UNIVERSITÀ DEGLI STUDI DI PADOVA

FACOLTÀ DI SCIENZE MM. FF. NN.
DIPARTIMENTO DI FISICA "GALILEO GALILEI"

LAUREA SPECIALISTICA IN FISICA

$N = 1$ effective supergravities for flux compactifications

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A.A. 2006/2007

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Introduction

Haud igitur penitus pereunt quaecumque videntur,
quando alid ex alio reficit natura nec ullam
rem gigni patitur nisi morte adiuta aliena[1].
TITUS LUCRETIUS CARUS,
DE RERUM NATURA, I, 262-264

The Standard Model (SM) of strong and electroweak interactions is probably the most outstanding achievement in theoretical physics over the last half century, since it is a correct, and sometimes much accurate, description of quantum phenomena at energies up to the Fermi scale, $G_F^{-1/2} \sim 293$ GeV, though its Higgs sector has not yet been completely verified.

The Standard Model is a renormalizable theory, but at energy scales comparable to the Planck scale, $M_P \equiv (8\pi G_N)^{-1/2} \sim 2.4 \times 10^{18}$ GeV, the effects of gravity cannot be neglected. The main challenge in theoretical physics is now to find a quantum theory of gravitation which unifies the Standard Model and General Relativity, describing phenomena at arbitrarily high energies. In the low-energy limit this theory should give back the Standard Model physics. Hence, in a bottom-up approach, we can look for extensions of the Standard Model.

Supersymmetry is a symmetry which relates bosons and fermions. The most convincing extensions of the Standard Model at the Fermi scale are theories with "softly" broken supersymmetry, because they provide a suitable framework to solve the hierarchy problem, the unification of gauge coupling constants and the problem of dark matter. However, globally supersymmetric theories need a large number of mass and coupling parameters, which describe how supersymmetry is "softly" broken to give the observed particle spectrum and leave many theoretical and phenomenological problems open. This leads to theories of local supersymmetry ($N = 1$, so to allow for chiral fermions), that include a supermultiplet with a spin-3/2 and a spin-2 particle, hence containing a description of gravitational interactions. For this reason they are called supergravity theories.

The most promising possibility for unifying, at the quantum level, the Standard Model with a theory of gravitation is string theory, whose basic elements are not point-like particles, as in the Standard Model, but one-dimensional objects, the strings.

[1]Therefore, what we see does not entirely perish \ because nature makes something from something and \ nothing is allowed to be generated, unless aided by an external death. \\

There are five different perturbative formulations of string theory that include both bosonic and fermionic degrees of freedom in space-time and are consistently defined in ten dimensions. There are also hints for an underlying theory, named M-theory, which, by taking appropriate limits, should give the five ten-dimensional formulations of string theory, called type I, type IIA, type IIB, heterotic with $E_8 \times E_8$ gauge group and heterotic with $SO(32)$ gauge group.

In the low-energy limit, string theory provide us with supergravity theories in ten dimensions (eleven for M-theory). However, since the four-dimensional description provided by the Standard Model agrees with experiments, it is important to understand how to obtain an effective field theory in four dimensions, possibly resembling the Standard Model or its minimal supersymmetric extension (MSSM). The obvious solution is to perform a compactification of the extra dimensions, factorizing the full space-time into the product of the four-dimensional Minkowski space-time (or better a de Sitter space, since recent measurements suggest a tiny positive vacuum energy density) and of a compact space, characterized by a mass scale at least of the order of 1 TeV, and probably much larger. In fact, if the compact space is small enough, the compactification scale must be higher than that explored in the high energy experiments, thus explaining why we have not found yet evidences for extra dimensions.

When a theory is compactified, many massless scalar fields, the moduli, arise in the effective lower-dimensional theory, which describe, among others, the size and the shape of the extra dimensions. This generates many problems. First of all, massless scalar fields can generate long-range forces in conflict with observations. Moreover, since the vacuum expectation values of the moduli control the masses and coupling coupling constants of the effective four-dimensional theory, the latter remain undetermined if the moduli are unconstrained by dynamics. It is then desirable to stabilize all the moduli through a potential which gives them a mass. The generation of a potential for the moduli is closely related to the problem of breaking supersymmetry and to that of the vacuum energy: in the path towards a realistic model, the goal would be to find a vacuum with spontaneously broken supersymmetry, all moduli stabilized and vanishing or tiny and positive vacuum energy.

There are two simple ways to introduce a potential for the moduli. The first is to turn on fluxes, i.e. non-trivial background values for the internal components (those in the extra dimensions) of the p-form field strengths in the higher-dimensional supergravity theory. The second way is to use the Scherk-Schwarz mechanism, which allows for a dependence on the internal coordinates of the fields in the original theory, compatible with the symmetries of the action. In some cases, this dependence can be interpreted as expectation values for the internal components of the spin connection. In this cases the fluxes are called geometric, since they are related to the geometry of the extra dimensions. These two mechanisms can work at the same time, but there are some consistency conditions that must be fulfilled, which do not permit arbitrary combination of fluxes.

Non-trivial fluxes are analogous to a magnetic field in classical electromagnetism. The sources for these fluxes are Dp-branes, extended objects in p spatial dimensions on which open strings can end that have a Ramond-Ramond (RR) charge, or other objects in the

Neveu-Schwarz-Neveu-Schwarz (NSNS) sector.

The study of supergravity compactifications obtained from string theory in the presence of fluxes and branes is thus important to understand if (and how) the low-energy limit of string theory can reproduce the Standard Model or one of its extensions, in particular the MSSM. In this sense, compactification characterizes physics at energies between the electroweak and the string scale.

The simplest compactifications are performed on tori, but the supergravity theories in ten and eleven dimension, when compactified on a torus, give an effective theory with four or eight supersymmetries, while the most physically interesting theories are those preserving just one supersymmetry, especially if spontaneously broken.

A possibility to obtain $N = 1$ supersymmetry in the effective four-dimensional theory is to compactify on an orbifold, i.e. the quotient space of a manifold with respect to a discrete symmetry. A simple and interesting example is that of the orbifold $\mathbb{T}^k/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $k = 6, 7$, which reduces the number of supersymmetries, with respect to the toroidal compactification, by a factor four. When necessary, to obtain an effective $N = 1$ theory, another $\mathbb{Z}_2$ projection can be introduced, which in type II theories corresponds to a particular operation called orientifold.

Compactifications of heterotic, type IIA and M-theory supergravities on $\mathbb{T}^k/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $k = 6, 7$ have been extensively discussed in literature, therefore we study here compactifications of type IIB supergravity on the orbifold $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, with two kinds of further orientifold projections to obtain a $N = 1$, $d = 4$ effective supergravity. From the point of view of the effective supergravity, orientifolds can be classified by the intrinsic parities of the fields and by the (hyper-) planes left invariant by the projection, which have p spatial dimensions and are called Op-planes. To obtain a $N = 1$, $d = 4$ effective theory from type IIB supergravity compactified on the $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold, the two orientifolds that we use define O3/O7 and O5/O9 orientifold planes, respectively.

The original part of this thesis begins in Chapter 5 with the study of six supergravity compactifications on the orbifold $\mathbb{T}^k/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $k = 6, 7$, with a further $\mathbb{Z}_2$ projection when necessary to obtain a $N = 1$, $d = 4$ effective supergravity. In each case, we concentrate on the scalars of the theory. We show that it is possible, through appropriate non-linear redefinitions of the fields of the reduced theory, to put the reduced action in the standard form for a $N = 1$, $d = 4$ supergravity, containing seven complex scalar fields. This allows us to compute the Kähler potential, which can be identified from the kinetic terms of the scalars. In these dimensional reductions there is no scalar potential. This corresponds to a vanishing superpotential.

The second original part is described in Chapter 6. There we study vacua in type IIB supergravity compactifications with O3/O7 and O5/O9 orientifolds, in the absence of sources for simplicity.

In type IIB supergravity with O3/O7, no geometric flux is compatible with the orbifold and orientifold projections. Thus, in this case, we turn on a subset of the allowed p-form fluxes and compute the scalar potential which is generated. Given the form of the Kähler potential, we compute the corresponding superpotential. Then we study the vacua of the theory. First, we show that there are only Minkowski vacua. Moreover, in the absence of sources, vacua can be obtained only for trivial values of the fluxes, in agreement with a no-go theorem, since the scalar potential is runaway.

The case of type IIB supergravity with O5/O9 orientifolds is more interesting. In this case there are allowed geometric fluxes, on which we concentrate. We compute the scalar potential which is obtained by generalized dimensional reduction when four geometric fluxes are turned on and we also compute the corresponding superpotential. In this case, we find again only Minkowski vacua, but not only with trivial fluxes. We find two non-trivial vacua, where two or four geometric fluxes are turned on, and in the simplest case we compute the spectrum of the scalars. However, the fluxes considered here (without sources) are not sufficient to stabilize all the moduli of the theory.

This is a contribute to the study of type IIB flux compactifications, in which we identified new vacua in the case with O5/O9 orientifolds. In perspective, the final goal would be to find a vacuum, even with sources for the fluxes and other more general "non-geometric" fluxes, which allows for a complete moduli stabilization and, when a suitable system of branes is introduced, contains the Standard Model or the MSSM.

In Chapter 1 we introduce $N = 1$ global supersymmetry in four dimensions. After a brief survey of the supersymmetry algebra, we give the Lagrangian for a generic supersymmetric gauge theory. Then we discuss how to break supersymmetry, which is necessary to obtain a mass spectrum compatible with observations. We will see that the breaking must be "soft". This will enable us to write the Lagrangian of the MSSM, which includes the superpartners of the SM particles and many soft breaking parameters.

In Chapter 2 we discuss theories of local $N = 1$, $d = 4$ supersymmetry. We begin by showing that a theory of local supersymmetry contains a theory of gravitation, then we introduce the component form for the $N = 1$, $d = 4$ supergravity Lagrangian. This will allow us to discuss supersymmetry breaking in supergravity models and in particular the super-Higgs effect. We conclude by studying two important supergravity models.

A brief introduction to compactification and dimensional reduction is given in Chapter 3, through some simple examples. The first is the dimensional reduction of a free and massless scalar field from $D = 5$ to $d = 4$ dimensions, both in the ordinary and in the generalized case. Next we discuss compactification of pure gravity, again from $D = 5$ to $d = 4$ dimensions, in the ordinary case. In the last examples, we discuss dimensional reduction of a free Dirac field from $D = 5$ to $d = 4$ dimensions, both in the ordinary and the generalized case, on the circle $\mathbb{S}^1$ and on the orbifold $\mathbb{S}^1/\mathbb{Z}_2$.

Chapter 4 contains a summary of the bosonic part of supergravity theories in ten and eleven dimensions. These are the starting point for performing orbifold compactifications down to four dimensions, which are studied in Chapter 5, where we discuss six specific examples.

Finally, in Chapter 6 we discuss the effect of p-form and geometric fluxes in four-dimensional orbifold compactifications of type IIB supergravity. In each of the two examples we consider, we derive the scalar potential by dimensional reduction, we identify the corresponding superpotential and its dependence on the fluxes, and we finally discuss the resulting vacuum structure, in relation with supersymmetry breaking and moduli stabilization.

Chapter 1

$N = 1$, $D = 4$ supersymmetry

In this chapter we introduce global (or rigid) supersymmetry in $D = 4$ space-time dimensions. We start by writing the supersymmetry algebra (concentrating on the $N = 1$ case, where N is the number of supersymmetry generators) and by describing some of its consequences. Then we build the most general Lagrangian for a supersymmetric renormalizable gauge theory (containing particles of spin 0, 1/2 and 1) and study the breaking of supersymmetry. Finally we give the Lagrangian of the minimal supersymmetric extension of the Standard Model. Since there are many reviews on the subject (see *e.g.* [1, 2, 3, 4, 5, 6, 7, 8]), we will only state the results which will be useful in the following.

1.1 The supersymmetry algebra

Symmetry principles play an essential role in our understanding of Nature. The best description of microscopic phenomena is the Standard Model (SM) of strong and electroweak interactions, whose symmetry group is a direct product of the Poincaré group and the compact gauge group $G = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. Since the composition of gauge and space-time symmetries in the Standard Model is a direct product, it is interesting to study symmetries which unify space-time and internal symmetries, therefore having generators which, in general, carry non-trivial spin.

According to the celebrated Coleman-Mandula theorem [9], the only possible Lie algebra symmetry generators compatible with the symmetries of the S-matrix consist of the generators of the Poincaré group[1], plus other "internal" symmetry generators which commute with the former, i.e. Lorentz scalar generators. Then, if we want to introduce new symmetry generators that carry half-integer spin, hence relating particles of different spin and statistics, i.e. connecting bosons and fermions, we must go beyond the concept of Lie algebra. This is the case of supersymmetry[2], whose generators belong to a graded

[1] Our notation is specified in Appendix A.

[2] Supersymmetry was discovered in 1971 (early works are [10] and [11]; reprints of the original (four-dimensional) literature and a historical introduction can be found in [12, 13]) and the building of a supersymmetric field theory, the Wess-Zumino model [14], led to a systematic study of the supersymmetry algebra, which was found to be the most general graded Lie algebra (see, e.g. [4]) compatible with local

Lie algebra, with commutation and anticommutation relations.

The supersymmetry algebra is an extension of the Poincaré algebra that includes spin-1/2 generators Q^i_α, $i = 1, \ldots, N$. In the most general case (in $D = 4$) it is:

$$\{Q^i_\alpha, \overline{Q}^j_{\dot\beta}\} = 2\,\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{ij}\,, \tag{1.1}$$

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij}\,, \quad \{\overline{Q}^i_{\dot\alpha}, \overline{Q}^j_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} Z^{\dagger ij}\,, \tag{1.2}$$

$$[P_\mu, Q^i_\alpha] = [P_\mu, \overline{Q}^j_{\dot\alpha}] = [P_\mu, Z^{ij}] = 0\,, \tag{1.3}$$

$$[Z^{ij}, \overline{Q}^k_{\dot\alpha}] = [Z^{ij}, Q^k_\alpha] = [Z^{ij}, Z^{mn}] = 0\,, \tag{1.4}$$

where $Z^{ij} = -Z^{ji}$ are a set of central charges of the algebra.

Since it is not possible to define chiral spinors belonging to a representation of *extended* ($N > 1$) supersymmetry algebra, we will restrict ourselves to the case $N = 1$, thus remaining without central charges. In this case the supersymmetry algebra becomes:

$$\begin{aligned} \{Q_\alpha, \overline{Q}_{\dot\beta}\} &= 2\,\sigma^\mu_{\alpha\dot\beta}\, P_\mu\,, \\ \{Q_\alpha, Q_\beta\} &= 0 = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\}\,. \end{aligned} \tag{1.5}$$

An irreducible representation of the supersymmetry algebra is called *supermultiplet*. Each supermultiplet contains both fermionic and bosonic states, which are said *superpartners* of each other.

$N = 1$, $d = 4$ supermultiplets have two fundamental properties:

1. *All particles belonging to an irreducible representation of supersymmetry have the same mass*, since P^2 is a Casimir operator of the supersymmetry algebra.

2. *Every representation of the supersymmetry algebra always contains an equal number of fermion and boson degrees of freedom*, because there is a symmetry connecting bosons and fermions.

1.2 Superfields and superspace

Superfields provide an elegant and compact description of supersymmetry representations, so we will introduce this formalism to obtain the most general supersymmetric and renormalizable Lagrangian which involves scalar and vector fields as well as their fermionic superpartners.

Let θ_α and $\overline{\theta}_{\dot\alpha}$ be two Grassmann variables. *Superspace* is the space whose coordinates have the form $X^M = \left(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha}\right)$.

relativistic quantum field theory and with the symmetries of the S-matrix (Haag-Łopuszański-Sohnius theorem, 1974; for a proof see [2, 4, 15]).

The supersymmetry algebra may be viewed as a Lie algebra with anticommuting parameters, therefore we are led to define a corresponding group element:

$$G\left(x, \epsilon, \overline{\epsilon}\right) \equiv e^{i\left(x_\mu P^\mu + \epsilon Q + \overline{\epsilon}\overline{Q}\right)}\,, \tag{1.6}$$

where ϵ_α ad $\overline{\epsilon}_{\dot\alpha}$ are spinorial (constant) parameters. The composition of group elements is[3]

$$G(x^\mu, \theta, \overline{\theta})\, G(0, \epsilon, \overline{\epsilon}) = G(x^\mu - i\epsilon\sigma^\mu\overline{\theta} + i\theta\sigma^\mu\overline{\epsilon}, \epsilon + \theta, \overline{\epsilon} + \overline{\theta})\,,$$

so the multiplication of group elements induces a motion in superspace which is generated by the differential operators Q and $\overline{Q}$

$$Q_\alpha = -i\left(\frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu\right)\,, \quad \overline{Q}_{\dot\alpha} = -i\left(-\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\right)\,, \tag{1.7}$$

which satisfy the algebra $\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2\,(\sigma^\mu)_{\alpha\dot\alpha}(-i\partial_\mu)$.

Superfields are functions of superspace coordinates $F(x, \theta, \overline{\theta})$. Since θ^α and $\overline{\theta}_{\dot\beta}$ are anticommuting parameters, the generic superfield can be expanded as:

$$\begin{aligned} F(x,\theta,\overline{\theta}) \;=\;& f(x) + \theta\,\phi(x) + \overline{\theta}\overline{\chi}(x) + \theta\theta\, m(x) + \overline{\theta\theta}\, n(x) + \\ &+ \theta\sigma^\mu\overline{\theta}\, v_\mu(x) + \theta\theta\overline{\theta}\overline{\lambda}(x) + \overline{\theta\theta}\theta\,\psi(x) + \theta\theta\overline{\theta\theta}\, d(x)\,, \end{aligned} \tag{1.8}$$

where

- f, m, n, d are complex scalar fields,
- $\phi_\alpha, \overline{\chi}^{\dot\alpha}, \overline{\lambda}^{\dot\alpha}, \psi_\alpha$ are two-component spinors,
- v_μ is a complex vector field.

A superfield has, in general, 16 real bosonic components and 16 real fermionic ones, so there is an equal number of fermions and bosons, but there are too many fields to describe supermultiplets with a small number of degrees of freedom. We can obtain irreducible representations of supersymmetry by imposing *covariant* constraints on superfields, i.e. restrictions to the general superfield which are invariant under supersymmetry transformations (they will *not* have any dynamical content, but only reduce the number of components).

The supersymmetry transformation of a superfield F is given by

$$\delta F = i\,\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right)\, F(x, \theta, \overline{\theta})\,; \tag{1.9}$$

then we can look for constraints which are preserved by supersymmetry transformations, introducing *covariant derivatives* D_α and $\overline{D}_{\dot\alpha}$ such that

$$D_\alpha(\delta F) = \delta(D_\alpha F)\,, \qquad \overline{D}_{\dot\alpha}(\delta F) = \delta(\overline{D}_{\dot\alpha} F)\,,$$

[3] We decide to act on the right, instead of on the left, otherwise we would obtain an algebra with a wrong sign.

and we obtain

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu\,, \quad \overline{D}_{\dot\alpha} = \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\,. \tag{1.10}$$

This allows us to define a *left-handed chiral* or *scalar* or *matter superfield* as a superfield which satisfies

$$\overline{D}_{\dot\alpha}\phi = 0\,. \tag{1.11}$$

Analogously, a *right-handed chiral* or *scalar* or *matter superfield* is a superfield which obeys

$$D_\alpha\overline{\phi} = 0\,. \tag{1.12}$$

Moreover, a *vector* or *gauge superfield* V is a superfield which satisfies the reality constraint

$$V(x,\theta,\overline{\theta}) = V(x,\theta,\overline{\theta})^\dagger\,. \tag{1.13}$$

Note that constants are chiral superfields, for $\phi = a$ is the solution to the equations $\overline{D}_{\dot\alpha}\phi = 0 = D_\alpha\phi$. Moreover, since Q_α and $\overline{Q}_{\dot\alpha}$ are derivatives, $\phi = a$ is invariant under supersymmetry transformations.

1.2.1 Chiral superfields

Introduce two new variables $y^\mu \equiv x^\mu - i\theta\sigma^\mu\overline{\theta}$, $\overline{y}^\mu \equiv x^\mu + i\theta\sigma^\mu\overline{\theta}$ such that $\overline{D}_{\dot\alpha}y^\mu = 0$ and $D_\alpha\overline{y}^\mu = 0$. Any function of the variables y and θ only is then a *left-handed chiral superfield*:

$$\begin{aligned}\phi(y,\theta) &= z(y) + \sqrt{2}\,\theta\,\psi(y) - \theta\theta\,f(y) = \\ &= z(x) + \sqrt{2}\,\theta\,\psi(x) - \theta\theta\,f(x) - i\,(\theta\sigma^\mu\overline{\theta})\,\partial_\mu z(x) + \\ &\quad + \frac{1}{\sqrt{2}}i\theta\theta\,(\partial_\mu\psi(x)\sigma^\mu\overline{\theta}) - \frac{1}{4}\theta\theta\overline{\theta\theta}\Box z(x)\,,\end{aligned} \tag{1.14}$$

where z, f are complex scalar fields, ψ_α is a left-handed Weyl spinor and in the second expression we have Taylor expanded ϕ around x^μ. In particular, f is introduced as an auxiliary field which is needed to equate the off-shell bosonic and fermionic degrees of freedom in the supermultiplet, while it does not carry any physical degree of freedom.

Analogously, any function of the variables $\overline{y}$ and $\overline{\theta}$ only is a *right-handed chiral superfield* and we have $\overline{\phi}(x,\theta,\overline{\theta}) = \phi(x,\theta,\overline{\theta})^\dagger$.

Under a supersymmetry transformation $\delta\phi(y,\theta) = i\,(\epsilon Q + \overline{\epsilon}\overline{Q})\,\phi(y,\theta)$, the components of a chiral superfield transform as:

$$\left\{\begin{array}{lcl}\delta z &=& \sqrt{2}\,\epsilon\psi \\ \delta\psi_\alpha &=& -\sqrt{2}\,f\epsilon_\alpha - \sqrt{2}\,i(\sigma^\mu\overline{\epsilon})_\alpha\partial_\mu z \\ \delta f &=& -\sqrt{2}\,i(\partial_\mu\psi\sigma^\mu\epsilon)\end{array}\right. .$$

The highest components of ϕ and $\overline{\phi}$ are f and $\overline{f}$ respectively; all higher powers in $\theta, \overline{\theta}$ are space-time derivatives. Thus the f or $\overline{f}$ component of a scalar superfield always transforms into a space-time derivative.

