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Precision Tests of Quantum Mechanics

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It is proposed to set stringent limits on possible nonlinear corrections to ordinary quantum mechanics by searching for the detuning of resonant transitions. A suggested nonlinear generalization of quantum mechanics is used to show that such detuning would be expected in the rf transition in $^9Be^+$ ions that is used to set frequency standards. Measurements at the National Bureau of Standards already set limits of order 10^{-21} on the fraction of the energy of the ^9Be nucleus that could be due to nonlinear corrections to quantum mechanics, with good prospects of improving this by 2–3 orders of magnitude.

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It seems long overdue to find precision tests of quantum mechanics itself, that would be more stringent than existing tests of specific quantum-mechanical theories. One sensitive way to test the linearity of quantum mechanics is to look for the detuning phenomenon characteristic of any system of nonlinear oscillators. The resonant frequency at which a weak external field can strongly perturb a nonlinear system will in general depend on the amplitudes of the various modes excited, so no matter how we set this frequency, the resonance will be detuned as the amplitudes change. This will prevent us from being able to drive the system slowly from one mode to another, say in a time T, unless the change in the resonant frequency during this transition is within a natural width of order $1/T$. This is a particularly promising way to look for departures from the linearity of quantum mechanics, because in the effort to set frequency standards, experimentalists[1] have already been able to drive hyperfine transitions in ions such as $^9Be^+$ with T as long as several seconds, and with good prospects of increasing this to several minutes. Thus such experiments (or related NMR or ESR experiments) can potentially set very stringent limits on any frequency shifts that might arise from nonlinearities in the equations of motion of the wave function.

In order to see what sort of nonlinearities might actually be expected to show up in such an experiment, and to make all this quantitative, it is essential to formulate some sort of nonlinear generalization of quantum mechanics. Recently there has been proposed[2,3] a generalized version of quantum mechanics that seems physically satisfactory, at least nonrelativistically. For the purposes of the present paper, it is sufficient to consider a discrete system, like the hyperfine states mentioned above, for which the wave function can be taken to depend on only a discrete variable k. The time dependence of the wave function is assumed to be given by an equation of the form

$$i\frac{d\psi_k}{dt}=\frac{\partial h(\psi,\psi^*)}{\partial\psi_k^*}\,, \tag{1}$$

where h is a real function of ψ and ψ^*, satisfying the homogeneity requirement, that for any complex number λ,

$$h(\lambda\psi,\psi^*)=h(\psi,\lambda\psi^*)=\lambda h(\psi,\psi^*)\,. \tag{2}$$

[This requirement ensures that if $\psi_k(t)$ is a solution of (1), then so is $\lambda\psi_k(t)$. It also plays an essential role in many aspects of the theory, including the proper treatment of separated systems.[4]] These equations are of the Hamiltonian form, with $q_k\equiv\sqrt{2}\,\mathrm{Re}\psi_k$ and $p_k\equiv\sqrt{2}\,\mathrm{Im}\psi_k$. Equation (1) reduces to the usual linear time-dependence equation of quantum mechanics if we take $h(\psi,\psi^*)$ as bilinear, $h=\psi_k^*H_{kl}\psi_l$, but it is not necessary for h to be bilinear to be homogeneous. Small nonbilinear terms in h will produce small nonlinearities in the

equation of motion (1).

Let us first consider a system like the $^9Be^+$ ion, but in the absence of time-varying external fields. For the moment, we will also restrict ourselves to two-component systems, with $k=1,2$. The Hamiltonian function h will be taken to have the form

$$h=n\bar{h}(a), \tag{3}$$

where n is the norm,

$$n=|\psi_1|^2+|\psi_2|^2, \tag{4}$$

and $\bar{h}$ is an arbitrary real function of the convenient action variable,

$$a=|\psi_2|^2/n. \tag{5}$$

[The form (3) will be automatic for the specific system to be studied here. More generally, it can be shown[3] that by a "canonical" transformation $\psi_k \rightarrow \psi_k'$, any two-component system satisfying (1) and (2) can be given a Hamiltonian function of the form (3), while still preserving the equation of motion (1).] Nonlinear terms in $\bar{h}(a)$ yield nonbilinear (though homogeneous) terms in h.

