. 2 of Ref. 7 yields $P_L(0) \simeq 0.93L^{-y}$, giving $C_2 \simeq 1-0.764x$, $x \ll 1$.

The data for C_1,C_2 are presented in scaled form in Fig. 2, where the solid curves are the small-x results derived above. The scatter in the data for larger x is no greater than the statistical error on the points, and so the data are consistent with the postulated scaling form within the statistical error. The vanishing of both C_1 and C_2 at large length scales shows that the *sign* of J', as well as its *magnitude*, is sensitive to weak perturbations to the bonds. The implication is that the block couplings at scale L decorrelate significantly for $J_0 L^{\zeta} \gtrsim 1$, and that the ground state of a macroscopic system is therefore unstable to a weak perturbation on length scales $L \gtrsim J_0^{-1/\zeta}$,

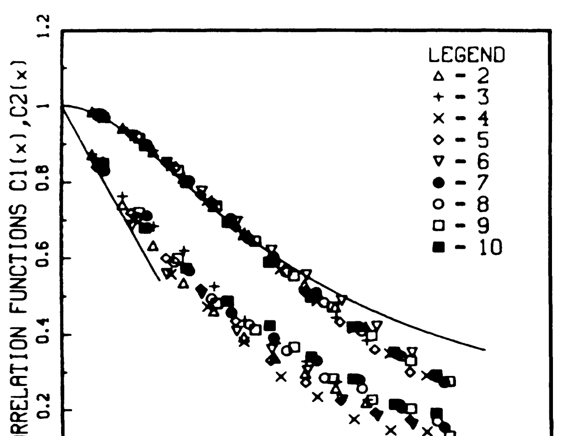

FIG. 2. Correlation functions measuring the effect of weak bond perturbations on the effective coupling at length scale L: upper data, C_1; lower data, C_2. The abscissa is the scaling variable $x=J_0[L_{\rm int}]_{\rm av}^{1/2}[J'(0)^2]_{\rm av}^{1/2}$, the averages being taken over the 10^3 samples used to obtain each point. The legend gives the value of L corresponding to each symbol. The solid lines give the small-x behavior derived in the text.

in agreement with the Imry-Ma argument.

For $d=3$, where $y>0$, the response of a system with fixed interactions to a change in T, for any $T<T_C$, may be analyzed in a similar manner. The block coupling for general T is the interface free energy, $J(T)=F_{\rm int}(T)$, giving $\delta J(T)=-S_{\rm int}(T)\delta T$, where $S_{\rm int}$ is the interface entropy, as the response to a temperature change δT. While $F_{\rm int}=E_{\rm int}-TS_{\rm int}$ is of order L^y, the separate energetic and entropic contributions are expected to be much larger, of order $L^{d_S/2}$, since they are the sum of L^{d_S} essentially independent contributions of random sign.[6] Thus $\delta J(T) \propto L^{d_S/2}\delta T$, and $\delta J(T) \sim J(T)$ at length scale $L^* \sim (\delta T)^{-1/\zeta}$ as claimed. At low temperatures it is expected[14] that (for continuous bond distributions) $S_{\rm int} \sim TL^{d_S/2}$ and therefore $L^* \sim (T\delta T)^{-1/\zeta}$. In particular, as the temperature is increased from zero, $L^* \sim T^{-2/\zeta}$ is the length scale at which the entropy first plays an important role, i.e., the length scale at which the ordering pattern $\{\langle S_i \rangle_T\}$ (where $\langle\,\rangle_T$ indicates a thermal average) loses coherence with the ground state. This latter length scale has also been noted by Huse and Fisher.[14] For discrete bond distributions, (e.g., the "$\pm J$ model") one expects[14] $S_{\rm int} \sim L^{d_S/2}$ for $T \to 0$, and $L^* \sim T^{-1/\zeta}$.

The spin-glass phase is "chaotic" in the following sense. The calculation of the scale-dependent coupling can be regarded as a *mapping from of order L^d variables* (the bonds in the sample) *onto one* (the block coupling, i.e., the interface energy). We have shown that a small random perturbation of strength J_0 to the "initial conditions" (i.e., the bonds) leads to a change in the output J' of relative size $\delta J'/J' \sim J_0 L^{\zeta}$, i.e., the small change in the initial conditions grows under "iteration" (increasing L) with "Lyapunov exponent" $\lambda=\zeta \equiv d_S/2-y$. This sensitivity to initial conditions is the defining property of chaotic behavior. Chaos in spin systems has heretofore been observed only in frustrated hierarchical models.[15] It is not clear to us whether the chaos mechanism discussed here is fundamentally diffeerent in its origin. We note that our mechanism for chaos operates even for $d=1$ where it is due to disorder rather than frustration: $\zeta>0$ is the necessary condition for the chaotic behavior discussed here.

Generalizing to finite temperatures for a fixed sample we argue that spin correlations in the ordered phase should be a chaotic function of (i) spin separation for fixed temperature, and (ii) temperature, for a given pair of spins, provided they are sufficiently far apart, $L \gtrsim (T\delta T)^{-1/\zeta}$. Hence there is no "hidden order parameter" for spin-glasses: No single "frozen pattern" describes the spin-glass order for all $T<T_C$.

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