Products of chiral superfields are again chiral superfields, since $\overline{D}_{\dot{\alpha}}(\phi^n) = n\phi^{n-1}\overline{D}_{\dot{\alpha}}\phi = 0$ and analogously for right-handed chiral superfields. The product $\phi^\dagger\phi$, however, is *not* a chiral superfield, but still the $\theta\theta\overline{\theta\theta}$ component transforms into a space-time derivative[4]; since it contains the kinetic terms for the fields, we define

$$\begin{aligned}
\mathcal{L}_K \equiv & \left[\phi^\dagger\phi\right]_{\theta\theta\overline{\theta\theta}} = \\
& = f^\dagger f - \frac{1}{4}\left(z\,\Box z^\dagger + z^\dagger\,\Box z\right) + \frac{1}{2}(\partial_\mu z^\dagger)\,(\partial^\mu z) + \\
& + \frac{i}{2}\,(\psi\sigma^\mu\partial_\mu\overline{\psi}) - \frac{i}{2}\,((\partial_\mu\psi)\sigma^\mu\overline{\psi})\,.
\end{aligned} \tag{1.15}$$

The supersymmetric Lagrangian involving only chiral superfields is then

$$\mathcal{L}_c = \left[\phi_i^\dagger\phi_i\right]_{\theta\theta\overline{\theta\theta}} + [W]_{\theta\theta} + [\overline{W}]_{\overline{\theta\theta}}\,, \tag{1.16}$$

where we defined the *superpotential*

$$W(z_i) \equiv a_i z_i + \frac{1}{2}\,m_{ij}\,z_i\,z_j + \frac{1}{3}\,\lambda_{ijk}\,z_i\,z_j\,z_k\,, \tag{1.17}$$

(sum over repeated indices is implicit) and the couplings m_{ij} and λ_{ijk} are symmetric in their indices. If we eliminate the auxiliary fields f_i by substituting their equations of motion:

$$f_i^\dagger = a_i + m_{ij}\,z_j + \lambda_{ijk}\,z_j\,z_k = \frac{\partial W}{\partial z_i}\,,$$

we obtain the expression for (1.16) in components:

$$\begin{aligned}
\mathcal{L}_c = & \frac{i}{2}\left(\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i - (\partial_\mu\psi_i)\sigma^\mu\overline{\psi}_i\right) + (\partial_\mu z_i^\dagger)\,(\partial^\mu z_i) + \\
& - \frac{1}{2}\,\frac{\partial^2 W(z)}{\partial z_i \partial z_j}\,\psi_i\,\psi_j - \frac{1}{2}\,\frac{\partial^2\overline{W}(z)}{\partial z_i^\dagger\partial z_j^\dagger}\,\overline{\psi}_i\,\overline{\psi}_j - V_c\,,
\end{aligned} \tag{1.18}$$

where we defined the scalar potential (involving only chiral superfields) $V_c = \sum_i \left|\frac{\partial W}{\partial z_i}\right|^2$.

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[4]We denote by $[\ldots]_{\theta\theta}$, $[\ldots]_{\overline{\theta\theta}}$, $[\ldots]_{\theta\theta\overline{\theta\theta}}$ the $\theta\theta$, $\overline{\theta\theta}$ and $\theta\theta\overline{\theta\theta}$ components of a superfield, respectively. Equivalently, it is possible to define an integration over Grassmann variables such that $\int d\theta = 0$, $\int d\theta\,\theta = 1$ and the brackets are replaced by $\int d^2\theta$, $\int d^2\overline{\theta}$, $\int d^4\theta \equiv \int d^2\,\theta d^2\overline{\theta}$.

1.2.2 Vector superfields

Let $V(x, \theta, \overline{\theta})$ be a vector superfield. Then, taken a chiral superfield ϕ, it is possible to define the following supersymmetric generalization of a gauge transformation:

$$V \to V + \phi + \phi^\dagger \,, \tag{1.19}$$

and by a suitable choice of ϕ it is possible to eliminate some of the unphysical fields. This is called the *Wess-Zumino (WZ) gauge*. In this gauge, the vector multiplet reduces to 4 bosonic and 4 fermionic degrees of freedom:

$$V_{WZ} = \theta\sigma^\mu\overline{\theta}v_\mu + i\,\theta\theta\overline{\theta\lambda} - i\,\overline{\theta\theta}\theta\lambda + \frac{1}{2}\,\theta\theta\overline{\theta\theta}\,D\,. \tag{1.20}$$

To build a Lagrangian for the vector multiplet, one introduces the superfields

$$W_\alpha \equiv -\frac{1}{4}\,\left(\overline{DD}\right)\,D_\alpha V\,, \quad \overline{W}_{\dot\alpha} \equiv -\frac{1}{4}\,(DD)\,\overline{D}_{\dot\alpha}V\,. \tag{1.21}$$

which are chiral and gauge invariant. Since W_α is chiral, the $\theta\theta$ component of $W^\alpha W_\alpha$ transforms into a space derivative, therefore we may write the supersymmetric gauge invariant kinetic Lagrangian for a vector field (in the WZ gauge)

$$\begin{aligned} \mathcal{L}_V &= \frac{1}{4}\,[W^\alpha W_\alpha]_{\theta\theta} + \frac{1}{4}\,\left[\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\right]_{\overline{\theta\theta}} = \\ &= \frac{i}{2}\,\left(\lambda\sigma^\mu\partial_\mu\overline{\lambda} - (\partial_\mu\lambda)\,\sigma^\mu\overline{\lambda}\right) + \frac{1}{2}\,D^2 - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}\,. \end{aligned} \tag{1.22}$$

1.3 Supersymmetric gauge theories

Consider a set of left-handed chiral superfields ϕ^i, transforming according to an arbitrary representation $\mathbf{R}$ of a (compact) gauge group G, and a set of vector fields V^a, belonging to the adjoint representation of G ($a = 1, \ldots, \dim G$).

Global invariance. We start from the case of a theory invariant under global transformations of G. Chiral superfields transform as

$$\phi'^i = \left[\exp\left(i\Lambda^a(T^a)\right)\right]^i{}_j\;\phi^j\,, \tag{1.23}$$

where the generators T^a of G in the representation $\mathbf{R}$ are hermitian, Λ^a are real constants. The Lagrangian for a chiral superfield is given by (1.16). The kinetic terms $\left[\phi_i^\dagger\phi^i\right]_{\theta\theta\overline{\theta\theta}}$ are invariant under the transformations (1.23), but in general the superpotential (1.17) is not, so we must require that each of its terms is invariant; in particular $a_i \neq 0$ only for fields ϕ^i which are singlets.

Local invariance. When the theory is gauged, the fields $\Lambda^a(x)$ are no longer superfields. But when they are promoted to chiral superfields in general one has $\Lambda^{a\dagger} \neq \Lambda^a$, so the kinetic terms $\left[\phi_i^\dagger\phi_i\right]_{\theta\theta\overline{\theta\theta}}$ are not gauge invariant: $(\phi^\dagger\phi)' = \phi^\dagger e^{-i\Lambda^\dagger}e^{i\Lambda}\phi$, where in matrix notation $\Lambda = \Lambda^a T^a$. To restore gauge invariance, we introduce a vector superfield $V = V^a T^a$ with the transformation

$$e^V \to e^{i\Lambda^\dagger} e^V e^{-i\Lambda}\,, \tag{1.24}$$

so that the new kinetic terms

$$\mathcal{L}_{KIN} = \left[\phi^\dagger e^V \phi\right]_{\theta\theta\overline{\theta\theta}} \tag{1.25}$$

are invariant under *local* transformations of G.

The chiral superfields W_α and $\overline{W}_{\dot\alpha}$ defined in (1.21) are no longer invariant under gauge transformations (1.24), when G is not abelian. The form for W_α and $\overline{W}_{\dot\alpha}$ which transforms as

$$W_\alpha \to e^{i\Lambda} W_\alpha e^{-i\Lambda}\,, \quad \overline{W}_{\dot\alpha} \to e^{i\Lambda} \overline{W}_{\dot\alpha} e^{-i\Lambda} \tag{1.26}$$

is:

$$W_\alpha = -\frac{1}{4}\,\overline{DD} e^{-V} D_\alpha e^V\,, \quad \overline{W}_{\dot\alpha} = -\frac{1}{4}\,DD e^{-V} \overline{D}_{\dot\alpha} e^V\,. \tag{1.27}$$

Fayet-Iliopoulos terms. If the gauge group G contains abelian $U(1)$ factors, the $\theta\theta\overline{\theta\theta}$ component of the corresponding vector multiplets V^a is invariant under the gauge group and transforms under supersymmetry with a total derivative. Therefore, in the presence of abelian vector fields, other terms are allowed in the Lagrangian:

$$\mathcal{L}_{FI} = \sum_a \xi^a\ [V^a]_{\theta\theta\overline{\theta\theta}}\,, \tag{1.28}$$

called *Fayet-Iliopoulos terms.*

We conclude that the gauge-invariant and globally supersymmetric Lagrangian is, in the WZ gauge and considering only terms of mass dimension $d \leq 4$,

$$\begin{aligned}\mathcal{L} \;=\; & \left[\phi_i^\dagger \left(e^{2gV}\right)^i{}_j\, \phi^j\right]_{\theta\theta\overline{\theta\theta}} + \\ & + \left[W(\phi^i) + \frac{1}{16 g^2 \tau_{\mathbf{R}}}\mathrm{Tr}\ \left(W^\alpha W_\alpha\right)\right]_{\theta\theta} + \\ & + \left[\overline{W}(\phi_i^\dagger) + \frac{1}{16 g^2 \tau_{\mathbf{R}}}\mathrm{Tr}\ \left(\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\right)\right]_{\theta\theta} + \\ & + \sum_a g\,\xi^a V^a\,, \end{aligned} \tag{1.29}$$

where the superpotential is

$$W(\phi^i) = a_i\,\phi^i + \frac{1}{2}\,m_{ij}\,\phi^i\,\phi^j + \frac{1}{3}\,\lambda_{ijk}\,\phi^i\,\phi^j\,\phi^k\,, \tag{1.30}$$

g is the gauge coupling constant and the index $\tau_{\mathbf{R}}$ of the representation $\mathbf{R}$ is defined by

$$\mathrm{Tr}\ (T^a T^b) = \tau_{\mathbf{R}}\,\delta^{ab}\,. \tag{1.31}$$

In components, eliminating the auxiliary fields through their equations of motion

$$f_i^\dagger = \frac{dW(z^k)}{dz^i}\,, \quad D^a = -g\,\left[\left(z_i^\dagger\,(T^a)^i{}_j\,z^j\right) + \xi^a\right]\,, \tag{1.32}$$

gives:

$$\begin{aligned}\mathcal{L} \;=\; & (D_\mu z)_i^\dagger\,(D^\mu z)^i + \frac{i}{2}\,\psi^i\sigma^\mu\,(D_\mu\overline{\psi})_i - \frac{i}{2}\,(D_\mu\psi)^i\,\sigma^\mu\overline{\psi}_i + \\ & -\frac{1}{4}\,F^a_{\mu\nu}F^{a\mu\nu} + \frac{i}{2}\,\lambda^a\sigma^\mu(D_\mu\overline{\lambda})^a - \frac{i}{2}\,(D_\mu\lambda)^a\sigma^\mu\overline{\lambda}^a + \\ & + \sqrt{2}\,ig\,(\overline{\psi}_i\overline{\lambda}^a)\,(T^a)^i{}_j z^j - \sqrt{2}\,ig\,z_i^\dagger\,(T^a)^i{}_j(\psi^j\lambda^a) + \\ & -\frac{1}{2}\,\frac{d^2W(z^k)}{dz^i dz^j}\,\psi^i\,\psi^j - \frac{1}{2}\,\frac{d^2\overline{W}(z_k^\dagger)}{dz_i^\dagger dz_j^\dagger}\,\overline{\psi}^i\,\overline{\psi}^j - V(z^i, z_j^\dagger)\,,\end{aligned} \tag{1.33}$$

where the scalar potential is

$$V(z^i, z_j^\dagger) = \sum_i |f^i|^2 + \frac{1}{2}\sum_a (D^a)^2\,. \tag{1.34}$$

The covariant derivatives are:

$$(D_\mu z)^i = \partial_\mu z^i + ig\,v_\mu^a\,(T^a)^i{}_j\,z^j\,, \tag{1.35}$$

$$(D_\mu\psi)^i = \partial_\mu\psi^i + ig\,v_\mu^a\,(T^a)^i{}_j\,\psi^j\,, \tag{1.36}$$

$$(D_\mu\lambda)^a = \partial_\mu\lambda^a - g\,f^{abc}\,v_\mu^b\,\lambda^c\,, \tag{1.37}$$

and the gauge field strengths are:

$$F^a_{\mu\nu} = \partial_\mu v^a_\nu - \partial_\nu v^a_\mu - g\,f^{abc}\,v^b_\mu\,v^c_\nu\,. \tag{1.38}$$

Observe that the scalar potential is a sum of positive terms. Moreover, the representation of the chiral superfield is arbitrary, but the "gauginos" λ^a have to belong to the *adjoint* representation of G, because they are the supersymmetric partners of the gauge fields v_μ^a.

1.4 Supersymmetry breaking

Particles belonging to the same supermultiplet have the same mass but, since we have not (yet) experimentally detected any supersymmetric partner of the SM particles, if supersymmetry is an underlying symmetry of the real world then it must be broken. In particular, for a sensible physical theory which includes supersymmetry, the mass of each of the SM particles has to be smaller than that of the corresponding superpartner. Thus the main question becomes: *how is supersymmetry broken?*

We will see that the previous requirement on the masses of the SM particles and their superpartners gives strong constraints on the answer.

Let us characterize supersymmetry breaking. *In global supersymmetry, $N = 1$ supersymmetry is unbroken if and only if the vacuum energy vanishes.* In facts, let $|\Omega\rangle$ be the vacuum state. If it is supersymmetric, then

$$Q_\alpha|\Omega\rangle = 0\,,$$

which implies, through the supersymmetry algebra,

$$\langle\Omega|H|\Omega\rangle = \frac{1}{4}\langle\Omega|\left(\overline{Q}_{\dot 1}Q_1 + Q_1\overline{Q}_{\dot 1} + \overline{Q}_{\dot 2}Q_2 + Q_2\overline{Q}_{\dot 2}\right)|\Omega\rangle = 0\,.$$

In globally supersymmetric theories the scalar potential is (1.34). Consequently, *in any rigid theory supersymmetry is broken if and only if there exists an auxiliary field which acquires a non-trivial vacuum expectation value (v.e.v.).* This last criterion is indeed the one that can be safely extended to the local case, as will be discussed in the next chapter.

There is another important consequence of supersymmetry breaking, the analogous of the Goldstone theorem of field theory.
Consider a globally supersymmetric theory in which supersymmetry is spontaneously broken. Then *there exists a massless spin $1/2$ state which is a Goldstone spinor, called Goldstino*, given by

$$\lambda_G = \langle f_i\rangle\,\psi^i - \frac{i}{\sqrt{2}}g\,\langle D^a\rangle\,\lambda^a\,. \tag{1.39}$$

1.4.1 Tree-level supersymmetry breaking

In supersymmetric theories an important quantity is the *supertrace* of the tree-level mass matrices, defined in general as

$$\mathrm{STr}\,\mathcal{M}^2 \equiv \sum_j (-1)^j(2j+1)\,\mathrm{Tr}\,\mathcal{M}_j^2\,, \tag{1.40}$$

where the index j runs over the different spins.

An important result, valid for arbitrary $\langle z^i\rangle$ at the tree level, is that for a renormalizable supersymmetric gauge theory the following *mass formula* holds (see, e.g. [1]):

$$\mathrm{STr}\,\mathcal{M}^2 = -2\sum_a g^a\,\langle D^a\rangle\,\mathrm{Tr}\,(T^a)\,. \tag{1.41}$$

There is supersymmetry breaking (if and) only if the auxiliary fields receive a v.e.v., so there are two possible limiting cases (in general a combination of them occurs):

1. *F-term supersymmetry breaking*, when $\exists\, i \mid \langle f^i\rangle \neq 0$ and $\langle D^a\rangle = 0\;\forall\, a$;

2. *D-term supersymmetry breaking*, when $\exists\, a \mid \langle f^i\rangle = 0$ and $\langle D^a\rangle \neq 0\;\forall\, i$.

These two cases are realized in the O'Raifertaigh and Fayet-Iliopoulos models.

O'Raifertaigh model

In the case of pure F-term breaking in a renormalizable theory, from (1.41) we have in general STr $\mathcal{M}^2 = 0$.

In this case one finds that F-term supersymmetry breaking splits each complex scalar into its real and imaginary part, lifting the mass of the former and reducing that of the latter by an equal amount. This means that *pure F-term breaking in a renormalizable theory cannot provide for a realistic tree-level spectrum.*

As an explicit example of F-term breaking, we can consider a model which contains three chiral superfields Φ_i, $i = 1, 2, 3$, and has a superpotential (with λ, g, m constant parameters)

$$W(\Phi_i) = \lambda\Phi_1\Phi_2 + g\,\Phi_3\,(\Phi_2^2 - m^2)\,.$$

The F-terms read $f_1 = \lambda z_2$, $f_2 = \lambda z_1 + 2gz_2z_3$ and $f_3 = g(z_3^2 - m^2)$. There is not a configuration of the scalar fields which leaves supersymmetry unbroken. Notice that this mechanism always requires at least three fields.

D-term breaking

When the gauge group contains an abelian $U(1)$ subgroup, from (1.41) we have in general STr $\mathcal{M}^2 \neq 0$, since the generators are not necessarily traceless. Take the fields z^i with charges q_i^a (in the following equation *no sum* on i is implied):

$$\sum_j (T^a)^i{}_j z^j = q_i^a z^i\,.$$

Then the scalar fields receive a mass square contribution $g_a q_i^a \langle D^a \rangle$ and in principle we could raise the mass of the scalar partners of the SM fermions as long as $-\sum_a g_a q_i^a \langle D^a \rangle > 0$. But the weak hypercharge Y of the SM does *not* satisfy this requirement and introducing another abelian gauge group can be very dangerous, generating anomalies and unwanted quadratic divergences.

As an example of this mechanism, we consider the Fayet-Iliopoulos model. It is a $U(1)$ supersymmetric gauge theory, which contains two superfields E ad $\overline{E}$ with charges $+1$ and -1, respectively. The D-term is $D = -e(z_{\overline{E}} z^\dagger_{\overline{E}} - z_E z^\dagger_E + \xi)$. We take the superpotential $W = mE\overline{E}$. If $m \neq 0$, the F-terms can vanish only if $\langle z_E \rangle = 0 = \langle z_{\overline{E}} \rangle$, but in this case $\langle D \rangle = \xi \neq 0$.

1.4.2 Non renormalization theorems

The most striking feature of supersymmetric theories is their renormalization behavior[5].

It has been shown that the counterterm Lagrangian which should be added to (1.33) can also be expressed in superfield formalism. There are two kinds of counterterms: *F-counterterms* have the form of a $\theta\theta$ component of a chiral superfield, while *D-counterterms* have the form of a $\theta\theta\overline{\theta}\overline{\theta}$ component of a vector superfield. It can be shown that:

[5]Since these renormalization properties are demonstrated in the most concise way in the superfield formalism, which goes far beyond our purposes, we will only state the results. An introduction to the superfield formalism can be found *e.g.* in [2, 4, 5, 6].

- Any perturbative quantum contribution to the effective action must be expressible as one integral over the whole superspace.

This means that quantum corrections are *always* D-contributions. The main reason for that is the presence of "miraculous" cancellations in supersymmetric theories between loop diagrams with the same external lines but with fields on the loops replaced by their supersymmetric partners.

This theorem implies that the *only* necessary renormalization constants are those needed to renormalize the wave functions of chiral and vector multiplets, while the superpotential is not renormalized at all.

Therefore, in the absence of a Fayet-Iliopoulos term $[\xi^a V^a]_{\theta\theta\overline{\theta}\overline{\theta}}$, a renormalizable supersymmetric theory contains only logarithmic divergences. Otherwise:

- If ξ^a is associated with a traceless generator T^a ($\mathrm{Tr}\, T^a = 0$), then there is no quadratic divergence and ξ^a is only multiplicatively renormalized.

Consequently, if $\mathrm{Tr}\, T^a = 0$ and $\xi^a = 0$ at tree level, then $\xi^a = 0$ at all orders. Moreover, when $\mathrm{Tr}\, T^a = 0$ a renormalizable supersymmetric theory does not have quadratic divergences at all.

1.4.3 The hierarchy problem

One of the main theoretical reason for introducing supersymmetry is the solution that it provides to the (technical) hierarchy problem.

The Standard Model provides the best known description for the so-far observed microscopic phenomena. It is based on the spontaneous breaking of the $SU(2)_L \otimes U(1)_Y$ gauge symmetry to $U(1)_{e.m.}$ at an energy scale of the order of $10^2\ GeV$. The gravitational coupling constant supplies us with another scale, the Planck mass scale $M_P \sim 10^{19}\ GeV$, at which we surely have to take into account quantum-gravitational effects, so the SM has to be considered as an effective quantum field theory which is valid at most up to the scale where a quantum theory of gravitation must be included. Therefore there will be a mass scale Λ at which the SM will no longer be valid[6] and we can use this mass scale as a momentum cutoff. All quantities in our renormalizable field theory can be regularized using the parameter Λ. In general, there is no symmetry to protect the scalar masses and they will, in general, diverge as Λ^2:

$$\mu^2_{(r)}(\Lambda) = \mu^2 + \Lambda^2\ (c_1\alpha_1 + c_2\alpha_2 + \ldots) \tag{1.42}$$

where α_i represent some coupling constants and c_i numerical factors. If, for example, we want to obtain a renormalized mass of the order of $10^2\ GeV$, while $\Lambda \sim M_P$, then we have to fine tune μ^2 so that $\mu^2_{(r)}$ is of the right order of magnitude, canceling almost exactly the Λ^2 term. Moreover, we should fine tune μ^2 at every order in perturbation theory, which is considered *unnatural*. This is the *technical hierarchy problem*. A deeper question is why there should be such different scales in nature.

[6]And we will have $\Lambda \leq M_P$ at least.

Supersymmetric theories provide a solution for the technical hierarchy problem because a tree-level generated hierarchy is stable under quantum corrections: realistic supersymmetric models (with Tr $T^a = 0$) have only logarithmic divergences, while being free of quadratic divergences, thus there is no more need for such an order-by-order fine tuning of the parameters. However, the physical question remains unsolved.

1.4.4 Soft supersymmetry breaking

Now we would like to introduce a phenomenologically acceptable supersymmetric field theory which can solve the technical hierarchy problem.

The non-renormalization theorems tell us that renormalizable supersymmetric theories (if Tr $T^a = 0$) are free of quadratic divergences. Is the opposite true? In other words, is a theory which does not have quadratic divergences always supersymmetric?

The answer is negative, since there exist terms which explicitly break supersymmetry without introducing quadratic divergences. The classification of those terms which break explicitly supersymmetry without introducing quadratic divergences, called *soft breaking terms*, was carried out by Girardello and Grisaru in [16] and the result, which can be motivated in the superfield formalism, is that the *only* soft breaking terms are of the form

$$m^2|z|^2\,, \quad m^2\,(z^2 + h.c.)\,, \quad \mu\,(z^3 + h.c.)\,, \quad \mu\lambda\lambda\,, \tag{1.43}$$

where z is a complex scalar and λ a gauge fermion.

These are the only non-supersymmetric terms allowed. Since their origin can be found in the low-energy limit of supergravity theories, we will come back to this point at a later stage.

1.5 The Minimal Supersymmetric Standard Model

Having studied how supersymmetry can be broken, we can build the minimal supersymmetric gauge theory which contains the Standard Model, the Minimal Supersymmetric Standard Model (MSSM).

First of all, we decide to use the gauge group $G = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. The vector bosons of the SM belong to vector supermultiplets and their spin 1/2 superpartners are called *gauginos*. In particular, the names are assigned in the following manner:

SM vector boson	superpartner
gluon g	gluino $\widetilde{g}$
W^0, $W^\pm$	winos $\widetilde{W}^0$, $\widetilde{W}^\pm$
B^0	bino $\widetilde{B}^0$
Z^0	zino $\widetilde{Z}^0$
photon γ	photino $\widetilde{\gamma}$

We decide to give them exactly the same quantum numbers as in the SM, as shown in Table 1.1.

Name	spin 1/2	spin 1	$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$
V_3^A	$\widetilde{g}^A$	g^A	$(8,1,0)$
V_2^i	$(\widetilde{W}^\pm)$, $(\widetilde{W}^0)$	$(W^\pm)$, (W^0)	$(1,3,0)$
V_1	$\widetilde{B}^0$	B^0	$(1,1,0)$

Table 1.1: Vector superfields in the MSSM. $A = 1,\ldots,8$, $i = 1,2,3$.

All the SM left-handed fermions belong to chiral supermultiplets. Their superpartners, the *sfermions*, are called *squarks* and *sleptons* and have the usual SM quantum numbers with respect to the gauge group G[7].