In general, energy eigenstates are solutions of (1) for which the whole wave function oscillates with a common factor $\exp(-iEt)$. In our case, we easily find them to be (*A*)

$$\psi_2=0, \quad a=0, \quad E=E_A=\bar{h}(0), \tag{6}$$

and (*B*)

$$\psi_1=0, \quad a=1, \quad E=E_B=\bar{h}(1). \tag{7}$$

Now let us turn on a small time-varying perturbation Δh, of the sort that might drive a transition between states *A* and *B*. Since Δh is assumed very small, we ignore the possibility that it might include even smaller nonbilinear terms, and take it as the bilinear

$$\Delta h=g\psi_2^*\psi_1 e^{-i\Omega t}+\text{c.c.}, \tag{8}$$

where g is a small coupling parameter, and Ω is a frequency at our disposal. The time dependence of $\psi_k(t)$ is then given by using Eq. (1), with $h+\Delta h$ in place of h. These two complex equations can conveniently be boiled down to a single real equation for the action (5):

$$\frac{da}{dt}=\{4|g|^2a(1-a)-[\Omega a-\bar{h}(a)+C]^2\}^{1/2}, \tag{9}$$

where C is an integration constant.

Now let us see if the perturbation can drive the transition $A \rightarrow B$ between the energy eigenstates (6) and (7). In order that the square root in (9) not have a negative argument at the starting point $a=0$, it is necessary to take $C=\bar{h}(0)$. Then, in order that the square root not have a negative argument at the ending point $a=1$, it is necessary also that

$$\Omega=\bar{h}(1)-\bar{h}(0)=E_B-E_A. \tag{10}$$

It is comforting to see that this result, related to energy conservation, holds here for arbitrary $\bar{h}(a)$ just as in ordinary quantum mechanics. With Ω tuned to this value, the transition will go all the way from $a=0$ to $a=1$ if and only if the argument of the square root is positive-definite for all intermediate a, i.e., if and only if

$$\{[\bar{h}(1)-\bar{h}(0)]a+\bar{h}(0)-\bar{h}(a)\}^2/a(1-a)<4|g|^2 \tag{11}$$

for all a with $0<a<1$. In ordinary quantum mechanics $\bar{h}(a)$ is linear, so the left-hand side vanishes, and the transition does occur. More generally, seeing the transition occur provides an upper bound on the nonlinearities in $\bar{h}(a)$. Note that it is *not* necessary to verify with high precision that the transition goes *all* the way from $a=0$ to $a=1$; if the inequality (11) is violated, the transition will typically get no further than half way. As mentioned earlier, in place of $|g|^2$ this bound can be written in terms of the time $T=\pi/2|g|$ that the transition would take in ordinary quantum mechanics. [Even when the inequality (11) is satisfied, nonlinearities in $\bar{h}(a)$ can show up as an asymmetry of the resonance line.]

As a variation on the detuning approach, one can try to make use of a technique due to Ramsey[5] that is used in setting frequency standards. As presently used, one first observes that for a given external rf field of frequency Ω, the transition $A \rightarrow B$ is driven in a certain time T. One then repeats the experiment, but now driving the transition only for a time $T/2$, then blocking the external rf field for a time $T' \gg T$, during which the system oscillates freely, and finally driving the transition again for a time $T/2$. In effect, this shifts the constant C in Eq. (9) by an amount $|g|\sin\{[\Omega-\bar{h}'(\frac{1}{2})]T'\}$, preventing the transition $A \rightarrow B$ unless $|\Omega-\bar{h}'(\frac{1}{2})|T' \ll 1$. This in itself does not set any new limits on nonlinearities in $\bar{h}(a)$; all that is learned is that not only is Ω equal to $\bar{h}'(a)$ within an accuracy of order $1/T$ over the range $0<a<1$, but also $\Omega=\bar{h}'(\frac{1}{2})$ to within a greater accuracy, of order $1/T'$. However, we can try blocking the external rf field at several different times, thus verifying that $\Omega=\bar{h}'(a)$ at various *different* values of a.