We also expect the SM Higgs doublet to belong to a chiral supermultiplet, since it corresponds to a physical scalar particle, but, to give mass to all quarks and charged leptons we need two Higgs doublets H_1 and H_2. In the SM, quarks and leptons receive their mass through their Yukawa couplings to a unique Higgs doublet H, which breaks $SU(2)_L \otimes U(1)_Y$ to $U(1)_{e.m.}$; charged leptons and charge $-1/3$ quarks couple to H, while charge $2/3$ quarks couple to $H^\dagger$. In a supersymmetric theory, the Yukawa couplings arise from the superpotential, which is an analytic function of the chiral superfields only (not of their conjugates). Therefore, we need a second Higgs doublet H_2 to give mass to charge $2/3$ quarks. The spin-1/2 superpartners of the Higgs scalars are called *higgsinos*.

The chiral superfield content of the MSSM is summarized in Table 1.2, in the basis in which all leptons and quarks are left-handed.

Name	spin 0	spin 1/2	$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$	description
Q^a	$\binom{\widetilde{u}_L^a}{\widetilde{d}_L^a}$	$\binom{u_L^a}{d_L^a}$	$\left(3,2,\frac{1}{6}\right)$	quark doublets
U_c^a	$(\widetilde{u}_c^a)_L$	$(u_c^a)_L$	$\left(\overline{3},1,-\frac{2}{3}\right)$	antiquarks, charge $-2/3$
D_c^a	$(\widetilde{d}_c^a)_L$	$(d_c^a)_L$	$\left(\overline{3},1,\frac{1}{3}\right)$	antiquarks, charge $1/3$
L^a	$\binom{\widetilde{\nu}^a}{\widetilde{e}_L^a}$	$\binom{\nu_L^a}{e_L^a}$	$\left(1,2,-\frac{1}{2}\right)$	lepton doublets
E_c^a	$(\widetilde{e}_c^a)_L$	$(e_c^a)_L$	$(1,1,1)$	charged antileptons
H_1	$\binom{H_1^0}{H_1^-}$	$\binom{\widetilde{H}_1^0}{\widetilde{H}_1^-}$	$\left(1,2,-\frac{1}{2}\right)$	"down" Higgs doublet
H_2	$\binom{H_2^+}{H_1^0}$	$\binom{\widetilde{H}_1^+}{\widetilde{H}_1^0}$	$\left(1,2,+\frac{1}{2}\right)$	"up" Higgs doublet

Table 1.2: Chiral superfields in the MSSM. The index $a = 1,2,3$ runs over the generations.

The next step is to decide the superpotential of the theory. Taking into account the

[7]We will not consider right-handed neutrinos, although they should be included in a non-minimal version of the SM to take neutrino masses into account.

minimal Higgs sector described above, the MSSM superpotential is

$$W = L^a\,\lambda_E^{ab}\,E_c^b\,H_1 + Q^a\,\lambda_D^{ab}\,D_c^b\,H_1 + Q^a\,\lambda_U^{ab}\,U_c^b\,H_2 + \mu\,H_1\,H_2\,, \tag{1.44}$$

with summations over the generation indices a, b. Moreover, $SU(3)_c$ and $SU(2)_L$ indices are omitted. It is understood that gauge-invariant combinations are constructed in the appropriate way, for example $\mu\,H_1\,H_2 = \mu\,\epsilon_{ij}\,H_1^i\,H_2^j$, where $\epsilon_{ij} = i\,\sigma^2$. The matrices λ_E, λ_D and λ_U contain the (complex) Yukawa couplings.

Finally, we must include soft breaking terms to have a physically acceptable theory. These include soft scalar masses, soft gaugino masses and scalar trilinear couplings. Thus the complete MSSM Lagrangian reads

$$\mathcal{L} = \mathcal{L}_{SUSY} + \mathcal{L}_{SOFT}\,, \tag{1.45}$$

where

$$\begin{aligned}
\mathcal{L}_{SUSY} \;=\; & \Big[(Q^a)^\dagger\, e^{V_3} e^{V_2} e^{\frac{1}{6}V_1} Q^a + (U_c^a)^\dagger\, e^{-V_3} e^{-\frac{2}{3}V_1} U_c^a + \\
& + (D_c^a)^\dagger\, e^{-V_3} e^{\frac{1}{3}V_1} D_c^a + (L^a)^\dagger\, e^{V_2} e^{-\frac{1}{2}V_1} L^a + \\
& + (E_c^a)^\dagger\, e^{V_1} E_c^a + H^\dagger e^{V_2} e^{-\frac{1}{2}V_1} H + \overline{H}^\dagger e^{V_2} e^{\frac{1}{2}V_1} \overline{H} \Big]_{\theta\theta\overline{\theta\theta}} + \\
& + [W]_{\theta\theta} + [W^\dagger]_{\overline{\theta\theta}} + \\
& + \frac{1}{8g_3^2} \left(\mathrm{Tr}\ [W_3^\alpha W_{3\alpha}]_{\theta\theta} + \mathrm{Tr}\ [\overline{W}_{3\dot\alpha} \overline{W}_3^{\dot\alpha}]_{\overline{\theta\theta}} \right) + \\
& + \frac{1}{8g_2^2} \left(\mathrm{Tr}\ [W_2^\alpha W_{2\alpha}]_{\theta\theta} + \mathrm{Tr}\ [\overline{W}_{2\dot\alpha} \overline{W}_2^{\dot\alpha}]_{\overline{\theta\theta}} \right) + \\
& + \frac{1}{8g_1^2} \left(\mathrm{Tr}\ [W_1^\alpha W_{1\alpha}]_{\theta\theta} + \mathrm{Tr}\ [\overline{W}_{1\dot\alpha} \overline{W}_1^{\dot\alpha}]_{\overline{\theta\theta}} \right)\,,
\end{aligned} \tag{1.46}$$

and

$$\begin{aligned}
\mathcal{L}_{SOFT} \;=\; & \sum_a \Big[\left(m_Q^a\right)^2\, |z_Q^a|^2 + \left(m_{U_c}^a\right)^2\, |z_{U_c}^a|^2 + \\
& + \left(m_{D_c}^a\right)^2\, |z_{D_c}^a|^2 + (m_L^a)^2\, |z_L^a|^2 + \left(m_{E_c}^a\right)^2\, |z_{E_c}^a|^2 \Big] + \\
& + m_H^2\, |z_H^2|^2 + m_{\overline{H}}^2\, |z_{\overline{H}}|^2 + \\
& + \sum_{a,b} \left(A_U^{ab} z_Q^a z_{U_c}^b z_{\overline{H}} + A_D^{ab} z_Q^a z_{D_c}^b z_H + A_L^{ab} z_L^a z_{E_c}^b z_H + h.c. \right) + \\
& + \frac{1}{2} M_3 \sum_{A=1}^{8} \left(\lambda_3^A \lambda_3^A + \overline{\lambda}_3^A \overline{\lambda}_3^A \right) + \frac{1}{2} M_2 \sum_{i=1}^{3} \left(\lambda_2^i \lambda_2^i + \overline{\lambda}_2^i \overline{\lambda}_2^i \right) + \\
& + \frac{1}{2} M_1 \left(\lambda_1 \lambda_1 + \overline{\lambda}_1 \overline{\lambda}_1 \right)\,.
\end{aligned} \tag{1.47}$$

Chapter 2

$N = 1,\ D = 4$ supergravity

In this chapter we describe supergravity, which is the theory of local supersymmetry. We start by showing that local supersymmetry contains a theory of gravitation. Then we describe the particle content of the supergravity multiplet, which includes a fermionic spin-3/2 partner for the spin-2 graviton. Next we give the Lagrangian for supergravity coupled to chiral and vector superfields. This will allow us to discuss supersymmetry breaking in supergravity theories and, especially, the super-Higgs effect. We conclude by presenting two supergravity models, the Polonyi model and the no-scale supergravity model, and by showing how supersymmetry breaking in supergravity theories can generate soft breaking terms.

2.1 Local supersymmetry

The Standard Model is based on *local* invariance under the symmetry group $G = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, while so far we studied theories which exhibit global invariance under supersymmetry transformations. Then it is natural to study the theory of local supersymmetry[1].

From the supersymmetry algebra (1.5) we deduce that the product of two supersymmetry transformations corresponds to a translation in space-time, generated by P^μ:

$$[\epsilon Q, \overline{Q}\overline{\epsilon}] = 2\,\epsilon\sigma^\mu\overline{\epsilon}P_\mu\,. \tag{2.1}$$

So far we considered global supersymmetry, hence ϵ was space-time independent ($\partial_\mu\epsilon = 0$). We now want to consider *local supersymmetry*, allowing ϵ to depend on the space-time coordinates:

$$\epsilon = \epsilon(x)\,.$$

The product of two supersymmetry transformations (2.1) now depends on the space-time point x^μ,

$$[\epsilon(x)Q, \overline{Q}\overline{\epsilon}(x)] = 2\,\epsilon(x)\sigma^\mu\overline{\epsilon}(x)P_\mu\,,$$

[1]For an introduction to General Relativity see [17]. Introductions to supergravity can be found in [4, 7, 18, 19, 20].

thus we obtain space-time translations which differ from point to point, which are equivalent to general coordinate transformations (GCT). Therefore we can have local supersymmetry only in theories that also exhibit GCT invariance. Since General Relativity can be thought of as the gauge theory of the Lorentz group and is invariant under GCT, we expect the theory of local supersymmetry to include a description of gravity and we call it *supergravity.*

Before showing this, we would like to illustrate the Nöther procedure, that can be used to build a theory which is invariant under a *local* symmetry out of one which is invariant only under *global* symmetry transformations, in a simple example (QED).

Consider the action for a massless Dirac field ψ in flat space-time (in four-component spinor notation, see Appendix A.1 for our conventions):

$$S_0 = i \int d^4x \, \overline{\psi} \gamma^\mu \partial_\mu \psi \, . \tag{2.2}$$

It is invariant under the *global* $U(1)$ symmetry transformation

$$\psi \rightarrow e^{-i\alpha} \psi \, ,$$

where α is a constant (scalar) parameter. If we allow α to depend on the space-time coordinates, $\psi(x) \rightarrow e^{-i\alpha(x)}\psi(x)$, the action (2.2) is no more invariant under symmetry transformations, because the variation of S_0 is:

$$\delta S_0 = \int d^4x \, (\partial_\mu \alpha) \, \overline{\psi} \gamma^\mu \psi = \int d^4x \, j^\mu \, \partial_\mu \alpha \, , \tag{2.3}$$

where $j^\mu \equiv \overline{\psi}\gamma^\mu\psi$ is the Nöther current associated with the $U(1)$ symmetry of S_0.

To cancel this term, we introduce a gauge field A_μ with the transformation property

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha \tag{2.4}$$

and modify the action

$$S = S_0 + S_g = i \int d^4x \, \overline{\psi} \gamma^\mu D_\mu \psi \, , \tag{2.5}$$

where we defined a *covariant derivative* $D_\mu \psi \equiv (\partial_\mu + iA_\mu)\,\psi$. Now the action (2.5) is invariant under *local* $U(1)$ transformations, provided that the gauge field transforms correctly:

$$\begin{cases} \psi & \rightarrow \quad e^{-i\alpha}\psi \\ A_\mu & \rightarrow \quad A_\mu + \partial_\mu \alpha \end{cases} \, .$$

In general, the promotion of global symmetry to a local one requires the introduction of new degrees of freedom, the gauge fields. Under local symmetries, the variation of these gauge fields in general depends on the *derivative* of the symmetry parameter (in this case $\partial_\mu\alpha$). In our example, the gauge field is thus a spin-1 field, because the parameter $\alpha(x)$ is a scalar. In particular, in this minimal formulation *only* the gauge fields transform with terms that are proportional to $(\partial_\mu \alpha)$, while the transformation of the other field contains α, but *not* $\partial_\mu \alpha$.

If we want to build a locally supersymmetric theory we must then introduce a gauge field Ψ_μ which allows us to define covariant derivatives. As in the previous case, the gauge field will transform, under a local supersymmetry transformation, with parameter $\epsilon(x)$, as

$$\delta_\epsilon \Psi_\mu \sim \partial_\mu \epsilon \, .$$

Since ϵ is a spinor, consistency requires that Ψ_μ is a spinor-vector, which has spin-3/2. Pure gravity, i.e. gravity without coupling to matter fields, is described by a spin-2 particle, the graviton. Supersymmetry connects particles that have spins differing by one-half, then Ψ_μ is our candidate superpartner of the graviton in a supermultiplet which describes supergravity. For this reason we will call it *gravitino*.

2.2 Pure supergravity

We now want to identify the $N = 1$ supergravity multiplet by writing the locally supersymmetric action which describes pure supergravity, i.e. supergravity no coupled to chiral or vector supermultiplets.

This goal can be achieved using the Nöther procedure.
The first step is to identify a globally supersymmetric Lagrangian describing the graviton and the gravitino.

- A free, massless spin-3/2 field can be described by the Rarita-Schwinger (RS) action which, in four-component notation, reads

$$S_{RS} = -\frac{1}{2} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} \, \overline{\Psi}_\mu \gamma_5 \gamma_\nu \partial_\rho \Psi_\sigma \, , \tag{2.6}$$

  where $\epsilon^{\mu\nu\rho\sigma}$ denotes the completely antisymmetric symbol[2]. This gives the field equations

$$\epsilon^{\mu\nu\rho\sigma} \, \gamma_5 \gamma_\nu \partial_\rho \Psi_\sigma = 0 \, . \tag{2.7}$$

  The Rarita-Schwinger action is invariant under the gauge transformation

$$\Psi_\mu \to \Psi_\mu + \partial_\mu \epsilon \, , \tag{2.8}$$

  where ϵ is an arbitrary Majorana spinor.

[2]We adopt the convention $\epsilon^{0123} = 1$.

- The Einstein-Hilbert (EH) action of General Relativity is not polynomial in the metric tensor $g_{\mu\nu}$, because it describes a theory which is invariant under general coordinate transformations. Since we are now looking for an action which is also invariant under global $N = 1$ supersymmetry, to simplify our task we can consider the Einstein-Hilbert action in the weak-field approximation [4]. This corresponds to writing

  $$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}\,, \tag{2.9}$$

  where $\eta_{\mu\nu}$ is the Minkowski metric, $h_{\mu\nu}$ is a small perturbation of $\eta_{\mu\nu}$ and κ is related to the Newtonian gravitational constant G_N by

  $$\kappa^2 \equiv 8\pi G_N\,. \tag{2.10}$$

  If we use $\eta_{\mu\nu}$ and its inverse to lower and raise indices, we obtain a linearized Ricci tensor

  $$R^L_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} + \partial_\mu\partial_\lambda h^\lambda{}_\nu + \partial_\nu\partial_\lambda h^\lambda{}_\mu - \partial_\mu\partial_\nu h^\lambda{}_\lambda\right) \tag{2.11}$$

  and the Einstein-Hilbert action in the weak-field approximation is

  $$S^L_{EH} = -\frac{1}{2\kappa^2}\int d^4x\,\left(R^L_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^L\right)\,h^{\mu\nu}\,, \tag{2.12}$$

  where the Ricci scalar is defined as $R^L \equiv \eta^{\mu\nu}R^L_{\mu\nu}$. The corresponding field equations are

  $$R^L_{\mu\nu} = 0\,. \tag{2.13}$$

  This action (2.12) is invariant under the transformation

  $$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\,, \tag{2.14}$$

  where ϵ_μ is an arbitrary four-vector parameter.

It can be shown [19] that the action

$$S_{GLOBAL} = S_{EH} + S_{RS}$$

is invariant under the following $N = 1$ global supersymmetry transformations:

$$\delta_\xi h_{\mu\nu} = -\frac{i}{2}\overline{\xi}\,\left(\gamma_\mu\Psi_\nu + \gamma_\nu\Psi_\mu\right)\,, \tag{2.15}$$

$$\delta_\xi\Psi_\mu = -i\sigma^{\rho\tau}\partial_\rho h_{\tau\mu}\xi\,, \tag{2.16}$$

where

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right], \tag{2.17}$$

provided that we use the field equations and the gauge invariances of $h_{\mu\nu}$ and Ψ_{μ}.

Using the Nöther procedure, we can arrive at the pure $N = 1$ supergravity action. It has the form of the original Einstein-Hilbert action plus the Rarita-Schwinger action, where the derivatives have been replaced by covariant derivatives that now contain some quadratic terms in the spin-3/2 field.

In a supersymmetric theory we need to describe fermionic as well as bosonic fields. Since it is not possible to define fermions in curved space, we will adopt a formulation which can describe both curved space-time and the tangent space to each point of space-time: this is the vielbein formalism, which is described in Appendix B.1. Using this technology, we can write the locally supersymmetric action for pure supergravity as

$$S_{pure} = -\frac{1}{2\kappa^2}\int d^4x\, |\det e|\, R - \frac{1}{2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\, \overline{\Psi}_{\mu}\gamma_5\gamma_{\nu}\widetilde{D}_{\rho}\Psi_{\sigma}\,, \tag{2.18}$$

where ${e_{\mu}}^{\alpha}$ denotes the vierbein ($D = 4$), with a world index μ and a Lorentz index α such that $h_{\mu\nu} = {e_{\mu}}^{\alpha}{e_{\nu}}^{\beta}\eta_{\alpha\beta}$, and $\widetilde{D}_{\mu}$ is the covariant derivative

$$\widetilde{D}_{\mu} \equiv \partial_{\mu} - \frac{i}{4}\widetilde{\omega}_{\mu\alpha\beta}\,\sigma^{\alpha\beta}\,, \tag{2.19}$$

where $\widetilde{\omega}_{\mu\alpha\beta}$ is called spin connection, given by

$$\widetilde{\omega}_{\mu\alpha\beta} = \omega_{\mu\alpha\beta} + \frac{i\kappa^2}{4}\left(\overline{\Psi}_{\mu}\gamma_{\alpha}\Psi_{\beta} + \overline{\Psi}_{\alpha}\gamma_{\mu}\Psi_{\beta} - \overline{\Psi}_{\mu}\gamma_{\beta}\Psi_{\alpha}\right) \tag{2.20}$$

and

$$\omega_{\mu\alpha\beta} = \frac{1}{2}\,{e_{\alpha}}^{\nu}\left(\partial_{\mu}e_{\beta\nu} - \partial_{\nu}e_{\beta\mu}\right) + \frac{1}{2}\,{e_{\alpha}}^{\rho}{e_{\beta}}^{\sigma}\,\partial_{\sigma}e_{\rho\gamma}{e_{\mu}}^{\gamma} - (\alpha \leftrightarrow \beta)\,. \tag{2.21}$$

The action (2.18) is invariant under the following local $N = 1$ supersymmetry transformations (of parameter ξ)

$${e_{\mu}}^{\alpha} \to {e_{\mu}}^{\alpha} - i\kappa\,\overline{\xi}\gamma^{\alpha}\Psi_{\mu}\,, \tag{2.22}$$

$$\Psi_{\mu} \to \Psi_{\mu} + \frac{2}{\kappa}\,\widetilde{D}_{\mu}\xi\,. \tag{2.23}$$

This shows that the supergravity multiplet (without auxiliary fields) is made of the graviton and its spin-3/2 superpartner, the gravitino, which are described by a $N = 1$ locally supersymmetric theory. In the case of N-extended supersymmetry, which we will not consider, there will be N gravitinos and possibly additional vector, spin-1/2 and scalar fields in order to complete an N-extended supersymmetric multiplet.

From now on we will adopt the convention, customary in supergravity, $\kappa \equiv 1$.

2.3 Coupling of to matter and gauge multiplets

Pure supergravity does not take into account chiral and vector superfields, which must be included in a realistic physical model to describe matter and gauge fields.

The computation of the action in which the gravity supermultiplet is coupled to chiral and vector superfields and which is invariant under local supersymmetry transformations can be done, in principle, using the Nöther procedure. However, it will be more useful to use a geometrical approach [2].

2.3.1 Kähler geometry

A $2d$-dimensional *Kähler manifold* is a Riemannian manifold of $U(d)$ holonomy. This implies that it is complex and that we can express its metric as the mixed second derivative of a scalar function $K(z^i, \overline{z}^{\overline{\jmath}})$

$$g_{\overline{\jmath}i} \equiv \frac{\partial}{\partial z^i}\frac{\partial}{\partial \overline{z}^{\overline{\jmath}}} K(z^i, \overline{z}^{\overline{\jmath}})\,. \tag{2.24}$$

The Kähler metric $g_{\overline{\jmath}i}$ is invariant under analytic shifts $F(z)$ of the Kähler potential

$$K(z^i, \overline{z}^{\overline{\jmath}}) \to K(z^i, \overline{z}^{\overline{\jmath}}) + F(z) + \overline{F}(\overline{z}^{\overline{\jmath}})\,, \tag{2.25}$$

called *Kähler transformations.*

It is possible to define Kähler covariant derivatives ∇_i, using a connection compatible with both the analytic structure of the manifold and with the metric.
Useful results are the expression of the connection as a function of the Kähler metric

$$\Gamma^k_{ij} = g^{k\overline{l}}\frac{\partial}{\partial z^i} g_{\overline{l}j} \tag{2.26}$$

and that of the curvature of the Kähler manifold:

$$\begin{aligned} R_{i\overline{\jmath}k\overline{l}} &= g_{\overline{l}m}\frac{\partial}{\partial \overline{z}^{\overline{\jmath}}}\Gamma^m_{ik} = \\ &= \frac{\partial}{\partial z^i}\frac{\partial}{\partial \overline{z}^{\overline{\jmath}}} g_{\overline{l}k} - g^{m\overline{n}}\left(\frac{\partial}{\partial \overline{z}^{\overline{\jmath}}} g_{\overline{l}m}\right)\left(\frac{\partial}{\partial z^i} g_{\overline{n}k}\right)\,. \end{aligned} \tag{2.27}$$

A Killing vector field X is a field such that the Lie derivative of the metric with respect to X vanishes, for a Kähler manifold this condition reads

$$(\mathcal{L}_X g)_{\overline{\jmath}i} = \nabla_i X_{\overline{\jmath}} + \nabla_{\overline{\jmath}} X_i = 0\,. \tag{2.28}$$

Compatibility with the complex structure requires that the Killing vectors be holomorphic functions

$$\begin{aligned} X^{(b)} &= X^{i(b)}(z)\,\frac{\partial}{\partial z^i}\,, \\ \overline{X}^{(b)} &= \overline{X}^{\overline{\imath}(b)}(z)\,\frac{\partial}{\partial \overline{z}^{\overline{\imath}}}\,, \end{aligned}$$

where the index $(b) = 1, \ldots, d$ labels the Killing vectors. The Lie bracket of two Killing vectors gives another Killing vector, according to

$$[X^{(a)}, X^{(b)}] = -f^{abc} X^{(c)} , \tag{2.29}$$

where f^{abc} are structure constants. Since the Killing vectors are holomorphic, (2.28) reduces to

$$\nabla_i X_j^{(a)} + \nabla_j X_i^{(a)} = 0 ,$$
$$\nabla_{\bar{\imath}} X_j^{(a)} + \nabla_j \overline{X}_{\bar{\imath}}^{(a)} = 0 .$$

On a Kähler manifold, the first equation is automatically satisfied. Locally, the second equation is satisfied if and only if there exist d real scalar functions $D^{(a)}$, *Killing potentials*, such that

$$g_{\bar{\jmath}i} \overline{X}^{\bar{\jmath}(a)} = i \frac{\partial}{\partial z^i} D^{(a)} , \quad g_{\bar{\jmath}i} X^{i(a)} = -i \frac{\partial}{\partial \overline{z}^{\bar{\jmath}}} D^{(a)} . \tag{2.30}$$

Each of the Killing potentials is defined up to an additive constant. Inverting the previous relations, we can solve for the Killing vectors:

$$X^{i(a)} = -i g^{i\bar{\jmath}} \frac{\partial}{\partial \overline{z}^{\bar{\jmath}}} D^{(a)} , \quad \overline{X}^{\bar{\jmath}(a)} = i g^{i\bar{\jmath}} \frac{\partial}{\partial z^i} D^{(a)} . \tag{2.31}$$

Since chiral supermultiplets contain complex scalar fields, it is natural to associate them with a Kähler manifold, endowed with a metric that can be derived from a Kähler potential contained in the theory.

Notation. Since in the following we will often use derivatives of scalar functions with respect to the scalar fields z^i and their conjugates $\overline{z}^{\bar{\jmath}}$, we will denote them by a lower index. For example, the Kähler metric will be

$$g_{\bar{\jmath}i} = \frac{\partial^2}{\partial z^i \partial \overline{z}^{\bar{\jmath}}} K \equiv K_{\bar{\jmath}i} \tag{2.32}$$

and its inverse will be denoted $K^{i\bar{\jmath}}$,

$$K_{\bar{\jmath}i} K^{i\bar{k}} = \delta_{\bar{\jmath}}^{\bar{k}} . \tag{2.33}$$

In general, given an arbitrary function of the scalar fields ϕ,

$$\phi_i(z^i, \overline{z}^{\bar{\jmath}}) \equiv \frac{\partial}{\partial z^i} \phi(z^i, \overline{z}^{\bar{\jmath}}) , \tag{2.34}$$

and similarly

$$\phi_{\bar{\imath}}(z^i, \overline{z}^{\bar{\jmath}}) \equiv \frac{\partial}{\partial \overline{z}^{\bar{\imath}}} \phi(z^i, \overline{z}^{\bar{\jmath}}) . \tag{2.35}$$

2.3.2 Component form of the supergravity Lagrangian

We now consider a supersymmetric theory which includes chiral superfields $\Phi = (z^i, \psi^i)$, $i = 1, \dots, n$, and vector superfields $V^{(a)} = (\lambda^{(a)}, v_\mu^{(a)})$, which transform in the adjoint representation of a gauge group, where the index in brackets is a gauge group index. We denote by g the gauge coupling constant. The superfield formalism gives a very compact form for the most general *globally* supersymmetric and gauge invariant Lagrangian with such a field content:

$$
\begin{aligned}
\mathcal{L}_{GLOBAL} \;=\; & \int d^4\theta\, K(\overline{\Phi}\, e^{2gV}, \Phi) + \int d^2\theta\, \left(W(\Phi) + h.c.\right) + \\
& + \int d^2\theta\, \left(f_{(ab)}(\Phi)\, W^{\alpha(a)} W_\alpha^{(b)} + h.c.\right) ,
\end{aligned} \tag{2.36}
$$

where the $W^{\alpha(a)}$ are given by (1.27). The Lagrangian is characterized by three functions:

- the *superpotential* $W(\Phi)$,
- the *Kähler potential* $K(\overline{\Phi}\, e^{2gV}, \Phi)$,
- the *gauge kinetic function* $f_{(ab)}(\Phi)$.

The superpotential must be analytic and gauge-invariant, the Kähler potential has to be real and gauge-invariant, modulo Kähler transformations, while the gauge kinetic function must be analytic and transforms as the symmetric product of adjoint representations of the gauge group.

The theory is renormalizable only if

- $W(\Phi)$ has mass dimension not greater than three,
- $K(\overline{\Phi}\, e^{2gV}, \Phi) = \overline{\Phi}\, e^{2gV}\, \Phi$,
- $f_{(ab)}(\Phi)$ is a constant.

We now discuss the structure of the most general *locally* supersymmetric Lagrangian, which includes the supergravity multiplet, in component form and without auxiliary fields. We will write it in the two-component spinor notation[3].

It depends only on *two* arbitrary functions, the gauge kinetic function $f_{(ab)}$ and the combination

$$
G \equiv K + \log |W|^2 , \tag{2.37}
$$

which sometimes is also called Kähler potential. However, we will use this name only to refer to the function K. Observe that they generate the same Kähler metric, since this definition corresponds to a Kähler transformation (2.25).

In the following we will consider upper and lower gauge indices (a): the gauge fields and the gauginos are defined to have upper indices, while the Killing vectors and Killing

[3] As in [2]. It can also be found, in four-component notation, in [7].

potentials have lower indices. These gauge indices can be lowered and raised by using $(\mathrm{Re}\, f)_{(ab)}$ and it inverse. The other notations will be given after the complete Lagrangian.

The complete supergravity Lagrangian can be decomposed in

$$\mathcal{L}_{SUGRA} = \mathcal{L}_{pure} + \mathcal{L}_B + \mathcal{L}_F \,, \tag{2.38}$$

where $\mathcal{L}_{pure}$ consists of the gravity supermultiplet, $\mathcal{L}_B$ is the purely bosonic part of the Lagrangian and $\mathcal{L}_F$ contains fermionic fields.

The pure supergravity Lagrangian (2.18) is the sum of the Einstein-Hilbert action and the Rarita-Schwinger action which, in two-component notation, are given by

$$\mathcal{L}_{pure} = \mathcal{L}_{EH} + \mathcal{L}_{RS} \,, \tag{2.39}$$

$$\mathcal{L}_{EH} = -\frac{1}{2}\, e\, R \,, \tag{2.40}$$

$$\mathcal{L}_{RS} = e\, \epsilon^{\mu\nu\rho\sigma}\, \overline{\Psi}_\mu \overline{\sigma}_\nu \widetilde{\mathcal{D}}_\rho \Psi_\sigma \,, \tag{2.41}$$

where $e \equiv |\det e|$ is the determinant of the vierbein.

The bosonic part can be split in terms containing derivatives and in a scalar potential V:

$$\mathcal{L}_B = \mathcal{L}_{KB} - V(z^i, \overline{z}^{\overline{\jmath}}) \,, \tag{2.42}$$

$$\begin{aligned} \mathcal{L}_{KB} \;\;=\;\; & e\, K_{\overline{\jmath}i} (\widetilde{\mathcal{D}}_\mu z^i)(\widetilde{\mathcal{D}}^\mu \overline{z}^{\overline{\imath}}) - \frac{1}{4} e\, (\mathrm{Re}\, f)_{(ab)}\, F_{\mu\nu}{}^{(a)} F^{\mu\nu(b)} + \\ & + \frac{1}{8} e\, (\mathrm{Im}\, f)_{(ab)}\, \epsilon^{\mu\nu\rho\sigma}\, F_{\mu\nu}{}^{(a)} F_{\rho\sigma}{}^{(b)} \,, \end{aligned} \tag{2.43}$$

$$V(z^i, \overline{z}^{\overline{\jmath}}) = e\, e^K \left[K^{i\overline{\jmath}}\, (D_i W)(D_{\overline{\jmath}} \overline{W}) - 3|W|^2 \right] + \frac{1}{2} e\, g^2 (\mathrm{Re}\, f^{-1})^{(ab)}\, D_{(a)} D_{(b)} \,. \tag{2.44}$$

We can divide the fermionic terms in those which contain space-time derivatives and those which do not. These can be divided in those which are quadratic in the fermionic fields and in those which contain terms with four fermion fields:

$$\mathcal{L}_F = \mathcal{L}_{KF} + \mathcal{L}_{MF} + \mathcal{L}_{4F} \,, \tag{2.45}$$

$$\begin{aligned} \mathcal{L}_{KF} \;\;=\;\; & ie\, K_{\overline{\jmath}i} \overline{\psi}^{\overline{\jmath}} \overline{\sigma}^\mu \widetilde{\mathcal{D}}_\mu \psi^i + \frac{i}{2} e\, (\mathrm{Re}\, f)_{(ab)} \left[\lambda^{(a)} \sigma^\mu \widetilde{\mathcal{D}}_\mu \overline{\lambda}^{(b)} + h.c. \right] + \\ & - \frac{1}{2} (\mathrm{Im}\, f)_{(ab)} \widetilde{\mathcal{D}}_\mu \left[e\, \lambda^{(a)} \sigma^\mu \overline{\lambda}^{(b)} \right] + \\ & - \frac{1}{4} \sqrt{2} e\, \left[f_{i(ab)} \psi^i \sigma^{\mu\nu} \lambda^{(a)} F_{\mu\nu}{}^{(b)} + h.c. \right] + \\ & - \frac{\sqrt{2}}{2} e \left[K_{\overline{\jmath}i} (\widetilde{\mathcal{D}}_\nu \overline{z}^{\overline{\jmath}}) \psi^i \sigma^\mu \overline{\sigma}^\nu \Psi_\mu + h.c. \right] + \\ & - \frac{i}{4} e\, (\mathrm{Re} f)_{(ab)} \left[\Psi_\mu \sigma^{\nu\lambda} \sigma^\mu \overline{\lambda}^{(a)} + h.c. \right] \left[F_{\nu\lambda}{}^{(b)} + \hat{F}_{\nu\lambda}{}^{(b)} \right] \,, \end{aligned} \tag{2.46}$$

$$
\begin{aligned}
\mathcal{L}_{MF} \;=\;& -e\, e^{K/2}\Big\{ \overline{W}\, \Psi_\mu \sigma^{\mu\nu} \Psi_\nu + W \overline{\Psi}_\mu \overline{\sigma}^{\mu\nu} \overline{\Psi}_\nu + \\
& - \frac{i}{2}\sqrt{2}\, \left[(D_i W)\, \psi^i \sigma^\mu \overline{\Psi}_\mu + h.c. \right] + \\
& + \frac{1}{2} \left[\mathcal{D}_i D_j W \psi^i \psi^j + h.c. \right] + \\
& - \frac{1}{4} \left[K^{i\overline{\jmath}} D_{\overline{\jmath}} \overline{W}\, f_{i(ab)} \lambda^{(a)} \lambda^{(b)} + h.c. \right] \Big\} + \\
& + \sqrt{2} e g \, \left[K_{\overline{\jmath} i}\, \overline{X}_{(a)}{}^{\overline{\jmath}}\, \psi^i \lambda^{(a)} + h.c. \right] + \\
& + \frac{1}{2} e g D_{(a)} \Psi_\mu \sigma^\mu \overline{\lambda}^{(a)} - \frac{1}{2} e g D_{(a)} \overline{\Psi}_\mu \overline{\sigma}^\mu \lambda^{(a)} + \\
& - \left[\frac{i}{4} \sqrt{2} e g\, f_{i(ab)} D^{(a)} \psi^i \lambda^{(b)} + h.c. \right] .
\end{aligned}
\tag{2.47}
$$

Finally, for completeness, we write the four-fermion terms:

$$
\begin{aligned}
\mathcal{L}_{4F} \;=\;& \frac{1}{4} e\, K_{\overline{\jmath} i} \left[i \epsilon^{\mu\nu\rho\tau} \Psi_\mu \sigma_\nu \overline{\Psi}_\rho + \Psi_\mu \sigma^\tau \overline{\Psi}^\mu \right] \, \psi^i \sigma_\tau \overline{\psi}^{\overline{\jmath}} + \\
& - \frac{1}{8} e \, \left[K_{\overline{\jmath} i} K_{\overline{l} k} - 2 R_{i\overline{\jmath} k \overline{l}} \right] \, \psi^i \psi^k \overline{\psi}^{\overline{\jmath}} \overline{\psi}^{\overline{l}} + \\
& - \frac{1}{16} e \, \left[2 K_{\overline{\jmath} i} (\mathrm{Re}\, f)_{(ab)} + (\mathrm{Re}\, f^{-1})^{(cd)}\, f_{i(bc)} \overline{f}_{\overline{\jmath}(ad)} \right] \overline{\psi}^{\overline{\jmath}} \overline{\sigma}^\mu \psi^i \overline{\lambda}^{(a)} \overline{\sigma}_\mu \lambda^{(b)} + \\
& + \frac{1}{8} e \, \left[\nabla_i f_{j(ab)} \psi^i \psi^j \lambda^{(a)} \lambda^{(b)} + h.c. \right] + \\
& + \frac{1}{16} e \left[(\mathrm{Re}\, f^{-1})^{(cd)} f_{i(ac)} f_{j(bd)} \psi^i \lambda^{(a)} \psi^j \lambda^{(b)} + h.c. \right] + \\
& - \frac{1}{16} e\, K^{i\overline{\jmath}} f_{i(ab)}\, \overline{f}_{\overline{\jmath}(cd)} \lambda^{(a)} \lambda^{(b)} \overline{\lambda}^{(c)} \overline{\lambda}^{(d)} + \\
& + \frac{3}{16} e\, (\mathrm{Re}\, f)_{(ab)} (\mathrm{Re}\, f)_{(cd)} \lambda^{(a)} \sigma^\mu \overline{\lambda}^{(b)} \lambda^{(c)} \sigma_\mu \overline{\lambda}^{(d)} + \\
& + \frac{i}{4} \sqrt{2}\, e \, \left[f_{i(ab)} \left[\psi^i \sigma^{\mu\nu} \lambda^{(a)} \Psi_\mu \sigma_\nu \overline{\lambda}^{(b)} - \frac{1}{4} \overline{\Psi}_\mu \overline{\sigma}^\mu \psi^i \lambda^{(a)} \overline{\lambda}^{(b)} \right] + h.c. \right] .
\end{aligned}
\tag{2.48}
$$

We used the convention

$$
\sigma^{\mu\nu} = \frac{1}{4} \left(\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu \right) ,
\tag{2.49}
$$

and introduced the following covariant derivatives:

$$
\begin{aligned}
\widetilde{D}_\mu z^i \;=\;& \partial_\mu z^i - g v_\mu^{(a)}\, X^i_{(a)} \, , \\
\widetilde{D}_\mu \psi^i \;=\;& \partial_\mu \psi^i + \psi^i \omega_\mu + \Gamma^i_{jk} (\widetilde{D}_\mu z^j) \psi^k - g v_\mu^{(a)} \partial_j X^i_{(a)}\, \psi^j + \\
& - \frac{1}{4} \left(K_j \widetilde{D}_\mu z^j - K_{\overline{\jmath}} \widetilde{D}_\mu \overline{z}^{\overline{\jmath}} \right) \psi^i - \frac{i}{2} g v_\mu^{(a)} \left(\mathrm{Im}\, F_{(a)} \right) \psi^i \, , \\
\widetilde{D}_\mu \lambda^{(a)} \;=\;& \partial_\mu \lambda^{(a)} + \lambda^{(a)} \omega_\mu - g f^{(abc)} v_\mu^{(b)}\, \lambda^{(c)} +
\end{aligned}
$$

$$
\begin{aligned}
&& + \frac{1}{4}\left(K_j \widetilde{D}_\mu z^j - K_{\bar{\jmath}} \widetilde{D}_\mu \overline{z}^{\bar{\jmath}}\right) \lambda^{(a)} + \frac{i}{2} g v_\mu^{(b)} \left(\operatorname{Im} F_{(b)}\right) \lambda^{(a)} \,, \\
\widetilde{D}_\mu \Psi_\nu &=& \partial_\mu \Psi_\nu + \Psi_\nu \omega_\mu + \\
&& + \frac{1}{4}\left(K_j \widetilde{D}_\mu z^j - K_{\bar{\jmath}} \widetilde{D}_\mu \overline{z}^{\bar{\jmath}}\right) \Psi_\nu + \frac{i}{2} g v_\mu^{(a)} \left(\operatorname{Im} F_{(a)}\right) \Psi_\nu \,, \\
D_i W &=& W_i + K_i W \,, \\
\mathcal{D}_i D_j W &=& W_{ij} + K_{ij} W + K_i D_j W + K_j D_i W - K_i K_j W - \Gamma^k_{ij} D_k W \,, \\
\nabla_i f_{j(ab)} &=& f_{ij(ab)} - \Gamma^k_{ij} f_{k(ab)} \,.
\end{aligned} \tag{2.50}
$$

Moreover, we defined the modified field strengths

$$
\hat{F}_{\mu\nu}{}^{(a)} = F_{\mu\nu}{}^{(a)} - \frac{i}{2}\left[\Psi_\mu \sigma_\nu \overline{\lambda}^{(a)} + \overline{\Psi}_\mu \overline{\sigma}_\nu \lambda^{(a)} - \psi_\nu \sigma_\mu \overline{\lambda}^{(a)} - \overline{\psi}_\nu \overline{\sigma}_\mu \lambda^{(a)}\right] , \tag{2.51}
$$

and

$$
F_{(a)} \equiv X_{(a)} K + i D_{(a)} \,. \tag{2.52}
$$

If we take G gauge invariant, then the $D_{(a)}$ have an explicit form

$$
D_{(a)} = i G_i X^i_{(a)} \,. \tag{2.53}
$$

Finally, we give the complete set of local supersymmetry transformations, under which the Lagrangian (2.38) is invariant:

$$
\delta_\xi z^i = \sqrt{2}\, \xi \psi^i \,, \tag{2.54}
$$

$$
\begin{aligned}
\delta_\xi \psi^i &=& -i\sqrt{2}\, \sigma^\mu \,\overline{\xi} \left(\widetilde{\mathcal{D}}_\mu z^i - \frac{1}{2}\sqrt{2} \psi_\mu \psi^i\right) - \Gamma^i_{jk} (\delta_\xi z^j) \psi^k + \\
&& + \frac{1}{4}\left(K_j \delta_\xi z^j - K_{\bar{\jmath}} \delta_\xi \overline{z}^{\bar{\jmath}}\right) \psi^i - \sqrt{2} e^{K/2} K^{i\bar{\jmath}} D_{\bar{\jmath}} \overline{W} \xi + \\
&& + \frac{1}{4}\sqrt{2}\, \xi K^{i\bar{\jmath}}\, \overline{f}_{\bar{\jmath}(ab)}\, \overline{\lambda}^{(a)} \overline{\lambda}^{(b)} \,, \\
\delta_\xi \lambda^{(a)} &=& \hat{F}_{\mu\nu}{}^{(a)} \sigma^{\mu\nu} \xi - \frac{1}{4}\left(K_j \delta_\xi z^j - K_{\bar{\jmath}} \delta_\xi \overline{z}^{\bar{\jmath}}\right) \lambda^{(a)} - i g (\operatorname{Re} f^{-1})^{(ab)} D_{(b)} \xi + \\
&& + \frac{1}{4}\sqrt{2} \xi (\operatorname{Re} f^{-1})^{(ab)}\, f_{i(bc)}\, \psi^i \lambda^{(c)} - \frac{1}{4}\sqrt{2} \xi (\operatorname{Re} f^{-1})^{(ab)}\, \overline{f}_{\bar{\imath}(bc)}\, \overline{\psi}^i \overline{\lambda}^{(c)} \,, \\
\delta_\xi v_\mu^{(a)} &=& i\left(\xi \sigma_\mu \overline{\lambda}^{(a)} + \overline{\xi} \overline{\sigma}_\mu \lambda^{(a)}\right) , \\
\delta_\xi \Psi_\mu &=& 2\, \widetilde{\mathcal{D}}_\mu \xi + \frac{i}{2} \sigma_{\mu\nu} \xi K_{\bar{\jmath} i} \psi^i \sigma^\nu \overline{\psi}^j - \frac{i}{2} (\operatorname{Re} f)_{(ab)} \left(g_{\mu\nu} + \sigma_{\mu\nu}\right) \xi \lambda^{(a)} \sigma^\nu \overline{\lambda}^{(b)} + \\
&& - \frac{1}{4}\left(K_j \delta_\xi z^j - K_{\bar{\jmath}} \delta_\xi \overline{z}^{\bar{\jmath}}\right) \Psi_\mu + i\, e^{K/2}\, W \sigma_\mu \overline{\xi} \,, \\
\delta_\xi e_\mu{}^\alpha &=& i\left(\xi \sigma^\alpha \overline{\Psi}_\mu + \overline{\xi} \overline{\sigma}^\alpha \Psi_\mu\right) .
\end{aligned} \tag{2.55}
$$

2.4 Spontaneous supersymmetry breaking

In global supersymmetry we saw that supersymmetry was broken when the auxiliary fields received a non-zero v.e.v., which was equivalent to have a positive definite energy, since the scalar potential (1.34) was quadratic in the auxiliary fields.

In *supergravity* the situation is quite different, because *the vacuum energy is no longer positive definite*, as follows from (2.44).

We want to characterize supersymmetry breaking in supergravity theories. Spontaneous supersymmetry breaking occurs when at least one of the fields in the theory has a v.e.v. which is *not* invariant under the supersymmetry transformations (2.55)[4]. In the following we assume that terms involving fermionic fields have zero v.e.v.'s[5]. Then, taking the v.e.v. of (2.55) and considering constant expectation values, the only non-trivial v.e.v.'s are

$$\begin{aligned} \langle \delta_\xi \psi^i \rangle &= -\langle \sqrt{2} e^{K/2} \, K^{i\bar{\jmath}} D_{\bar{\jmath}} \overline{W} \xi \rangle \, , \\ \langle \delta_\xi \lambda^{(a)} \rangle &= -\langle ig (\mathrm{Re}\, f^{-1})^{(ab)} D_{(b)} \xi \rangle \, . \end{aligned} \tag{2.56}$$

Using (2.37) and (2.53), these equations are equivalent to

$$\begin{aligned} \langle \delta_\xi \psi^i \rangle &= -\langle \sqrt{2} e^{G/2} \, G^{i\bar{\jmath}} G_{\bar{\jmath}} \xi \rangle \, , \\ \langle \delta_\xi \lambda^{(a)} \rangle &= \langle g (\mathrm{Re}\, f^{-1})^{(ab)} G_i X^i_{(b)} \xi \rangle \, , \end{aligned} \tag{2.57}$$

therefore supersymmetry is broken if and only if $\langle G_i \rangle \neq 0$ for some index i.

The scalar potential (2.44) can be written as:

$$V(z^i, \bar{z}^{\bar{\jmath}}) = e\, e^G \, \left[K^{i\bar{\jmath}} \, G_i G_{\bar{\jmath}} - 3 \right] + \frac{1}{2} e \, g^2 (\mathrm{Re}\, f^{-1})^{(ab)} \, D_{(a)} D_{(b)} \, . \tag{2.58}$$

Assume for simplicity that $D_{(a)} = 0$. This expression directly shows that, when supersymmetry is *unbroken* in supergravity theories, then the vacuum energy is *negative* definite:

$$\langle V \rangle = -3 \langle e^G \rangle \, . \tag{2.59}$$

When supersymmetry is broken, $\langle G_i \rangle \neq 0$, then the vacuum energy can be negative, zero or positive.

As can be directly seen from (2.58), the scalar potential is extremized, $\langle V_i \rangle = 0 \; \forall \, i$, and has zero cosmological constant, $\langle V \rangle = 0$, when the following conditions are satisfied:

$$\langle K^{i\bar{\jmath}} G_{\bar{\jmath}} \nabla_k G_i + G_k \rangle = 0 \, , \quad \langle K^{i\bar{\jmath}} G_{\bar{\jmath}} G_i \rangle = 3 \, , \tag{2.60}$$

where the covariant derivative is defined as in (2.50). Therefore we can have spontaneous supersymmetry breaking with vanishing cosmological constant in supergravity theories.

[4]The case $W = 0$ is peculiar and must be treated carefully, but we do enter into details, since they are not necessary for our purposes.

[5]In general this is not true and we can have supersymmetry breaking by gaugino condensates, in which $\langle \lambda^{(a)} \lambda^{(b)} \rangle \neq 0$ for some indices $(a), (b)$. We will not consider this case here (see [19]).