What form do we expect for $\bar{h}(a)$ in the experiments of Ref. 1? The ^9Be nucleus has spin $j=\frac{3}{2}$, and is in a magnetic field, mostly due to the valence electron of the $^9Be^+$ ion. Let us first consider the nuclear Hamiltonian function h_0 in the absence of the magnetic field. The only bilinear term in h_0 allowed by rotational invariance is just proportional to the norm n, and merely contributes a constant to $\bar{h}(a)$. The simplest nonbilinear term satisfying the homogeneity condition (2) would be proportional to a product of two ψ's and two ψ^*'s, divided by a single power of the norm. There are two rotationally in-

variant terms of this type (because the product of two ψ's can only have total spin $j=3$ or 1, and likewise for two ψ^*'s) but one linear combination just gives a term proportional to n again, so we have essentially only one possible rotationally invariant term of this type. This term is of the form

$$h_0=\frac{\epsilon}{n}\sum_\sigma\{2|\sqrt{3}\psi(\tfrac{3}{2},\sigma)\psi(-\tfrac{1}{2},\sigma)-\psi(\tfrac{1}{2},\sigma)\psi(\tfrac{1}{2},\sigma)|^2+3|\psi(\tfrac{3}{2},\sigma)\psi(-\tfrac{3}{2},\sigma)-\psi(\tfrac{1}{2},\sigma)\psi(-\tfrac{1}{2},\sigma)|^2$$
$$+2|\sqrt{3}\psi(\tfrac{1}{2},\sigma)\psi(-\tfrac{3}{2},\sigma)-\psi(-\tfrac{1}{2},\sigma)\psi(-\tfrac{1}{2},\sigma)|^2\}. \quad (12)$$

Here ϵ is a small coefficient with the dimensions of energy; n is the norm,

$$n=\sum_{m,\sigma}|\psi(m,\sigma)|^2, \quad (13)$$

and $\psi(m,\sigma)$ is the component of the wave function with nuclear spin z component equal to m and with quantum numbers for everything else in the problem labeled σ. For the $^9Be^+$ ion, σ can be taken as the z component of the valence electron spin, so that $\sigma=\pm\frac{1}{2}$. If $\psi(m,\sigma)$ can be factored as $\psi_N(m)\psi_e(\sigma)$ then, because of its homogeneity, the energy eigenvalues and time dependence obtained from (12) do not depend on ψ_e. (This is an example of what was referred to earlier as a proper treatment of separated systems. As we shall see, in our case this factorization is a fair approximation.)

In the absence of any other terms, Eq. (12) would have the energy eigenvalues $\frac{9}{4}\epsilon$, 2ϵ, $\frac{3}{4}\epsilon$, and zero, each corresponding to a rotationally invariant submanifold of eigenvectors. However, this pattern is completely changed by the external magnetic fields acting on the ^9Be nucleus.

Now let us include the hyperfine interaction of the ^9Be nucleus with the valence electron's spin, and the interaction of both nucleus and electron with an external magnetic field **B**. These are all small relative to typical nuclear energies, so here we ignore possible nonbilinear terms, and take h as just the expectation value of the usual Hamiltonian of quantum mechanics:

$$h_{QM}=\sum_{m,\sigma}\sum_{m',\sigma'}\psi^*(m',\sigma')\psi(m,\sigma)[\mu_e(\mathbf{J}_e)_{\sigma'\sigma}\cdot\mathbf{B}\delta_{m'm}+\mu_N(\mathbf{J}_N)_{m'm}\cdot\mathbf{B}\delta_{\sigma'\sigma}+\kappa(\mathbf{J}_e)_{\sigma'\sigma}\cdot(\mathbf{J}_N)_{m'm}], \quad (14)$$

with κ representing the strength of the hyperfine interaction. By itself, this would give eight states, with energies as a function of $|\mathbf{B}|$ represented by a typical Breit-Rabi diagram. For the relatively strong magnetic field used in Ref. 1, the energy eigenstates are nearly pure in m and σ, with admixtures limited to a few percent. The separation of these states is so much greater (presumably) than the small shifts due to nonlinearities that we can first solve for the energy eigenstates of (14) (by finding its stationary points on the surface of unit norm) and then evaluate the nonlinear term (12) for any mixture of these.