2.5 The super-Higgs mechanism

Consider the fermionic mass terms, which come from $\mathcal{L}_{MF}$ (2.47), written in terms of G and its derivatives:

$$\begin{aligned}\mathcal{L}_{fermion\,mass} &= -e\,\langle e^{G/2}\rangle\Big\{\Psi_\mu\sigma^{\mu\nu}\Psi_\nu + \Psi_\mu\overline{\sigma}^{\mu\nu}\overline{\Psi}_\nu + \\ &\quad -\frac{i}{2}\sqrt{2}\,\langle G_i\rangle\,\psi^i\sigma^\mu\overline{\Psi}_\mu - \frac{i}{2}\sqrt{2}\,\langle G_{\bar{\imath}}\rangle\,\overline{\psi}^i\overline{\sigma}^\mu\Psi_\mu + \\ &\quad +\frac{1}{2}\langle\nabla_i G_j + G_iG_j\rangle\psi^i\psi^j + \frac{1}{2}\langle\nabla_{\bar{\imath}}G_{\bar{\jmath}} + G_{\bar{\imath}}G_{\bar{\jmath}}\rangle\psi^i\psi^j\Big\}\,. \end{aligned} \tag{2.61}$$

There is a mixing between the gravitino Ψ_μ and the chiral fermions ψ^i which is proportional to $\langle G_i\rangle$, therefore when supersymmetry is broken we must diagonalize this expression.

Note that the combination

$$\eta \equiv \langle G_i\rangle\psi^i \tag{2.62}$$

is a Goldstone fermion, because under supersymmetry it transforms, in the vacuum, as

$$\langle\delta_\xi\eta\rangle = -\sqrt{2}\langle e^{G/2}\,G^{i\bar{\jmath}}G_{\bar{\jmath}}G_i\xi\rangle\,, \tag{2.63}$$

i.e. it transforms by a shift, so it can be gauged away. The existence of a Goldstone fermion is a manifestation of broken supersymmetry.

When supersymmetry is broken, then, we expect that the gravitino "eats" the Goldstone fermion, combining its $\pm 3/2$ helicity states with the $\pm 1/2$ helicity states from of the Goldstone fermion. This is the *super-Higgs mechanism*. It is realized when one performs the redefinition of the gravitino field

$$\Psi'_\mu = \Psi_\mu + \frac{1}{3}\sqrt{2}\,\langle e^{-G/2}\rangle\partial_\mu\eta + \frac{i}{6}\sqrt{2}\sigma_\mu\overline{\eta}\,, \tag{2.64}$$

diagonalizing the mass terms:

$$\begin{aligned}\mathcal{L}_{fermion\,mass} &= -e\,\langle e^{G/2}\rangle\Big\{\Psi'_\mu\sigma^{\mu\nu}\Psi'_\nu + \Psi'_\mu\overline{\sigma}^{\mu\nu}\overline{\Psi}'_\nu + \\ &\quad +\frac{1}{2}\langle\nabla_i G_j + \frac{1}{3}G_iG_j\rangle\psi^i\psi^j + \frac{1}{2}\langle\nabla_{\bar{\imath}}G_{\bar{\jmath}} + \frac{1}{3}G_{\bar{\imath}}G_{\bar{\jmath}}\rangle\psi^i\psi^j\Big\}\,. \end{aligned} \tag{2.65}$$

The gravitino mass is then

$$m_{3/2} = \langle e^{G/2}\rangle\,. \tag{2.66}$$

The scalar masses can be obtained from the second derivative of the scalar potential (2.44), so we can find a tree-level mass formula analogous to (1.41):

$$\begin{aligned}\mathrm{STr}\,\mathcal{M}^2 &= (n-1)\,\left[2m_{3/2}^2 - g^2\langle(D^{(a)})^2\rangle\right] + \\ &\quad -2\,\langle R_{i\bar{\jmath}}G^{i\bar{k}}G^{l\bar{\jmath}}G_{\bar{k}}G_l\rangle\,m_{3/2}^2 + 2g^2\langle K^{i\bar{\jmath}}D^{(a)}_{i\bar{\jmath}}D^{(a)}\rangle\,,\end{aligned}$$

where n is the number of chiral supermultiplets of the theory. We can consider a simplified situation in which $K_{\bar{\jmath}i} = \delta_{\bar{\jmath}i}$ and $f_{(ab)} = \delta_{ab}$, called *minimal kinetic terms*, and in which we do not consider the $D_{(a)}$ terms. In this case

$$\mathrm{STr}\,\mathcal{M}^2 = 2\,(n-1)\,m_{3/2}^2\,, \tag{2.67}$$

while, neglecting D-term breaking, in global supersymmetry we obtained a zero squared-mass formula. When $n > 1$, the bosons are, on the average, heavier than the fermions, proportionally to the gravitino mass, which therefore characterizes supersymmetry breaking.

2.6 Supergravity models

So far we did not explicitly introduce any superpotential, but it is interesting to study some particular supergravity models[6].

Supergravity models are usually made up of two sectors. The first, called *observable sector*, contains the MSSM particles and the second, the *hidden sector*, includes some extra fields which are used to break supersymmetry. These two sectors are assumed to interact only through gravitation, therefore the hidden sector fields must be gauge singlets with respect to all of the gauge interactions of the observable sector.

This can be realized by dividing the superpotential in the hidden and the observable sector:

$$W(Z^i, Y^r) = \hat{W}(Z^i) + \widetilde{W}(Y^r)\,, \tag{2.68}$$

where Z^i denotes the hidden-sector fields and Y^r the observable-sector fields. In realistic models, we will take $\widetilde{W}(Y^r)$ as in (1.44). The hidden-sector superpotential should be derived from a more fundamental theory, but it is useful to study two simple supergravity models. The first allows for non-supersymmetric vacua and also contains an arbitrary mass parameter which can be fine tuned to give a vanishing cosmological constant. The second gives a zero tree-level scalar potential, thus avoiding (at least at the classical level) the need for such a fine tuning. In the following we will not consider D-terms.

1. In the *Polonyi model*, the hidden sector consists of just one chiral superfield, Z, which is a gauge singlet and has a superpotential of the form

$$\hat{W} = m^2\,(Z + b)\,, \tag{2.69}$$

where m and b are parameters with the dimension of a mass. If we assume minimal kinetic terms $K = |Z|^2$, then

$$G_Z = \frac{1 + \overline{Z}(Z + b)}{Z + b}\,, \tag{2.70}$$

[6]for a broader discussion see [7].

so we can have supersymmetry breaking only if $|b| < 2$. The scalar potential (2.58) assumes the form

$$V = m^4 e^{|Z|^2} \left[|1 + \overline{Z}(Z+b)|^2 - 3|Z+b|^2 \right] \tag{2.71}$$

and it has a non-supersymmetric minimum with $\langle V \rangle = 0$ only if

$$b = 2 - \sqrt{3}\,, \tag{2.72}$$

corresponding to a v.e.v. $\langle Z \rangle = \sqrt{3} - 1$. Thus we can at least fine tune the parameter b to have a vanishing cosmological constant. The gravitino mass is then

$$m_{3/2} = m^2\, e^{\frac{(\sqrt{3}-1)^2}{2}} \tag{2.73}$$

and is uniquely determined by the parameter m.

2. The second class of models that we would like to cite are called *no-scale supergravity models*. Such models will arise, for instance, from effective theories obtained from string theories. The simplest example is made of a single chiral superfield, a gauge singlet T, with a Kähler potential

$$K = -3 \log(T + \overline{T})\,. \tag{2.74}$$

If we consider a constant (field-independent) superpotential $\hat{W} = k$, then we obtain

$$V = 0 \tag{2.75}$$

identically, with

$$G_T = G_{\overline{T}} = -\frac{3}{T + \overline{T}}\,, \tag{2.76}$$

so supersymmetry is broken and the tree-level scalar potential is flat, leaving the gravitino mass undetermined at the tree level.

We conclude by showing how hidden-sector supersymmetry breaking manifests itself in the observable sector.

In the following we assume for simplicity that the kinetic terms are minimal, i.e.

$$K = \sum_i |Z^i|^2 + \sum_r |Y^r|^2\,. \tag{2.77}$$

The scalar potential has then the form

$$
\begin{aligned}
V &= e^{\left(\sum_i |Z^i|^2+\sum_r |Y^r|^2\right)} \Big[\sum_i \Big| \hat{W}_i + \overline{Z}^{\overline{i}} (\hat{W} + \widetilde{W}) \Big|^2 + \\
&\quad + \sum_r \Big| \widetilde{W}_i + \overline{Y}^{\overline{r}} (\hat{W} + \widetilde{W}) \Big|^2 - 3|\hat{W} + \widetilde{W}|^2 \Big] . \qquad (2.78)
\end{aligned}
$$

Moreover, supposing that supersymmetry is broken, we can write the hidden-sector vacuum expectation values as

$$
\langle Z^i \rangle \equiv a_i \,, \quad \langle \hat{W} \rangle \equiv \mu \,, \quad \langle \hat{W}_i \rangle \equiv c_i \, \mu \,, \qquad (2.79)
$$

where μ is a mass scale and a_i, c_i are dimensionless constants.

The scalar potential is quite complicated, thus we consider the *flat limit* $M_P \to \infty$, $m_{3/2} = const.$, so called because it corresponds to the limit $k \to 0$, in which we expect to find a renormalizable theory. This also corresponds to a low-energy approximation, that is obtained by considering only terms of the first order in μ/M_P. The gravitino mass is of order μ:

$$
m_{3/2} = e^{\sum_i |a_i|^2/2} \, \mu \,, \qquad (2.80)
$$

thus the flat-limit effective potential is obtained by replacing Z^i, $\hat{W}$ and $\hat{W}_i$ by their v.e.v.'s and keeping only those terms which do not vanish when $M_P \to \infty$. We obtain, in leading order in μ/M_P, the flat-limit scalar potential

$$
V = e^{\sum_i |a_i|^2} \left[\sum_r \left[\left| \widetilde{W}_r \right|^2 + \mu^2 |Y_r|^2 \right] + \mu \sum_r \Big(Y_r \widetilde{W}_r + (A-3) \widetilde{W} + h.c. \Big) \right] , \qquad (2.81)
$$

where

$$
A = \sum_i (\overline{c}_{\overline{i}} + a_i) \, \overline{a}_{\overline{i}} \,. \qquad (2.82)
$$

The first term is a scalar potential for a globally supersymmetric model, with superpotential $\widetilde{W}$. The second term provides a common mass term for all the scalars particles of the observable sector. Finally, the third term gives some (at most trilinear) couplings between the observable-sector scalars. This shows that a spontaneously broken supergravity manifests itself in the observable sector generating the soft breaking terms which are required by realistic models in global supersymmetry.

Chapter 3

Compactification and dimensional reduction

In this chapter we give an introduction to compactification and dimensional reduction, by working through several simple and pedagogical examples. We begin with a brief survey of physics in more than four space-time dimensions. Then we describe "ordinary" and "generalized" dimensional reduction in the simple case of a free complex scalar field on the circle $\mathbb{S}^1$. Afterwards, we review ordinary dimensional reduction of pure gravity, i.e. the Kaluza-Klein model. Finally, we perform the dimensional reduction of a free five-dimensional spinor on $\mathbb{S}^1$ and $\mathbb{S}^1/\mathbb{Z}_2$, both in the ordinary and in the generalized case. We will discuss, in particular, how it is possible to obtain chiral fermions in the reduced theory by considering orbifold compactifications.

3.1 Physics with extra dimensions

One interesting possibility which has been considered in the effort towards a theory unifying the description of physical phenomena is the existence of more than four space-time dimensions. The first noticeable work (after some pioneering papers by Nordström [22]) was that of Kaluza and Klein [23, 24], where they proposed that the electromagnetic potential A_μ could be associated with the (5μ) component of the five-dimensional metric tensor, so that gauge invariance is inherited from the general coordinate transformation invariance of (five-dimensional) gravity. The idea of extra dimensions re-emerged in the mid 70s in the context of higher-dimensional supergravities and of superstring theory. The latter has five natural formulations in ten space-time dimensions, which, in the field-theory limit, provide us with supergravity theories in ten dimensions. Moreover, the idea of Dp-branes, which were discovered in string theories in the late 1990s, gave another reason to discuss the possibility of extra dimensions: they are localized $(p+1)$-dimensional objects, embedded inside the higher-dimensional 'bulk' space-time, where some of the SM fields can be confined, and provide new tools for realistic model building. Even with a small input from string theories, it is possible to work out interesting models, that address the hierarchy problem[1], which has been the main phenomenological motivation for supersymmetric extensions of the Standard Model.

All the experiments performed so far[2] give results compatible with the existence of four space-time dimensions, thus the extra dimensions should be hidden to our tests.

Consider a D-dimensional space-time ($D > 4$). If we suppose that the extra dimensions are compact, then the structure of the complete space-time is $\mathcal{M}_4 \times \mathcal{K}_{D-4}$, where $\mathcal{M}_4$ is the four-dimensional Minkowski space-time and $\mathcal{K}_{D-4}$ is a $(D-4)$-dimensional compact manifold[3]. The process of factorizing in this way the D-dimensional space-time is called *compactification*. Then, the simplest way to conceal the compact extra dimensions is to make them small with respect to the length scales which can be probed by high-energy experiments. This is not the only possibility, since, for example, the Standard Model particles may be tied to a D3-brane in a non-compact higher-dimensional space, however we will limit our study to compactifications.

When we compactify a field theory, the four dimensional theory consists of an infinite number of fields, which correspond to a mode expansion of the fields in the D-dimensional theory on the compact space. For each light field, there is a Kaluza-Klein (KK) tower of excitations with an increasing mass, characterized by the Kaluza-Klein scale M_{KK}, given by the inverse of the typical length of the compact directions (we suppose that they are of the same order of magnitude). For example, if the compact space is a circle of length L, $M_{KK} = 2\pi/L$. Dimensional reduction (in the ordinary case) corresponds to neglecting the dependence of the D-dimensional fields on the internal coordinates, or equivalently to ignore all the KK excitations (for a pedagogical review, see e.g. [27]).

In ordinary dimensional reductions, the D space-time coordinates are split into four *external* coordinates, which characterize the four-dimensional space-time, and $(D-4)$ *internal* coordinates, which describe the compact space. There are then essentially two kinds of dimensional reduction, called "ordinary" and "generalized" [28, 29]. In ordinary dimensional reductions, the fields of the original D-dimensional theory are supposed to be independent of the coordinates of the extra dimensions. This can be generalized, according to Scherk and Schwarz, to fields depending on the extra dimensions in some particular ways, which will be described in Chapter 6. Using generalized dimensional reduction, it is possible to introduce a potential, which allows for mass parameters. Here we describe some of the simplest dimensional reductions, both in the ordinary and in the generalized cases, so to understand how these mechanisms work.

3.2 Compactification of a complex scalar field on $\mathbb{S}^1$

To prepare the ground for compactifications of supergravity theories on factorisable manifolds, we begin by discussing the simplest example of compactification in a field theory

[1]See, for example, the reviews [25] and also [26].

[2]At the moment of writing the LHC collider at CERN is almost, but not yet, ready to start. In some models, extra dimensions may be revealed at energies ($\sqrt{s} \sim 14$ TeV) reached at this collider.

[3]In the following we will only consider space-like extra dimensions, because there are several theoretical problems with time-like extra dimensions. The most dangerous is a loss of causality in the full theory.

with extra dimensions.

Consider a free and massless complex scalar field φ in $D = 5$ space-time dimensions[4]:

$$S_5 = \int d^5x \, (\partial_M \varphi)^* (\partial^M \varphi) \, , \qquad M = 0, 1, 2, 3, 5 \, . \tag{3.1}$$

This theory has, among others, the following symmetries:

1. five-dimensional Poincaré transformations,

2. global $U(1)$ phase transformations

$$\varphi \to e^{i\alpha} \varphi \, , \tag{3.2}$$

3. constant translations (axionic symmetry)

$$\varphi \to \varphi + const \, . \tag{3.3}$$

Then we factorize the five-dimensional space-time, by requiring *four-dimensional Poincaré invariance*, as the product[5] of the four-dimensional Minkowski space-time and a circle of length L, $\mathcal{M}_4 \times \mathbb{S}^1$, where we denote by x^μ, $\mu = 0, \ldots, 3$, the coordinates in the four-dimensional space-time and by y the coordinate of the extra dimension. Thus we break the original five-dimensional Poincaré invariance, which is regained only in the limit $L \to \infty$, to the four-dimensional Poincaré invariance.

The circle of length L can be obtained from the straight line as follows. If we denote by x^μ the coordinates in $\mathcal{M}_4$ and by y the coordinate on the circle, setting $x^M = (x^\mu, y)$, the circle is defined as the quotient space of the straight line with respect to the equivalence relation

$$y \sim y + L \, . \tag{3.4}$$

We then describe the free complex scalar field by an action normalized as

$$S = \int d^4x \int_0^L \frac{dy}{L} \, \left[(\partial_M \varphi)^* (\partial^M \varphi) \right] \, , \tag{3.5}$$

where $x^M = (x^\mu, y)$.

We are going to perform two different kinds of dimensional reduction.

The first, called "ordinary" dimensional reduction [27], consists in taking the five-dimensional field φ to be independent of the internal coordinate y:

$$\frac{\partial \varphi}{\partial y} = 0 \, . \tag{3.6}$$

[4]In this example we do not consider the effects of gravitation, thus neglecting the fluctuations of the metric. We will simply use its background value, the flat metric. Gravity will be examined later.

[5]Usually we will omit $\mathcal{M}_4$, which will be understood.

This will be equivalent to considering the limit $L \to 0$ in which the size of the compact extra dimension vanishes. In this case we will obtain again a free and massless four-dimensional theory.

The second kind of dimensional reduction is the generalization of the ordinary case studied by Scherk and Schwarz [28, 29]. In this case, the field is allowed to depend on the internal coordinate in a specific way, which will be described below. This will enable us to introduce a *mass parameter* for the four-dimensional scalar field.

3.2.1 Periodic field

We can go to a frame of reference in which the extra dimension is a circle of length L, obtained from the real line by the identification:

$$y \equiv y + L\,. \tag{3.7}$$

We impose the periodicity conditions on the scalar field:

$$\varphi(x, y + L) \equiv \varphi(x, y)\,. \tag{3.8}$$

Since the extra dimension is compact, we can expand the field in Fourier series as[6]

$$\varphi(x, y) = \frac{1}{\sqrt{L}} \sum_{n \in \mathbb{Z}} \varphi_n(x)\, e^{i\left(\frac{2\pi n}{L}\right) y}\,. \tag{3.9}$$

Separating between internal and external components and integrating over y, we obtain the action (from now on we omit the dependence on x^μ, which will be understood):

$$S = \int d^4x \sum_{n \in \mathbb{Z}} \left[(\partial_\mu \varphi_n)^* (\partial^\mu \varphi_n) - \left(\frac{2\pi n}{L}\right)^2 |\varphi_n|^2 \right]\,. \tag{3.10}$$

This is the action for an infinite number of four-dimensional scalar fields, a *Kaluza-Klein* tower of states labeled by n, with masses

$$(m_n)^2 = \left(\frac{2\pi n}{L}\right)^2\,. \tag{3.11}$$

This four-dimensional theory with infinitely many fields is called compactification of the original five-dimensional theory. We will see in a moment how to define a four-dimensional effective theory for a finite number of light fields only, a procedure called dimensional reduction.

There are two interesting limits.

1. In the case $L \to \infty$ we recover the original five-dimensional theory, since in this limit the extra dimensions become non-compact.

[6] If φ were real, we would have to reduce by one half the number of independent Fourier components, or *modes*, since in that case $\varphi_m(x)^* = \varphi_{-m}(x)$.

2. In the limit $L \to 0$ the masses for the $n \neq 0$ states diverge, while the $n = 0$ state, the zero-mode, stays massless. Thus the massive modes decouple from the light sector of the effective theory and we can suppress the massive modes of the theory, performing a *truncation*. This leaves only the zero-mode in the four-dimensional effective theory. Equivalently, the KK excitations can be set to zero by considering only the fields independent of the internal coordinates.

   Another way to obtain an effective theory with a finite number of fields is through *consistent truncation*. The effective four-dimensional theory has a cutoff mass scale Λ, which is smaller than the compactification scale M_{KK}. Since the massive KK modes have masses $M_n^2 = (n\, M_{KK})^2$, we can use the equations of motion for the heavy fields in the original theory to eliminate the heavy sector from the four-dimensional effective theory, at order $O\,(\Lambda/M_{KK})$.

   In general, a truncation is not a consistent truncation, in the sense that the equations of motion of the four-dimensional effective theory are not consistent with those of the original theory.

After the truncation, the only remaining state is the zero-mode φ_0, described by the action

$$S = \int d^4x \, [(\partial_\mu \varphi_0)^*(\partial^\mu \varphi_0)] \,, \tag{3.12}$$

where the free complex scalar field in four dimensions φ_0 is *massless*.

The same result can be obtained by retaining only the mode independent of y in (3.10), as in (3.6). The resulting theory has four-dimensional Poincaré invariance, instead of the five-dimensional invariance, but still has a global $U(1)$ invariance and the axionic shift symmetry.

3.2.2 Scherk-Schwarz twist

Imposing the periodicity condition (3.8) on the complex scalar field, and applying dimensional reduction, we obtained a massless four-dimensional effective theory.

We build the circle by identifying points on a line, according to (3.7). The fields at the identified points have to be equal, *up to a transformation which is a symmetry of the action*, because fields differing by a symmetry transformation are physically equivalent. So we can impose *generalized* boundary conditions which break one of the two initial symmetries, but allow for a mass parameter. We require that:

$$\varphi(y + L) \equiv e^{i\mu L}\, \varphi(y) \,, \tag{3.13}$$

where μ is a real parameter with the dimension of a mass. This means that the field has a periodic identification *modulo a symmetry of the original theory*. In this case, the global $U(1)$ symmetry is preserved, but the reduced theory will not exhibit the axionic symmetry, since the $U(1)$ symmetry transformation, used for modifying the periodicity conditions, does not commute with it.

In general, it is possible to impose a condition of the form

$$\varphi(y+L) \equiv T\,\varphi(y)\,, \tag{3.14}$$

where T is a symmetry of the action. This breaks all the symmetries of the action which do not commute with T and allows for the introduction of mass parameters in the theory. It is the *Scherk-Schwarz twist*.

The general form of a field satisfying (3.13) is:

$$\varphi(y) = e^{i\mu y}\frac{1}{\sqrt{L}}\sum_{n\in\mathbb{Z}}\varphi_n(x)\,e^{i\left(\frac{2\pi n}{L}\right)y}\,. \tag{3.15}$$

As in the case of periodic conditions, we perform the ∂_y derivative and integrate over y, obtaining the reduced action

$$S = \int d^4x \sum_{n\in\mathbb{Z}}\left[(\partial_\mu\varphi_n)^*(\partial^\mu\varphi_n) - \left|\mu + \frac{2\pi n}{L}\right|^2 |\varphi_n|^2\right]\,, \tag{3.16}$$

which represents a tower of Kaluza-Klein states, with masses

$$m_n = \left|\mu + \frac{2\pi n}{L}\right|\,. \tag{3.17}$$

In the limit $L \to 0$, we are again left with only one finite-mass state, the zero-mode, but in this case it has a mass

$$m_0 = |\mu| \neq 0\,. \tag{3.18}$$

The corresponding four-dimensional reduced action is given by

$$S = \int d^4x\,\left[(\partial_\mu\varphi_0)^*(\partial^\mu\varphi_0) - \mu^2\,|\varphi_0|^2\right]\,. \tag{3.19}$$

As expected, the reduced action is still invariant under global $U(1)$ transformations (3.2), but the axionic symmetry (3.3), which does not commute with the $U(1)$ symmetry, is *broken.*

This is the simplest example of the Scherk-Schwarz mechanism for generating masses via dimensional reduction, since in this case we exploit a global symmetry, however it is also possible to use a local symmetry for the same purpose. This case will be studied in Chapter 6. Finally, we note that the dimensional reduction for the complex scalar field can be simply obtained by making the following local redefinition of the five-dimensional scalar field:

$$\varphi(x,y) = e^{i\mu y}\,\varphi(x)\,. \tag{3.20}$$

This is a general feature of the Scherk-Schwarz mechanism with global symmetries.

3.3 Compactification of pure gravity on $\mathbb{S}^1$

An important example is the dimensional reduction of pure gravity. In this section we study the compactification of Einstein $D = 5$ gravity on the circle $\mathbb{S}^1$ (Kaluza [23] and Klein [24]). The Scherk-Schwarz dimensional reduction will be presented in Chapter 6 in a more general framework.

Consider pure five-dimensional dimensional Einstein gravity, described by the action

$$S = -\frac{1}{4\kappa_5^2} \int d^5x \sqrt{g_5}\, R_5 \,, \tag{3.21}$$

where g_5 is the determinant of the five-dimensional metric, R_5 is the corresponding curvature scalar and κ_5 is Newton's gravitational constant in five dimensions.

In the frame in which the fifth coordinate corresponds to a circle of radius R on which we compactify our theory,

$$x^5 + 2\pi R \equiv x^5 \,, \tag{3.22}$$

we can write the action as

$$S = -\frac{1}{\kappa_5^2} \int d^4x \int_0^{2\pi R} \frac{dx^5}{2\pi R} \sqrt{g_5}\, R_5 \,. \tag{3.23}$$

It is invariant under general coordinate transformations (GCT) of parameter $\xi^M(x^\mu, x^5)$, which have the form

$$\delta_{GCT} g_{MN} = (\partial_M \xi^P)\, g_{PN} + (\partial_N \xi^P)\, g_{PM} + \xi^P\, \partial_P g_{MN} \,. \tag{3.24}$$

It is convenient to make the following redefinitions of the metric components:

$$g_{MN}(x^\mu, x^5) = \phi^{-1/3} \left(\begin{array}{c|c} g_{\mu\nu} + \kappa_5^2\, \phi\, A_\mu A_\nu & \kappa_5\, \phi\, A_\mu \\ \hline \kappa_5\, \phi\, A_\mu & \phi \end{array} \right) , \tag{3.25}$$

where $M, N = 0, 1, 2, 3, 5$ and $\mu, \nu = 0, 1, 2, 3$. Thus we associated to the $D = 5$ metric a four-dimensional metric, a four-dimensional vector field A_μ and a scalar field ϕ.

We assume periodicity in the metric tensor, $g_{MN}(x^\mu, x^5 + 2\pi R) = g_{MN}(x^\mu, x^5)$, therefore $g_{\mu\nu}$, A_μ and ϕ must be periodic (from now on, the x^μ dependence of the fields is understood):

$$\begin{aligned} g_{\mu\nu}(x^5) &= \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} g_{\mu\nu}^{(n)}\, e^{i \frac{n}{R} x^5} \,, \\ A_\mu(x^5) &= \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} A_\mu^{(n)}\, e^{i \frac{n}{R} x^5} \,, \\ \phi(x^5) &= \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} \phi^{(n)}\, e^{i \frac{n}{R} x^5} \,. \end{aligned} \tag{3.26}$$

Since the fields are real, only half of the Fourier components are independent:

$$g_{\mu\nu}^{(-n)} = (g_{\mu\nu}^{(n)})^* \,, \quad A_\mu^{(-n)} = (A_\mu^{(n)})^* \,, \quad \phi^{(-n)} = (\phi^{(n)})^* \,. \tag{3.27}$$

Substituting in the five-dimensional action, integrating over x^5 and truncating the massive sector, we are arrive at

$$S = -\int d^4x \sqrt{-g_4^{(0)}} \left[\frac{1}{\kappa_5^2} R_4^{(0)} - \frac{1}{4}\phi^{(0)} F_{\mu\nu}^{(0)} F^{\mu\nu(0)} - \frac{1}{6\kappa_5^2 \phi^{(0)2}} (\partial^\mu \phi^{(0)})(\partial_\mu \phi^{(0)}) \right] \,, \tag{3.28}$$

where we use only the zero-mode $g_{\mu\nu}^{(0)}$ from the four-dimensional metric and where we also defined $F_{\mu\nu}^{(0)} \equiv \partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)}$.

It is interesting to study how the original $D = 5$ GCT invariance manifests itself in the reduced theory. We begin by noting that the parameter ξ^M must be periodic:

$$\xi_M(x^5) = \frac{1}{\sqrt{2\pi R}} \sum_{n\in\mathbb{Z}} \xi_M^{(n)}\, e^{i\frac{n}{R}x^5} \,. \tag{3.29}$$

Moreover, we must keep only the zero-mode $\xi_M^{(0)}$. Then the inherited four-dimensional GCT invariance (3.24) is

$$\begin{aligned}
\delta_{GCT} g_{\mu\nu}^{(0)} &= (\partial_\mu \xi^{(0)\rho})\, g_{\rho\nu}^{(0)} + (\partial_\nu \xi^{(0)\rho})\, g_{\rho\mu} + \xi^{(0)\rho}\, \partial_\rho g_{\mu\nu}^{(0)} \,, \\
\delta_{GCT} A_\mu^{(0)} &= (\partial_\mu \xi^{(0)\rho})\, A_\rho^{(0)} + \xi^{(0)\rho}\, \partial_\rho A_\mu^{(0)} \,, \\
\delta_{GCT} \phi^{(0)} &= \xi^{(0)\rho} \partial_\rho \phi^{(0)} \,.
\end{aligned} \tag{3.30}$$

We can also perform gauge $U(1)$ transformations with parameter $\kappa_5^{-1}\, \xi^{(0)5}$:

$$\delta_{gauge} A_\mu^{(0)} = \kappa_5^{-1}\, \partial_\mu \xi^{(0)5} \,. \tag{3.31}$$

The action (3.28) is also invariant under the global scale transformation of parameter λ

$$\begin{aligned}
\delta A_\mu^{(0)} &= \lambda\, A_\mu^{(0)} \,, \\
\delta \phi^{(0)} &= -2\lambda\, \phi^{(0)} \,.
\end{aligned} \tag{3.32}$$

As for the complex scalar field, the zero-modes $g_{MN}^{(0)}$ appear in the action only under four-dimensional derivatives ∂_μ, hence being massless, while the $n \neq 0$ modes $g_{MN}^{(n)}$ have a mass $m_n = \frac{n}{R}$. This is a Kaluza-Klein tower of states.

The scalar field ϕ is called *radion*, since it is associated to fluctuations in the proper length L of the radius of compactification:

$$\delta L = \oint_{\mathbb{S}^1} \sqrt{g_{55}} = 2\pi R\, \phi^{1/3} \,. \tag{3.33}$$

The most striking feature of the model is the *graviphoton* A_μ, which behaves as a gauge field with gauge invariance (3.31) associated to local four-dimensional translations in the x^5 direction. The associated charge is the momentum along the x^5 direction, which in quantized: $p_n = \frac{n}{R}$.

We started with a theory of pure gravity, but the reduced action (3.28) describes three interacting massless particles, the radion, the graviphoton and the four-dimensional graviton. In the Kaluza-Klein picture (1920s), the introduction of an extra dimension unifies gravitation and electromagnetism, in the sense that the graviphoton represents the electromagnetic quadri-potential, with the right gauge invariance inherited from the five-dimensional GCT.

The radion has only derivative terms with respect to the first four dimensions ∂_μ, being independent of x^5, thus it can have an arbitrary v.e.v.

$$\langle \phi \rangle = \overline{\phi}\,, \tag{3.34}$$

while for the graviton and the graviphoton:

$$\langle g^{(0)}_{\mu\nu} \rangle = \eta_{\mu\nu}\,, \quad \langle A^{(0)}_\mu \rangle = 0\,. \tag{3.35}$$

The v.e.v. of the line element determined by the metric (3.25),

$$\langle ds^2 \rangle = \eta_{\mu\nu} dx^\mu dx^\nu - (\overline{\phi})^{2/3}\, dx^5 dx^5\,. \tag{3.36}$$

This shows that the five-dimensional space-time has the geometry of a flat cylinder of radius $\overline{\phi}^{1/3}$ and the radion represents the fluctuations of the radius of the extra dimension hence being related to the geometry of the compact space-time.

The symmetry of the vacuum is the four-dimensional Poincaré group times $\mathbb{R}$ (it is not $U(1)$, since the zero-mode does not depend on the internal coordinate). The four-dimensional graviton is massless due to GCT, as usually, and the masslessness of $A^{(0)}_\mu$ is due to the gauge symmetry. Instead, the radion is massless because of the scale symmetry (3.32).

The v.e.v. of the radion $\overline{\phi}$ labels physically inequivalent vacua, but it is not determined by classical dynamics. The masslessness of the radion is phenomenologically unacceptable, since it would contribute to Einstein's gravity, introducing a long-range fifth force and violating Newton's law. For this reason it is important to stabilize the v.e.v. of the radion $\overline{\phi}$ by introducing a potential which provides for a radion mass.

The radion is our first example of *modulus*, a massless field which arises in the reduced theory. One of the most important problems with dimensional reductions is then that of stabilizing the moduli of the reduced theory, i.e. to make the v.e.v. of the moduli not arbitrary. Moduli stabilization can be achieved by a potential for the scalar fields that, if sufficiently generic, can fix their v.e.v. and give them nonzero masses.

In Chapter 6 we will give more details on moduli stabilization, discussing supergravity compactifications.

3.4 Compactification of a free Dirac field on $\mathbb{S}^1$ and $\mathbb{S}^1/\mathbb{Z}_2$

Our last simple example is that of a free Dirac field in $D = 5$ dimensions. We will discuss compactifications on the circle $\mathbb{S}^1$ and on the orbifold $\mathbb{S}^1/\mathbb{Z}_2$, with periodic conditions, first, and then with generalized Scherk-Schwarz twist.

A brief introduction to spinors in a generic dimensionality is given in Appendix B.2.

Spinors in five space-time dimensions are represented by four-component Dirac spinors (in $D = 5$ it is not possible to impose a Weyl or a (pseudo) Majorana condition), which satisfy the symplectic (pseudo) Majorana condition, with the five gamma matrices

$$\Gamma^M = \{\gamma^\mu, -i\gamma_5\}\,, \tag{3.37}$$

where the four dimensional gamma matrices are taken as in Appendix A.1.

The action for a free (massless) five-dimensional Dirac field is

$$S = \int d^4x \int_0^L \frac{dy}{L}\, \overline{\Psi}\, \left(i\Gamma^M \partial_M\right)\, \Psi\,, \tag{3.38}$$

where we used the same set-up of the scalar field 3.2, namely we obtain a circle by the identification

$$y + L \equiv y\,, \tag{3.39}$$

where y is a coordinate in the extra dimension.

In this frame of reference, the spinor can be decomposed in two four-dimensional Weyl spinors as

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \overline{\psi}_2 \end{pmatrix}\,, \tag{3.40}$$

but when introducing the Scherk-Schwarz twist we will use a more convenient parametrization.

3.4.1 Periodic fields

We begin with the compactification on the circle $\mathbb{S}^1$ by imposing periodic conditions on the five-dimensional Dirac field (the dependence on the coordinates x^μ of the fields is understood throughout this section):

$$\Psi(y + L) \equiv \Psi(y). \tag{3.41}$$

Therefore we can expand in the usual Fourier series

$$\Psi(y) = \frac{1}{\sqrt{L}} \sum_{n\in\mathbb{Z}} e^{i\frac{2\pi n}{L} y}\, \Psi^{(n)}\,. \tag{3.42}$$

Performing the ∂_y derivative and integrating over y in (3.38), we obtain the action

$$S = \int d^4x \sum_{n\in\mathbb{Z}} \overline{\Psi}^{(n)} \left(i\gamma^\mu\partial_\mu + i\frac{2\pi n}{L}\gamma_5\right)\, \Psi^{(n)}\,, \tag{3.43}$$

where we take γ_5 as in (A.17), which represents an infinite tower of four-dimensional Dirac fermions with physical masses

$$(m_n)^2 = \left(\frac{2\pi n}{L}\right)^2 . \tag{3.44}$$

After dimensional reduction, performing the limit $L \to 0$ and considering only finite-mass states, we obtain the reduced action

$$S = \int d^4x\, \overline{\Psi}^{(0)} \left(i\gamma^\mu \partial_\mu\right) \Psi^{(0)} , \tag{3.45}$$

which is the massless zero-mode.

This is completely analogous to the free complex scalar field studied above. In the end, we obtain a single free and massless field in four dimensions. An unattractive feature of this model is that the reduced theory contains a *non-chiral* four-dimensional fermion. This is called the *chirality problem.*

Since physics is correctly described by the Standard Model up to energies of order 1 TeV, we must find a mechanism which allows us to find *chiral* fermions in the effective four-dimensional theory.

A simple solution to the chirality problem is to compactify on the orbifold $\mathbb{S}^1/\mathbb{Z}_2$, instead of on the circle $\mathbb{S}^1$. We build the orbifold by imposing the condition

$$y \equiv -y \tag{3.46}$$

on the coordinate on the circle. The extra dimension is now equivalent to the segment $[0, L/2]$. This identification is possible if we also assign a "parity" transformation P to the $D = 5$ Dirac field, which is respected by the action (3.38):

$$\Psi(-y) = P(\Psi)(y) . \tag{3.47}$$

Using the parametrization (3.40) for the Dirac field, we thus require

$$P(\Psi_L) = +\Psi_L , \quad P(\Psi_R) = -\Psi_R . \tag{3.48}$$

Moreover, we impose the periodicity conditions (3.41).

The orbifold was constructed by first identifying points on a straight line under the translation $\tau : \mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/\tau$, $y \mapsto y + L$, and then identifying points related by the $\mathbb{Z}_2$ reflection $\mathbb{Z}_2 : y \mapsto -y$. We therefore impose two boundary conditions, namely (3.41) and (3.47), which in general have the form

$$\Psi(y + L) = T\,\Psi(y) , \tag{3.49}$$
$$\Psi(-y) = Z\,\Psi(y) , \tag{3.50}$$

where T and Z are two matrices which act on the field of the theory and represent two symmetries of the action corresponding to the operations τ and $\mathbb{Z}_2$. In this case T is trivially the identity, while

$$Z = \begin{pmatrix} \mathbb{I}_2 & 0_2 \\ 0_2 & -\mathbb{I}_2 \end{pmatrix} . \tag{3.51}$$

Since in this case T is trivial, T and Z commute (we imposed periodic boundary conditions), we will obtain a massless reduced theory, even though we expect that the introduction of the orbifold symmetry reduces some of the degrees of freedom of the theory (3.45).

In the general case, however, T and Z must fulfill a *consistency condition*. Consider a point $\overline{y} \in [0, L]$. If we first apply a reflection $\mathbb{Z}_2$ and then a translation τ, we arrive at $L - \overline{y}$. But we can arrive there also by using the inverse translation τ^{-1}, which takes $\overline{y}$ to $\overline{y} - L$, and then using the $\mathbb{Z}_2$ reflection. Therefore, when applying this to the Dirac field, we obtain the relation:

$$ZTZ = T^{-1} \,. \tag{3.52}$$

Since Z represents a reflection, $Z^2 = 1$, but in general $T^2 \neq 1$. When T is not trivial, i.e. when applying a Scherk-Schwarz twist, the consistency condition tells us that the combined field transformation $Z' \equiv TZ$ must satisfy

$$(Z')^2 = (TZ)^2 = (TZ)(Z^{-1}T^{-1}) = 1 \,. \tag{3.53}$$

This shows that the orbifold $\mathbb{S}^1/\mathbb{Z}_2$ can be described by two reflection operators Z and Z', which in general do not commute with each other. We will return to this point when discussing the Scherk-Schwarz twist on the Dirac field.

Now we can discuss the solution to the chirality problem. Imposing just the periodic condition (3.41), we obtain the Fourier expansion (3.42), but not all of the Kaluza-Klein modes are consistent with the orbifold condition. If we consider the decomposition in Weyl spinors (3.40), with the parity assignment (3.48), we can parametrize the Kaluza-Klein modes as

$$\begin{aligned}
\Psi_L(y) &= \frac{2}{\sqrt{L}} \sum_{n=0}^{+\infty} \Psi_L^{(n)} \cos\left(\frac{2\pi n}{L}\, y\right) \,, \\
\Psi_R(y) &= \frac{2i}{\sqrt{L}} \sum_{n=0}^{+\infty} \Psi_R^{(n)} \sin\left(\frac{2\pi n}{L}\, y\right) \,.
\end{aligned} \tag{3.54}$$

We see that the zero-mode $\Psi_R^{(0)}$ has been modded out by the orbifold projection. Hence the reduced theory contains only one finite-mass state, namely $\Psi_L^{(0)}$, which is a Weyl four-dimensional spinor: we obtained a chiral $D = 4$ theory out of a (non-chiral) $D = 5$ theory. In facts, the reduced action is

$$S_4 = \int d^4x \, \overline{\Psi}_L^{(0)} (i\gamma^\mu \partial_\mu) \Psi_L^{(0)} \,. \tag{3.55}$$

As claimed, it contains a free and massless field, but it is *chiral*.

As in the case of the scalar field, we expect that a non-trivial Scherk-Schwarz twist will allow for a mass parameter (smaller than the KK scale $1/L$).

3.4.2 Scherk-Schwarz twist

Instead of the periodicity condition (3.41), we would like to impose a twisted condition. This is easier with another parametrization for the Dirac field. So far we decomposed the four-dimensional Dirac spinor in two-component Weyl spinors

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \overline{\psi}_2 \end{pmatrix}, \tag{3.56}$$

but now it is more convenient to parametrize it with two two-component spinors as in [30]

$$\Phi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3.57}$$

We also define $\overline{\Phi} = \left(\overline{\psi}_1\, \overline{\psi}_2\right)$. Moreover, we define the three matrices $\hat{\sigma}^n$, $n = 1, 2, 3$, as those which act like the corresponding sigma matrix on the components of Φ. Then the five-dimensional action reads

$$S = \int d^4x \int_0^L \frac{dy}{L} \left[i\overline{\Phi}\, \overline{\sigma}^M \partial_M \Phi - \frac{1}{2} \left[i\Phi^T \hat{\sigma}^2 \partial_y \Phi + h.c. \right] \right]. \tag{3.58}$$

The action is invariant under the global $SU(2)$ symmetry

$$\Phi(y) \to U\, \Phi(y)\,, \qquad U \in SU(2)\,. \tag{3.59}$$

We compactify this theory on the circle $\mathbb{S}_1$, first.

As we have seen in the case of the complex scalar field, a twist has to use a symmetry of the full-dimensional action, breaking all of those which do not commute with the former. We can use a $U(1)$ subgroup of the $SU(2)$ symmetry for introducing a generalized Scherk-Schwarz twist:

$$\Phi(y + L) \equiv U_{\vec{\beta}}\, \Phi(y)\,, \qquad U_{\vec{\beta}} \equiv e^{i\vec{\beta}\cdot\vec{\sigma}} = \mathbb{I}\, \cos\beta + i \left(\vec{\beta} \cdot \vec{\sigma} \right) \frac{\sin\beta}{\beta}\,, \tag{3.60}$$

where $\vec{\beta} = (\beta^1, \beta^2, \beta^3) \in \mathbb{R}^3$ is a triplet of real parameters, $\beta = \sqrt{(\beta^1)^2 + (\beta^2)^2 + (\beta^3)^2}$ and $\vec{\sigma} = (\hat{\sigma}^1, \hat{\sigma}^2, \hat{\sigma}^3)$. It is not restrictive to assume $\beta \leq \pi$.

The general form of a field which satisfies the twist condition is

$$\Phi(y) = e^{-i(\vec{\beta}\cdot\vec{\sigma})\, \mu y} \frac{1}{\sqrt{L}} \sum_{n\in\mathbb{Z}} e^{i\frac{2\pi n}{L} y}\, \Phi^{(n)}\,, \tag{3.61}$$

where μ is a mass parameter and the Kaluza-Klein modes are defined as

$$\Phi^{(n)} = \begin{pmatrix} \psi_1^{(n)} \\ \psi_2^{(n)} \end{pmatrix}. \tag{3.62}$$

We can fix part of the $SU(2)$ gauge symmetry (3.59), leaving a residual $U(1)$ gauge invariance preserved by transformations of parameter

$$\vec{\beta} = (0, \beta, 0)\,. \tag{3.63}$$

This breaks the $SU(2)$ symmetry to $U(1)$. Then $U_{\vec{\beta}} = e^{i\beta\mu\hat{\sigma}^2} = \mathbb{I}\cos\beta + i\mu\hat{\sigma}^2\sin\beta$. Substituting the mode expansion (3.61) in (3.58) and integrating in y we obtain the action

$$S = \int d^4x \sum_{n\in\mathbb{Z}} \left[i\overline{\Phi}^{(n)}\,\overline{\sigma}^M \partial_M\,\Phi^{(n)} + \left(-\beta\mu + \frac{2\pi n}{L}\right)(\Phi^{(n)})^T\Phi^{(n)} \right], \tag{3.64}$$

which describes an infinite tower of Kaluza-Klein modes, labeled by n, with masses

$$m_n = \left| -\beta\mu + \frac{2\pi n}{L} \right|\,. \tag{3.65}$$

Taking the limit $L \to 0$ and considering only the states with a finite mass, we finally obtain the action:

$$S = \int d^4x \left[i\overline{\Phi}^{(0)}\,\overline{\sigma}^M \partial_M\,\Phi^{(0)} - \beta\mu\,(\Phi^{(0)})^T\Phi^{(0)} \right]. \tag{3.66}$$

We obtained an action describing a four-dimensional Dirac field, which now has a mass

$$m_0 = |\beta|\,. \tag{3.67}$$

The reduced theory, as expected, has no more the original $SU(2)$ global invariance, but only a $U(1)$ global invariance. The $U(1)$ symmetry corresponds to the choice of $\vec{\beta}$ made in (3.63), while the $SU(2)$ transformations which do not preserve that choice correspond to a broken symmetry. Again, the original symmetry has been broken into a smaller one, but there is a mass term for the fields of the theory, since there is a non-trivial overall dependence of the Fourier expansion (3.61) on y.

Finally, we compactify on the orbifold $\mathbb{S}^1/\mathbb{Z}_2$. In this case, we impose two conditions: the Scherk-Schwarz twist (3.60) and the orbifold condition

$$\Phi(-y) \equiv Z\,\Phi(y)\,, \quad Z \equiv \hat{\sigma}^3\,, \tag{3.68}$$

discussed above. This means that

$$\psi_1(-y) \equiv \psi_1(y)\,, \quad \psi_2(-y) \equiv -\psi_2(y)\,. \tag{3.69}$$

$U_{\vec{\beta}}$ and Z must satisfy the consistency condition (3.52), which reads

$$U_{\vec{\beta}} Z U_{\vec{\beta}} = Z \iff e^{i\vec{\beta}\cdot\vec{\sigma}}\hat{\sigma}^3 e^{i\vec{\beta}\cdot\vec{\sigma}} = \hat{\sigma}^3\,. \tag{3.70}$$

Using the expression for $U_{\vec{\beta}}$ in (3.60), we find the solution (for $\beta = \pi$ there is the trivial solution $U_{\vec{\beta}} = -\mathbb{I}$)

$$\vec{\beta} = (\beta^1, \beta^2, 0)\,. \tag{3.71}$$

As before, we can fix part of the $SU(2)$ gauge symmetry, leaving an unbroken $U(1)$ symmetry preserved by transformations of parameter

$$\vec{\beta} = (0, \beta, 0)\,. \tag{3.72}$$

This gauge fixing is achieved through a $SU(2)$ transformation of parameter $(0, 0, \beta^3)$, generated by $e^{i\beta^3\hat{\sigma}^3}$, thus it commutes with $Z = \hat{\sigma}^3$.

In this case, not all of the modes (3.61) are allowed, since we must impose the orbifold condition, which on the Kaluza-Klein modes reads

$$\psi_1^{(n)} = \psi_1^{(-n)}\,, \qquad \psi_2^{(n)} = -\psi_2^{(-n)}\,. \tag{3.73}$$

As in the untwisted case, the reduced action contains only one *chiral* fermion, but now there is a mass parameter, i.e. we obtained one chiral massive four-dimensional fermion, out of a five-dimensional massless one.