In the measurements of Ref. 1, one starts with the $^9Be^+$ ion in an energy eigenstate with $m=-\frac{3}{2}$, $\sigma=\frac{1}{2}$, and drives a single-photon transition to the eigenstate with $m=-\frac{1}{2}$, $\sigma=\frac{1}{2}$ with an rf field tuned to this transition. No other components of the wave function are appreciably excited, so we can use our previous results for the two-component system. In our previous notation, taking $\psi_1\equiv\psi(-\frac{3}{2},\frac{1}{2})$ and $\psi_2\equiv\psi(-\frac{1}{2},\frac{1}{2})$, and dropping other components, Eq. (12) yields a nonlinear term in $\bar{h}(a)$,

$$\bar{h}(a)=2\epsilon a^2. \quad (15)$$

[There are also much larger linear terms in $\bar{h}(a)$, arising from h_{QM}.] From Eq. (11), we see that if the transition $A\rightarrow B$ is observed to be driven in a time T, then $|\epsilon|<2|g|\simeq\pi/T$. The measurements already performed,[1] with T of order 1 s, set a bound on $|\epsilon|$ of order 10^{-15} eV, less than the binding energy per nucleon of the ^9Be nucleus[6] by a factor of order 10^{-21}. This may be improved by 1 or 2 orders of magnitude by the reduction of the rf power to lengthen the transition time T, or by use of the Ramsey trick, with several free-precession times of several minutes at various stages in the transition. Line-splitting methods might allow one to do even better.

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[1]J. J. Bollinger, J. D. Prestage, S. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **54**, 1000 (1985). The familiar cesium atomic clock operates with a smaller value for T, and would give a less stringent absolute limit on nonlinearities.

[2]S. Weinberg, in Proceedings of the International Symposium on Spacetime Symmetries, Maryland, 24 May 1988 (to be published) [University of Texas Report No. UTTG-15-88, 1988]. There are problems with the construction in this reference of $h(\psi,\psi^*)$ for position-dependent wave functions, to be discussed in Ref. 3. However, these problems will not bother us in the present paper, where we only discuss discrete systems.

[3]S. Weinberg, to be published.

[4]See Ref. 3. The problem of dealing with separated systems has led other authors to limit possible nonlinear terms in the Schrödinger equation to a logarithmic form; see I. Bialynicki-Birula and J. Mycielski, Ann. Phys. (N.Y.) **100**, 62 (1976).

The homogeneity assumption (2) makes this unnecessary.

[5]N. F. Ramsey, *Molecular Beams* (Oxford Univ. Press, London, 1956), Chap. V, Sec. 4.

[6]The question naturally arises, whether to compare the limit on $|\epsilon|$ with the binding energy of the ^{9}Be nucleus, or the larger rest mass of the nucleon, or the much smaller hyperfine energy in the $^9Be^+$ ion. The departures from quantum mechanics discussed here would arise from the internal energy of the *free* ^{9}Be nucleus, as given in Eq. (12), not from its interaction with external magnetic fields, so it would be pointless to compare $|\epsilon|$ with the hyperfine energy. Also, rotational invariance does not allow any nonbilinear homogeneous terms in the Hamiltonian function for a free particle of spin $\frac{1}{2}$, so it seems inappropriate to compare $|\epsilon|$ with the nucleon rest masses. On the other hand, because the ^{9}Be nucleus has spin $\frac{3}{2}$, rotational invariance does not prevent departures from quantum mechanics in the internal dynamics of the nucleus from producing nonbilinear terms in the Hamiltonian function of a free nucleus, such as that shown in Eq. (12). Thus it seems reasonable to compare $|\epsilon|$ with some energy characteristic of the internal dynamics of the nucleus, such as the binding energy per nucleon.