This shows how the Scherk-Schwarz generalized dimensional reduction is compatible with the orbifold projection. Moreover, we have given an explicit example of how we can break a symmetry by introducing an orbifold and also truncate some of the fields of the original theory, in this case the zero-mode $\psi_2^{(0)}$.

In summary, we illustrated various aspects of dimensional reduction from $D = 5$ to $d = 4$, making use of three simple examples. Ordinary dimensional reduction on a circle gives a four-dimensional effective theory which is simply a consistent truncation of the compactified theory, containing the full tower of Kaluza-Klein states: only the massless zero-mode is kept. If we introduce twisted periodicity condition, consistently with some symmetry of the theory, we can introduce a mass parameter, at the price of breaking, in the lower-dimensional theory, the symmetries that do not commute with the one that has been used for the twist. Finally, we introduce an orbifold projection, which truncates away some states from the effective reduced theory. When we also impose a twist, we must be careful to respect the consistency conditions, which make the twist and the orbifold projection compatible.

Chapter 4

Supergravity in ten and eleven dimensions

Supergravity theories can be formulated in more than four space-time dimensions. In this chapter we describe the supergravity theories that arise as low-energy limits of superstring theories or M-theory in $D = 10$ and $D = 11$ space-time dimensions. These theories will be the starting point for our study on compactifications with fluxes. In detail, we discuss the bosonic part of the effective actions for supergravity in $D = 11$ and for type IIA, IIB and $\mathcal{N} = 1$ supergravities in $D = 10$, where $\mathcal{N}$ is the number of D-dimensional supersymmetry generators, which equals the number of gravitinos in the theory.

From now on, we mostly follow Polchinski's notation [31, 32]. The only difference is that we denote p-forms with a round bracket (p) subscript, when indices are not explicitly displayed, instead of using an italicized subscript. In particular, in the previous chapters we used the "mostly minus" convention on the Minkowski metric, while from now on we will adopt the "mostly plus" convention:

$$\eta_{MN} = diag(-,+,+,+,+,\ldots,+)\,, \qquad M,N = 0,1,2,3,5,\ldots,D\,. \tag{4.1}$$

4.1 $D = 11$ supergravity

The best candidate theories unifying quantum mechanics with gravity are string theories, formulated in ten space-time dimensions (when including both space-time bosons and fermions). There are also hints for a more fundamental theory, M-theory, which should give, in appropriate limits, the five different string theories. In the low-energy limit, string theories provide us with supergravity models in ten and eleven dimensions. A detailed study and a review of string theories goes far beyond our purposes, thus we will consider these effective supergravity theories as our starting point.

The maximum dimension in which it is possible to formulate a supersymmetric theory with spin not greater than two is eleven. Heuristically, the reason is as follows. Supersymmetry charges carry spin-1/2. All states in an irreducible representation of the supersymmetry algebra are obtained by applying creation operators, defined as combinations of the supersymmetry charges, on the lowest helicity state. A spinor in D dimensions has $2^{[D/2]}$ components, as explained in Appendix B.2, therefore, when there are too many independent spin-1/2 creation operators, it is unavoidable to have all supermultiplets containing states with spin greater than two. The critical dimension is found to be $D = 11$ (see, e.g., [4]).

The $\mathcal{N} = 1$, $D = 11$ supergravity action was constructed by Cremmer, Julia and Scherk in [33] and is called today *M-theory supergravity*, since it is thought to be the low-energy limit of M-theory.

The field content of the $D = 11$ theory is quite simple. Since we are in a supergravity theory, we must have the supergravity multiplet, which contains the vielbein ${e_M}^A$ and the gravitino field Ψ_M. The vielbein has $\frac{(D-2)(D-1)}{2} - 1 = 44$ physical degrees of freedom, since it corresponds to a symmetric and traceless metric tensor. The gravitino is a Majorana spin-3/2 field, which in $D = 11$ corresponds to $(1/2)\, 2^{[D/2]}\,(D-2-1) = 128$ degrees of freedom. The factor 1/2 comes from the Majorana condition (which can consistently be defined in $D = 11$, see Appendix B.2 and also [4, 27]), the $(D-2)$ is due to the classification of massless physical states in D dimensions with $SO(D-2)$ (in $D = 4$ it corresponds to helicity), while the -1 is due to the additional invariance (2.8) of the Rarita-Schwinger action. There is a difference of $128 - 44 = 84 = \binom{11-2}{3}$ physical degrees of freedom, which corresponds to the degrees of freedom of a 3-form in $D = 11$, that we call $A_{(3)}$. This is the field content of maximal supergravity.

The complete action is:

$$\begin{aligned} S \;=\; & \int d^{11}x \Big[-\frac{e_{11}}{4k_{11}^2}\, R(\omega) - \frac{ie_{11}}{2}\, \overline{\Psi}_M \Gamma^{MNR} D_N \left(\frac{\omega + \hat{\omega}}{2}\right) \Psi_R + \\ & - \frac{e_{11}}{48}\, F_{MNRS} F^{MNRS} + \\ & + \frac{k_{11} e_{11}}{192} \left(\overline{\Psi}_M \Gamma^{MNRSTP} \Psi_N + 12 \overline{\Psi}^R \Gamma^{ST} \Psi^P \right) \left(F_{RSTP} + \hat{F}_{RSTP} \right) + \\ & + \frac{2k_{11}}{(144)^2}\, \epsilon^{S_1 S_2 S_3 S_4 T_1 T_2 T_3 T_4 MNR} F_{S_1 S_2 S_3 S_4} F_{T_1 T_2 T_3 T_4} A_{MNR} \Big] , \end{aligned} \tag{4.2}$$

where

$$F_{(4)} = dA_{(3)} \,, \tag{4.3}$$

e_{11} is the determinant of the vielbein, k_{11} is the eleven-dimensional gravitational constant, the product of n gamma matrices is defined as

$$\Gamma^{M_1 \cdots M_n} \equiv \Gamma^{[M_1} \cdots \Gamma^{M_n]} \,, \tag{4.4}$$

$$\hat{F}_{MNRS} = F_{MNRS} - 3k_{11} \overline{\Psi}_{[M} \Gamma_{NR} \Psi_{S]} \tag{4.5}$$

and the covariant derivative is

$$\widetilde{D}_M \left(\widetilde{\omega}\right) \Psi_N \equiv \partial_M \Psi_N + \frac{1}{4} \widetilde{\omega}_{MAB}\, \Gamma^{AB} \Psi_N \,, \tag{4.6}$$

with

$$\widetilde{\omega}_{MAB} = \omega_{MAB} + K_{MAB}\,, \tag{4.7}$$

where K_{MAB} is the contorsion tensor

$$K_{MAB} = \frac{ik_{11}^2}{4}\left[-\overline{\Psi}_C \Gamma_{MAB}{}^{CD}\Psi_D + 2\left(\overline{\Psi}_M\Gamma_A\Psi_B + \overline{\Psi}_A\Gamma_M\Psi_B - \overline{\Psi}_M\Gamma_B\Psi_A\right)\right]\,, \tag{4.8}$$

$$\omega_{MAB} = \frac{1}{2}\,e_A{}^N\left(\partial_M e_{BN} - \partial_N e_{BM}\right) + \frac{1}{2}\,e_A{}^R e_B{}^S\left(\partial_S e_{RC}\right)e_M{}^C - (A \leftrightarrow B) \tag{4.9}$$

and

$$\hat{\omega}_{MAB} = \omega_{MAB} + \frac{ik_{11}^2}{4}\,\overline{\Psi}_C\Gamma_{MAB}{}^{CD}\Psi_D\,. \tag{4.10}$$

The maximal supergravity action, up to total divergences, is invariant under both general coordinate transformations and local supersymmetry transformations.

In the following, we will consider only the bosonic part of this theory, i.e. the one containing only the fields G_{MN} and $A_{(3)}$. For this reason, we give the bosonic part of the action separately, using a more compact notation [32]:

$$S_{11} = \frac{1}{2k_{11}^2}\int d^{11}x\,(-G_{11})^{1/2}\left[R_{11} - \frac{1}{2}\left|F_{(4)}\right|^2\right] - \frac{1}{6}\int A_{(3)}\wedge F_{(4)}\wedge F_{(4)}\,. \tag{4.11}$$

In general, given a p-form $F_{(p)}$, we use the convention

$$|F_{(p)}|^2 = \frac{1}{p!}\,G^{M_1N_1}\cdots G^{M_pN_p}\,F_{M_1\cdots M_p}\,F_{N_1\cdots N_p}\,. \tag{4.12}$$

This will be our starting point for compactifications of eleven-dimensional supergravity.

4.2 $D = 10$ supergravities

There are three ten-dimensional effective supergravities, which are the low-energy limit of string theories. Two of these can be obtained from M-theory supergravity by dimensional reduction on $\mathbb{R}^{10}\times\mathbb{S}^1$ or on the orbifold $\mathbb{R}^{10}\times\mathbb{S}^1/\mathbb{Z}_2$ and are related to type IIA and heterotic string theories, respectively, as explained in [34]. Since our interest is the compactification of the effective supergravity actions, we will not give a derivation of these, which goes far beyond our purposes, but simply describe the bosonic part of ten-dimensional effective supergravity theories.

Before this, we must discuss a subtle point which we must take into account both in writing the actions and in performing dimensional reductions.

Consider a p-form $A_{(p)}$ in d dimensions and let $F_{(p+1)} \equiv dA_{(p)}$. The corresponding free action is

$$-\frac{1}{2}\int d^dx\,\sqrt{-G_d}\,|F_{(p+1)}|^2\,. \tag{4.13}$$

The equations of motion which can be deduced from this action are

$$d * F_{(p+1)} = 0\,, \tag{4.14}$$

while the Bianchi identity is

$$dF_{(p+1)} = ddA_{(p)} = 0\,. \tag{4.15}$$

If we define

$$F'_{(d-p-1)} \equiv *F_{(p+1)}\,, \tag{4.16}$$

then the equations of motion and the Bianchi identities become, respectively,

$$dF'_{(d-p-1)} = 0\,, \quad d * F'_{(d-p-1)} = 0\,, \tag{4.17}$$

i.e. they are exchanged with respect to (4.14) and (4.15): the field strength has been dualized. We have obtained an equivalent description of the $(p+1)$-form through a $(d-p-1)$-form, in d dimensions.

It is possible to solve, at least locally, the new Bianchi identity in (4.17), which gives

$$F'_{(d-p-1)} = dA'_{(d-p-2)}\,. \tag{4.18}$$

We can add to the action (4.13) a term

$$\int A'_{(d-p-2)} \wedge dF_{(p+1)}\,. \tag{4.19}$$

Here $A'_{(d-p-2)}$ is thought as a Lagrange multiplier which enforces the Bianchi identity for $F_{(p+1)}$. Integrating by parts, we obtain the action

$$-\frac{1}{2}\int d^dx\,\sqrt{-G_d}\,|F'_{(d-p-1)}|^2\,, \tag{4.20}$$

so we have shown that there exists a one-to-one correspondence between the p-form $A_{(p)}$ and its dual $A'_{(d-p-2)}$[1]. Later we will see some consequences of this correspondence.

4.2.1 IIA supergravity

The first $\mathcal{N} = 2$, $D = 10$ effective supergravity theory that we consider corresponds to the low-energy limit of type IIA string theory. The massless and bosonic part of this theory can be obtained from M-theory supergravity through compactification on $\mathbb{R}^{10} \times \mathbb{S}^1$, but we will give the complete bosonic action for type IIA effective supergravity.

In all ten-dimensional effective supergravities, there is a universal sector, called NSNS sector because of the way it originates from the underlying string theory (NS $\equiv$ Neveu-Schwarz, R $\equiv$ Ramond), as we will see comparing the various theories. It consists of the ten-dimensional metric G_{MN}, the dilaton Φ and a 2-form $B_{(2)}$.

The type IIA theory contains also a peculiar sector, called RR sector again for reasons that have to do with its string theory origin: it contains p-form potentials with p *odd*. Since we are in $D = 10$, there are the following duality relations in the RR sector:

[1]The number of physical degrees of freedom for a p-form in d space-time dimensions is $\binom{d-2}{p}$, which is the same as those of its dual form, $\binom{d-2}{d-2-p}$.

$C_{(1)}$	$\leftrightarrow$	$C_{(7)}$
$C_{(3)}$	$\leftrightarrow$	$C_{(5)}$
$C_{(9)}$	$\leftrightarrow$	$C_{(-1)}$

where the last relation means that we can take a 9-form, which has a 10-form field strength, and consider its dual. Formally it is a (-1)-form, since its field strength is a 0-form, that we can denote $M \equiv *F_{(10)}$. M is then a well-defined mass parameter, since the equation of motion for $F_{(10)}$ is $d*F_{(10)} = 0$ and, being $*F_{(10)}$ a scalar, it corresponds to $*F_{(10)} = const.$

Considering only one representative for each couple of dual fields, we describe our theory in terms of the following fields, which we divide into the NSNS and the RR sector:

Field content of **type IIA** supergravity	
NSNS sector	G_{MN}, Φ, $B_{(2)}$
RR sector	$C_{(1)}$, $C_{(3)}$, $C_{(-1)}$.

The corresponding action is given by [32]:

$$S_{IIA} = S_{NS} + S_R + S_{CS} \,, \tag{4.21}$$

where

$$S_{NS} = \frac{1}{2k_{10}^2} \int d^{10}x \; (-G_{10})^{1/2} \; e^{-2\Phi} \; \left[R_{10} + 4(\partial^\mu \Phi)(\partial_\mu \Phi) - \frac{1}{2} \left| H_{(3)} \right|^2 \right] , \tag{4.22}$$

$$S_R = -\frac{1}{4k_{10}^2} \int d^{10}x \; (-G_{10})^{1/2} \; \left[M^2 + \left| F_{(2)} \right|^2 + \left| \widetilde{F}_{(4)} \right|^2 \right] , \tag{4.23}$$

$$S_{CS} = -\frac{1}{4k_{10}^4} \int B_{(2)} \wedge F_{(4)} \wedge F_{(4)} \,, \tag{4.24}$$

and with

$$H_{(3)} = dB_{(2)} \,, \tag{4.25}$$

$$F_{(2)} = dC_{(1)} + MB_{(2)} \,, \tag{4.26}$$

$$F_{(4)} = dC_{(3)} + \frac{1}{2} MB_{(2)} \wedge B_{(2)} \,, \tag{4.27}$$

$$\widetilde{F}_{(4)} = F_{(4)} - C_{(1)} \wedge H_{(3)} \,. \tag{4.28}$$

G_{10} is the determinant of the ten-dimensional metric.

4.2.2 IIB supergravity

The second $\mathcal{N}=2$, $D=10$ effective supergravity theory is the low-energy limit of type IIB string theory. This theory cannot be obtained by dimensional reduction of M-theory.

The NSNS sector consists of the ten-dimensional metric G_{MN}, the dilaton Φ and the 2-form $B_{(2)}$.

In the type IIB theory, the RR sector contains p-form potentials with p *even*. Since we are in $D=10$, there are the following duality relations in the RR sector:

$C_{(0)}$	$\leftrightarrow$	$C_{(8)}$
$C_{(2)}$	$\leftrightarrow$	$C_{(6)}$
$C_{(4)}$	$\leftrightarrow$	$C_{(4)}$

The last relation means that the 4-form has a self-dual field strength, or

$$F_{(5)} = *F_{(5)} \,. \tag{4.29}$$

Because of its first-order equation of motion, the construction of a consistent Lorentz-invariant action for this supergravity theory requires the use of auxiliary degrees of freedom, which we do not want to introduce here (see [35] for further details). For the sake of simplicity, we will consider a non self-dual $F_{(5)}$ in the action, imposing the self-duality constraint by hand on the solutions. When writing the action, we will thus insert a factor 1/2, otherwise we would count the same physical degrees of freedom twice.

The field content for type IIB effective supergravity is as follows:

Field content of **type IIB** supergravity	
NSNS sector	G_{MN}, Φ, $B_{(2)}$
RR sector	$C_{(0)}$, $C_{(2)}$, $C_{(4)}$.

We write the action as:

$$S_{IIB} = S_{NS} + S_R + S_{CS} \,, \tag{4.30}$$

where

$$S_{NS} = \frac{1}{2k_{10}^2}\int d^{10}x\, (-G_{10})^{1/2}\, e^{-2\Phi}\left[R_{10} + 4(\partial^\mu\Phi)(\partial_\mu\Phi) - \frac{1}{2}\left|H_{(3)}\right|^2\right] , \tag{4.31}$$

$$S_R = -\frac{1}{4k_{10}^2}\int d^{10}x\, (-G_{10})^{1/2}\left[\left|F_{(1)}\right|^2 + \left|\widetilde{F}_{(3)}\right|^2 + \frac{1}{2}\left|\widetilde{F}_{(5)}\right|^2\right] , \tag{4.32}$$

$$S_{CS} = -\frac{1}{4k_{10}^2}\int C_{(4)} \wedge H_{(3)} \wedge F_{(3)} \,, \tag{4.33}$$

and with

$$\widetilde{F}_{(3)} = F_{(3)} - C_{(0)} \wedge H_{(3)} \,, \tag{4.34}$$

$$\widetilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)} \,. \tag{4.35}$$

4.2.3 $\mathcal{N} = 1$ (heterotic) supergravity

The third $D = 10$ effective supergravity theory is the $\mathcal{N} = 1$ low-energy limit of the heterotic $E_8 \times E_8$ string theory. It can be obtained from M-theory supergravity through compactification on $\mathbb{R}^{10} \times \mathbb{S}^1/\mathbb{Z}_2$. The low-energy limit of type I string theory, which describes open strings as well as closed ones, has the same supersymmetry as the heterotic supergravity theory and its action differs from this only in the gauge sector, through a different dependence on the dilaton Φ [32] and the presence of the gauge group $SO(32)$. Indeed, there is also another consistent version of the $N = 1$ heterotic superstring, and of the corresponding supergravity, with gauge group $SO(32)$. Since we will not consider the Yang-Mills sector in our examples of compactifications, we will just discuss the universal non-gauge sector of heterotic supergravity.

As before, the NSNS sector consists of the ten-dimensional metric G_{MN}, the dilaton Φ and the 2-form $B_{(2)}$.

The theory does not have a RR sector, but it includes the $E_8 \times E_8$ or $SO(32)$ gauge fields $A_{(1)}$:

Field content of $\mathcal{N} = 1$ supergravity	
NSNS sector	G_{MN}, Φ, $B_{(2)}$
gauge sector	$A_{(1)}$.

The effective supergravity action, to lowest order in the metric derivatives, can be written as

$$S = \frac{1}{2k_{10}^2} \int d^{10}x \, (-G_{10})^{1/2} \, e^{-2\Phi} \Big[R_{10} + 4(\partial^\mu \Phi)(\partial_\mu \Phi) - \frac{1}{2} \left| \widetilde{H}_{(3)} \right|^2 + - \frac{k_{10}^2}{g_{10}^2} \, \mathrm{Tr}_{\,V} |F_{(2)}|^2 \Big] , \tag{4.36}$$

where

$$\widetilde{H}_{(3)} = dB_{(2)} - \frac{k_{10}^2}{g_{10}^2} \, \omega_{(3)} \, , \tag{4.37}$$

the Chern-Simons 1-form is

$$\omega_{(3)} = \mathrm{Tr}_{\,V} \left(A_{(1)} \wedge dA_{(1)} - \frac{2i}{3} A_{(1)} \wedge A_{(1)} \wedge A_{(1)} \right) \tag{4.38}$$

and g_{10} is a gauge coupling parameter.

For simplicity, in the following examples we will not consider the Yang-Mills sector ($A_{(1)} = 0$).

Chapter 5

Simple $N = 1$ orbifold compactifications

In this chapter, we begin the study of some simple compactifications of the $D = 10, 11$ supergravity theories introduced in Chapter 4. For technical and physical reasons, we are interested in those compactifications that preserve an exact or spontaneously broken $N = 1$ local supersymmetry in the resulting $d = 4$ effective theory. The simplest compactifications of this kind are those on the orbifold $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ or $\mathbb{T}^7/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, which reduce the number of supersymmetries, with respect to a torus compactification, from N to $N/4$. When needed, this will be supplemented by an additional projection, with respect to a $\mathbb{Z}_2$ symmetry, that will lead to $N/8$ residual supersymmetry. In our approximations, we will always end up with a $d = 4$, $N = 1$ supergravity theory containing, besides the gravitational multiplet, seven chiral supermultiplets and no vector multiplet.

The goal of this chapter is to study this effective theory when no fluxes are included, therefore no scalar potential is generated. In each of the six examples to be considered, we will derive the correspondence between the bosonic degrees of freedom of the effective $d = 4$ theory and those of the underlying $D = 10$ theory, and the resulting form of the Kähler potential.

5.1 Orbifold compactifications

The simplest compactifications are performed on tori. In the following we will study compactifications of the higher-dimensional supergravity theories described in Chapter 4 to four dimensions, on toroidal orbifolds/orientifolds. We will start by discussing toroidal compactifications in general, then introduce orbifolds and orientifolds. Thus it will be useful to discuss toroidal compactifications in general, first, and then introduce orbifolds and orientifolds.

5.1.1 Torus compactifications

Consider a supergravity theory in D dimensions. We want to compactify it to d dimensions on a k-torus, $k = D - d$. If we split the D-dimensional space-time coordinates as

$$x^M = (x^\mu, x^i)\,, \tag{5.1}$$

$\mu = 0, \ldots, d-1$, $i = d+1, \ldots, D$, then the k-dimensional torus on which we compactify is the quotient space obtained from $\mathbb{R}^k$ through the identification

$$x^i \equiv x^i + 2\pi R\,, \quad i = d+1, \ldots, D\,, \tag{5.2}$$

where we choose for simplicity the same periodicity $2\pi R$ for each coordinate on the torus (the physical radii depending on the background value of the metric).

Since we are interested in the effective theory for the massless modes only, we make the following reduction Ansatz for the metric:

$$\begin{aligned} ds^2 &= G_{MN}(x^\mu, x^i)\, dx^M dx^N \\ &= G_{\mu\nu}(x^\rho)\, dx^\mu dx^\nu + G_{ij}(x^\rho)\,(dx^i + A^i_\mu(x^\rho)dx^\mu)(dx^j + A^j_\nu(x^\rho)dx^\nu)\,, \end{aligned} \tag{5.3}$$

where in d dimensional $G_{\mu\nu}$ is a symmetric tensor, A^i_μ are k vectors and the G_{ij} are $k(k+1)/2$ scalar fields, called *metric moduli.*

In our concrete examples, we will consider orbifolds, i.e. we will introduce some discrete symmetries acting on the coordinates and retain only the components of the d-dimensional massless fields that are left invariant. The vectors A^i_μ will not be invariant under the orbifold action and will be truncated, so for simplicity we will not consider them from the beginning. Therefore, the D-dimensional metric reduces to two diagonal blocks, one containing the d-dimensional metric $G_{\mu\nu}$ and one the "internal" metric G_{ij}.

Since the ten-dimensional supergravities that we consider have the same NSNS sector, we perform the reduction of an action in D dimensions which has the form of the NSNS sector action for type II supergravities. When the theory is compactified on a k-torus the reduced action becomes

$$\begin{aligned} S &= \frac{1}{2\kappa_D^2}\int d^Dx\,\sqrt{-G_D}\,e^{-2\Phi}\left[R_D + 4(\partial^\mu\Phi)(\partial_\mu\Phi) - \frac{1}{2}\left|H_{(3)}\right|^2\right] = \\ &= \frac{1}{2\kappa_d^2}\int d^dx\,\sqrt{-G_d}\,\sqrt{G_k}\,e^{-2\Phi}\Big[R_d + \\ &\quad + 4\left(\partial_\mu\left(\Phi - \frac{1}{4}\log G_k\right)\right)\left(\partial^\mu\left(\Phi - \frac{1}{4}\log G_k\right)\right) + \\ &\quad - \frac{1}{4}G^{mn}G^{pq}\,\left((\partial_\mu G_{mp})(\partial^\mu G_{nq}) + (\partial_\mu B_{mp})(\partial^\mu B_{nq})\right) + \\ &\quad - \frac{1}{4}G^{mn}H_{m\mu\nu}H_n^{\mu\nu} - \frac{1}{12}\,H_{\mu\nu\rho}H^{\mu\nu\rho}\Big]\,, \end{aligned} \tag{5.4}$$

where the d-dimensional gravitational constant is given by

$$\frac{1}{\kappa_d^2} = \frac{(2\pi R)^k}{\kappa_D^2}\,, \tag{5.5}$$

$G_k \equiv \det G_{mn}$ is the determinant of the internal block of the metric and the 3-form field strength H is defined as

$$H_{(3)} = dB_{(2)}\,. \tag{5.6}$$

This result is obtained in the string frame, but we can express it in the Einstein frame. This is achieved by making the Weyl rescaling

$$G_{\mu\nu} = \hat{s}^{-1}\widetilde{G}_{\mu\nu}\,, \tag{5.7}$$

where the expression of the d-dimensional field $\hat{s}$ in terms of the dilaton (also assumed to depend only on x^μ) and of the internal components of the metric will be determined later. Under this rescaling, the determinant in d dimensions transforms as

$$\sqrt{-G_d} = \hat{s}^{-\frac{d}{2}}\sqrt{-\widetilde{G}_d}\,, \tag{5.8}$$

while the Ricci scalar transforms according to

$$\sqrt{-G_d}R \;=\; \sqrt{-\widetilde{G}_d}\,\hat{s}^{\frac{1}{2}(2-d)}\left[\widetilde{R}_d - \frac{1}{4}(d-2)(d-1)\hat{s}^{-2}\widetilde{G}^{\mu\nu}(\partial_\mu\hat{s})(\partial_\nu\hat{s})\right]. \tag{5.9}$$

The reduced action becomes then

$$\begin{aligned} S \;=\; & \frac{1}{2\kappa_d^2}\int d^dx\,\sqrt{-\widetilde{G}_d}\,\sqrt{G_k}\,\left(e^{-2\Phi}\hat{s}^{\left(1-\frac{d}{2}\right)}\right)\,\Big[\widetilde{R}_d + \\ & -\frac{1}{4}(d-2)(d-1)\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) + \\ & + 4\left(\partial_\mu\left(\Phi - \frac{1}{4}\log G_k\right)\right)\left(\partial^\mu\left(\Phi - \frac{1}{4}\log G_k\right)\right) + \\ & -\frac{1}{4}G^{mn}G^{pq}\,\left((\partial_\mu G_{mp})(\partial^\mu G_{nq}) + (\partial_\mu B_{mp})(\partial^\mu B_{nq})\right) + \\ & -\frac{1}{4}\hat{s}\,G^{mn}H_{m\mu\nu}H_n^{\mu\nu} - \frac{1}{12}\,\hat{s}^2\,H_{\mu\nu\rho}H^{\mu\nu\rho}\Big]\,, \end{aligned} \tag{5.10}$$

where the lowering and raising of the d-dimensional indices is done here with $\widetilde{G}_{\mu\nu}$ and its inverse, respectively.

We will also need to reduce the action containing a general p-form $A_{(p)}$, with field strength $F_{(p+1)} = dA_{(p)}$. In analogy with what was done with for metric and the dilaton, we make the hypothesis that the p-form $A_{(p)}$ depends only on the external coordinates, $A_{(p)}(x^\mu)$, so that $\partial_i A_{(p)}(x^\mu) \equiv 0$. Non-trivial background (p-form and geometric) fluxes will be studied in Chapter 6. For the moment we will consider only fluctuations around a zero mean value of the fluxes. This means that the all-internal component field strengths vanish $F_{m_1\cdots m_{p+1}} = 0$. Then we must consider field strengths with q external and $(p-q+1)$ internal components, $1 \le q \le (p+1)$.

In the case $q = 1$, the field strength can be thought as a derivative (with respect to the external coordinates) acting on a scalar field $A_{m_2\cdots m_{p+1}}(x^\mu)$: $F_{\mu m_2\cdots m_{p+1}} \propto \partial_\mu A_{m_2\cdots m_{p+1}}$.

The case $q=2$ describes a d-dimensional vector $A_{\mu m_3\cdots m_{p+1}}(x^\mu)$. The case $q=3$ is quite peculiar, since it describes an antisymmetric tensor $A_{\mu\nu m_4\cdots m_{p+1}}(x^\mu)$, which in $d=4$ is dual to a scalar field. The cases $q=4,\ldots,(p+1)$ then represent $(q-1)$-forms $A_{\mu_1\cdots\mu_q m_{q+1}\cdots m_{p+1}}$. The cases $q=2$ and $q\geq 4$, however, will be truncated by the orbifold projection in our examples, so we will not consider them (the same holds for the NSNS 2-form $B_{(2)}$). Recalling our convention (4.12), the action for a p-forms is reduced according to

$$\begin{aligned}
&\frac{1}{2\kappa_D^2}\int d^Dx\,\sqrt{-G_D}\,|F_{(p+1)}|^2 = \\
= \; &\frac{1}{2\kappa_D^2}\int d^Dx\,\sqrt{-G_D}\,\frac{1}{p!}\Big[G^{m_2n_2}\cdots G^{m_{p+1}n_{p+1}}\,(\partial_\mu A_{m_2\cdots m_{p+1}})\,(\partial^\mu A_{n_2\cdots n_{p+1}}) + \\
&+ G^{m_4n_4}\cdots G^{m_{p+1}n_{p+1}}(F_{\mu\nu\rho m_4\cdots m_{p+1}})\,(F^{\mu\nu\rho}{}_{n_4\cdots n_{p+1}})\Big]\,. \qquad (5.11)
\end{aligned}$$

With our simplifications, toroidal compactification of a supergravity theory from D to d dimensions, containing the universal NSNS sector and a p-form, leads to the d-dimensional Poincaré invariant action

$$\begin{aligned}
S \;=\; &\frac{1}{2\kappa_D^2}\int d^Dx\,\sqrt{-G_D}\,e^{-2\Phi}\left[R_D + 4(\partial^\mu\Phi)(\partial_\mu\Phi) - \frac{1}{2}e^{2\Phi}\left|F_{(p+1)}\right|^2\right] = \\
= \; &\frac{1}{2\kappa_d^2}\int d^dx\,\sqrt{-\widetilde{G}_d}\,\sqrt{G_k}\,\left(e^{-2\Phi}\hat{s}^{\left(1-\frac{d}{2}\right)}\right)\,\Big[\widetilde{R}_d + \\
&- \frac{1}{4}(d-2)(d-1)\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) + \\
&+ 4\left(\partial_\mu\left(\Phi - \frac{1}{4}\log G_k\right)\right)\left(\partial^\mu\left(\Phi - \frac{1}{4}\log G_k\right)\right) + \\
&- \frac{1}{4}G^{mn}G^{pq}\,\left((\partial_\mu G_{mp})(\partial^\mu G_{nq}) + (\partial_\mu B_{mp})(\partial^\mu B_{nq})\right) + \\
&- \frac{1}{12}\,\hat{s}^2\,H_{\mu\nu\rho}H^{\mu\nu\rho} + \\
&- \frac{1}{2(p!)}\hat{s}^2 e^{2\Phi}G^{m_4n_4}\cdots G^{m_{p+1}n_{p+1}}(F_{\mu\nu\rho m_4\cdots m_{p+1}})\,(F^{\mu\nu\rho}{}_{n_4\cdots n_{p+1}}) + \\
&- \frac{1}{2}e^{2\Phi}\,\frac{1}{p!}G^{m_2n_2}\cdots G^{m_{p+1}n_{p+1}}(\partial_\mu A_{m_2\cdots m_{p+1}})\,(\partial^\mu A_{n_2\cdots n_{p+1}})\Big]\,, \qquad (5.12)
\end{aligned}$$

which in $d=4$ becomes

$$\begin{aligned}
S \;=\; &\frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\sqrt{G_k}\,\left(e^{-2\Phi}\hat{s}^{-1}\right)\,\Big[\widetilde{R}_4 - \frac{3}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) + \\
&+ 4\left(\partial_\mu\left(\Phi - \frac{1}{4}\log G_k\right)\right)\left(\partial^\mu\left(\Phi - \frac{1}{4}\log G_k\right)\right) + \\
&- \frac{1}{4}G^{mn}G^{pq}\,\left((\partial_\mu G_{mp})(\partial^\mu G_{nq}) + (\partial_\mu B_{mp})(\partial^\mu B_{nq})\right) + \\
&- \frac{1}{12}\,\hat{s}^2\,H_{\mu\nu\rho}H^{\mu\nu\rho} +
\end{aligned}$$

$$- \frac{1}{2(p!)} \hat{s}^2 e^{2\Phi} G^{m_4 n_4} \cdots G^{m_{p+1} n_{p+1}} (F_{\mu\nu\rho m_4 \cdots m_{p+1}}) \, (F^{\mu\nu\rho}{}_{n_4 \cdots n_{p+1}}) +$$

$$- \frac{1}{2} e^{2\Phi} \frac{1}{p!} G^{m_2 n_2} \cdots G^{m_{p+1} n_{p+1}} (\partial_\mu A_{m_2 \cdots m_{p+1}}) \, (\partial^\mu A_{n_2 \cdots n_{p+1}}) \Big] \, . \tag{5.13}$$

Each term in the reduced action (5.13) contains two space-time derivatives, while there are no terms without derivatives. The reason of this is that we are now considering fluctuations of the fields around a zero background value, hence there is not a scalar potential. Such a dimensional reduction will allow us to compute the Kähler potential K for the effective supergravity theory, but the superpotential will be identically vanishing at this level. Finally, there will not be any gauge kinetic function in our effective four-dimensional theories, since our reductions do not contain fields belonging to vector multiplets.

5.1.2 Orbifolds

Given a manifold $\mathcal{M}$ and a discrete group $\mathcal{G}$ acting on its points, an *orbifold* is the quotient space $\mathcal{O} = \mathcal{M}/\mathcal{G}$. If the action of $\mathcal{G}$ has fixed points, the resulting orbifold has singular points and is no longer a manifold.

In general, supergravity theories can be compactified on quite complicated spaces[1]. Orbifolds provide a simple way of consistently truncating the massless spectrum of the effective theory[2] and reducing the number of supersymmetries, with respect to what we would obtain by compactifying on the original manifold $\mathcal{M}$.

In our main examples, simple toroidal compactifications give $4\mathcal{N}$ supersymmetries in $d = 4$ (with $\mathcal{N} = 2$ for type II and for M-theory supergravities, $\mathcal{N} = 1$ for the heterotic supergravity studied in Chapter 4), while we want to obtain an effective theory with simple local supersymmetry. This is achieved (in part) by considering orbifold projections generated by appropriate groups. In our examples, we will consider toroidal orbifolds, in which the manifold is $\mathbb{T}^k$ for $k = 6, 7$, and the discrete group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Toroidal compactifications of type II $D = 10$ supergravities on $\mathbb{T}^6$ and of $D = 11$ M-theory on $\mathbb{T}^7$ give $N = 8$ supersymmetries in $d = 4$, but the orbifold projection on $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ or on $\mathbb{T}^7/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ gives just $N = 2$ supersymmetries for these theories, while in the heterotic case we directly obtain a $N = 1$ supersymmetry in $d = 4$. Thus, for type II and M-theory supergravities, we need to reduce again the number of residual supersymmetries, in order to obtain a $N = 1$, $d = 4$ effective theory. In M-theory we will use a further orbifold projection, while in type II theories we will introduce another projection, called *orientifold* for reasons that have to do with its string theory origin, which will be described below.

[1]For example on a Calabi-Yau manifold, which is a Kähler manifold of $SU(3)$ holonomy. These objects are rather difficult to discuss and their introduction goes beyond our purposes.

[2]In general, additional massless fields can exist at the fixed points. These correspond to strings belonging to the *twisted sector*, i.e. strings which obey twisted boundary conditions like $X^i(\pi) = gX^i(0)$, $\forall\, g \in \mathcal{G}$, where the functions X^i describe the string coordinates on the orbifold, taking arguments ranging from 0 to π. We will not study the twisted sector of our effective supergravity theories, assuming that all its fields can be set to zero consistently with their equations of motion.

We now discuss some examples of orbifolds, showing the general features of this construction. We observe that the simple orbifolds to be considered in our examples have a purely geometrical action: the orbifold projection acts non-trivially only on the coordinates, while the action on the fields is trivial, in the sense that their transformation properties are the same as those of the combination of space-time indices they carry. Then consistency requires that only some components of the fields can be defined on the orbifold, namely those fields which are invariant with respect to the action of the discrete group $\mathcal{G}$. The other must be projected out. The consequence is that orbifolds consistently truncate some of the massless modes of the theory.

For simplicity we will always consider (with only one exception) the case of toroidal orbifolds. The discrete groups will be products of $\mathbb{Z}_M$ groups, where M is some integer. These groups are formed by M elements: if Θ_M is the M-th root of the unity, i.e. if M is the smallest integer such that $(\Theta_M)^M = 1$, then $\mathbb{Z}_M$ is generated by Θ_M, $\mathbb{Z}_M = \{1, (\Theta_M)^1, \ldots, (\Theta_M)^{M-1}\}$. Since $\mathbb{Z}_2 = \{1, \Theta_2\}$, in this case, with a little abuse of notation, we can use $\mathbb{Z}_2$ to indicate the generator Θ_2.

- The first simple example will be the only non-toroidal orbifold that we discuss. Consider the orbifold $\mathcal{O} = \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$. Let R_1 and R_2 be two (non-aligned) vectors of $\mathbb{R}^2$. The group $\mathbb{Z} \times \mathbb{Z}$ is generated by the discrete translations

  $$g_1 = e^{2\pi i P_1 R_1}\,, \quad g_2 = e^{2\pi i P_2 R_2}\,. \tag{5.14}$$

  In this case there are no fixed points, thus the orbifold is a smooth manifold, the *two-dimensional torus* $\mathbb{T}^2$ of radii R_1 and R_2. The resulting symmetry are lattice translations, hence P_1 and P_2 are not arbitrary, but must be quantized:

  $$P_1 R_1 \in \mathbb{Z}\,, \quad P_2 R_2 \in \mathbb{Z}. \tag{5.15}$$

  This is a general feature: orbifolds are defined by discrete groups which act only on the coordinates, but consistency requires that the fields of the theory must respect the discrete symmetry.

- As a second example, consider the orbifold $\mathbb{T}^2/\mathbb{Z}_2$, where $\mathbb{T}^2$ is a two-dimensional torus of radii R_1 and R_2, obtained as in the previous example. Since the action of $\mathbb{Z}_2$ is involutory, it acts as a rotation by π about the origin. We can represent its action on coordinates y^1, y^2 on the plane as

  $$\mathbb{Z}_2 : (y^1, y^2) \to (-y^1, -y^2)\,, \tag{5.16}$$

  thus there are four fixed points: $(0,0)$, $(R_1/2, 0)$, $(0, R_2/2)$ and $(R_1/2, R_2/2)$. The physical states must be invariant with respect to $\mathbb{Z}_2$, thus if, for example, we have a two-dimensional metric g_{ab}, then all of its components are invariant under $\mathbb{Z}_2$, but a vector field A_a cannot be invariant, therefore it must be projected out. This shows how to consistently eliminate the fields which are incompatible with the orbifold discrete symmetry, even though the projection itself does not act on the fields, but only on the coordinates.

- Another interesting example is the orbifold $\mathbb{T}^6/\mathbb{Z}_3$. The $\mathbb{Z}_3$ group generator has the property $(\Theta_3)^3 = 1$, which is satisfied by $\Theta_3 = e^{\frac{2\pi}{3}i}$. The generators are then complex. This suggests us to factorize the six-dimensional torus in three two-tori,

$$\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2\,, \tag{5.17}$$

and to describe each two-torus with a complex coordinate. We denote these complex coordinates on $\mathbb{T}^6$ by (Z^1, Z^2, Z^3).

The action of Θ_3 on the complex coordinates can be represented as

$$\Theta_3 : (Z^1, Z^2, Z^3) \to e^{\frac{2\pi}{3}i}(Z^1, Z^2, Z^3)\,. \tag{5.18}$$

In each two-torus there are three fixed points, corresponding to the three-roots of the unity, thus there are $3^3 = 27$ total fixed points. Θ_3 is now an element of $SU(3)$, hence the orbifold $\mathbb{T}^6/\mathbb{Z}_3$ has discrete holonomy in $SU(3)$, i.e. it is the singular limit of a Calabi-Yau manifold. The physical states must be invariant under the action of Θ_3, but a detailed description goes beyond our purposes.

- The last two examples will be relevant for our specific examples of supergravity compactifications. The fields of the various supergravity theories which are compatible with these discrete symmetries will be studied later, but we can begin with some geometry. Consider the orbifold $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2')$. The discrete group is $\mathbb{Z}_2 \times \mathbb{Z}_2' = \{1, \mathbb{Z}_2, \mathbb{Z}_2', \mathbb{Z}_2\mathbb{Z}_2'\}$. We can represent the action of the generators $\mathbb{Z}_2$, $\mathbb{Z}_2'$, $\mathbb{Z}_2\mathbb{Z}_2'$ on the six-torus, such that we can factorize the six-torus as $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$, leaving as in the previous case three invariant two-tori. If we introduce three complex coordinates on $\mathbb{T}^6$, denoted by (Z^1, Z^2, Z^3), we can choose the generators to act on the coordinates as

$\mathbb{Z}_2$	:	$(Z^1, Z^2, Z^3) \to (-Z^1, -Z^2, +Z^3)$
$\mathbb{Z}_2'$	:	$(Z^1, Z^2, Z^3) \to (+Z^1, -Z^2, -Z^3)$
$\mathbb{Z}_2\mathbb{Z}_2'$	:	$(Z^1, Z^2, Z^3) \to (-Z^1, +Z^2, -Z^3)$

where the combined action is consistent with the single definitions.

Fixed points can be found in analogy with the case $\mathbb{T}^2/\mathbb{Z}_2$. Every generator of $\mathbb{Z}_2 \times \mathbb{Z}_2'$ reverses the orientation of two two-tori, leaving the third unaltered. Each two-torus on which a symmetry generator acts non-trivially has 4 fixed points, thus every symmetry generator has 16 invariant fixed two-tori. In total, there are then 48 fixed two-tori.

- Our last example is a straightforward extension of the last case: $\mathbb{T}^7/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. It can be simply thought as the product $(\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2')) \times \mathbb{S}^1$. In this case we introduce complex coordinates (Z^1, Z^2, Z^3) and a real coordinate on the extra-dimension. The only geometrical difference from the previous case is that the fixed points are now three-tori, instead of two-tori.

In these examples we have learnt how to describe orbifolds from a geometrical point of view. We have also seen in an explicit example how an orbifold can truncate some fields in a theory: the orbifold acts only on the coordinates which describe a manifold, but the fields of the theory must be consistent with it, or projected out. In the next section we will study many examples of this mechanism.

5.1.3 Orientifolds

The orientifold is a particular operation that can be defined in type II string theories. Since we are interested only in supergravity theories, we will give just give a description of the corresponding operation at the level of the effective field theory.

For our purposes, an orientifold projection will correspond to modding out the four-dimensional fields of a type II theory that are not invariant under a parity operation, acting both on the internal spatial coordinates and on the fields. The fields will thus be characterized by an intrinsic orientifold parity, and the combined action of the orientifold parity on the fields and on their space-time indices will correspond to a symmetry of the action.

The $(p+1)$-dimensional subspaces of the ten space-time dimensions which are invariant with respect to the orientifold projection are called Op-planes. In particular, O9-planes are space-time-filling invariant planes. These invariant planes are determined, then, by the geometrical action of the orientifold projection on the space-time coordinates.

There are other aspects of Op-planes that are relevant in string theory, but for our purposes it will be sufficient to understand their geometrical role in relation with the orientifold parity, which will be used in our explicit examples to halve the number of supersymmetries and to project out some of the massless states.

5.2 Examples of compactifications on $\mathbb{T}^k/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

In the following we will consider the effective four-dimensional supergravities corresponding to compactifications on the orbifold $\mathbb{T}^k/(\mathbb{Z}_2 \times \mathbb{Z}_2')$, $k = 6, 7$, of the higher-dimensional supergravity theories presented in Chapter 4.

We choose to split the space-time coordinates as in Sec. 5.1.1. Referring to the complex coordinates on the 6-torus introduced in Sec 5.1.2, we define the orbifold action through the identification

$$Z^1 \equiv x^5 + ix^6\,, \quad Z^2 \equiv x^7 + ix^8\,, \quad Z^3 \equiv x^9 + ix^{10}\,. \tag{5.19}$$

Explicitly, in terms of the real coordinates on the torus,

$$\boxed{\begin{array}{ccccc} \mathbb{Z}_2 & : & (x^5, x^6, x^7, x^8, x^9, x^{10}) & \to & (-x^5, -x^6, -x^7, -x^8, +x^9, +x^{10}) \\ \mathbb{Z}_2' & : & (x^5, x^6, x^7, x^8, x^9, x^{10}) & \to & (+x^5, +x^6, -x^7, -x^8, -x^9, -x^{10}) \\ \mathbb{Z}_2\mathbb{Z}_2' & : & (x^5, x^6, x^7, x^8, x^9, x^{10}) & \to & (-x^5, -x^6, +x^7, +x^8, -x^9, -x^{10}) \end{array}}$$

while x^{11} remains unchanged, when considered.

This identifies three planes (x^5, x^6), (x^7, x^8) and (x^9, x^{10}), or more precisely three invariant two-tori $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$.

Consider now the D-dimensional metric, $D = 10, 11$. The internal components of the metric which do not correspond to the three two-tori are not invariant under all of the generators $\mathbb{Z}_2$, $\mathbb{Z}_2'$ and $\mathbb{Z}_2\mathbb{Z}_2'$, hence must be projected out. We can parametrize the remaining components, which belong to three 2×2 blocks, as

$$G_{MN} = \begin{cases} blockdiag\,(\hat{s}^{-1}\widetilde{G}_{\mu\nu}, G_{i_1j_1}, G_{i_2j_2}, G_{i_3j_3}) & D = 10 \\ blockdiag\,(\hat{s}^{-1}\widetilde{G}_{\mu\nu}, G_{i_1j_1}, G_{i_2j_2}, G_{i_3j_3}, \hat{v}) & D = 11 \end{cases} , \tag{5.20}$$

where we displayed the Weyl rescaling parameter $\hat{s}^{-1}$, which will be determined later. Since G_{ij} is symmetric, we can parametrize the 2×2 blocks $G_{i_Aj_A}$, $A = 1, 2, 3$, corresponding to the three invariant two-tori, with nine real scalar fields

$$G_{i_Aj_A} = \frac{\hat{t}_A}{\hat{u}_A} \begin{pmatrix} (\hat{u}_A^2 + \hat{\nu}_A^2) & \hat{\nu}_A \\ \hat{\nu}_A & 1 \end{pmatrix} . \tag{5.21}$$

The determinant of each 2×2 block is

$$\det G_{i_Aj_A} = \hat{t}_A^2 , \tag{5.22}$$

so that the k-dimensional determinant of the internal components of the metric becomes

$$G_k = \begin{cases} \left(\hat{t}_1\hat{t}_2\hat{t}_3\right)^2 & D = 10 \\ \left(\hat{t}_1\hat{t}_2\hat{t}_3\right)^2 \hat{v} & D = 11 \end{cases} . \tag{5.23}$$

Finally, the inverse of $G_{i_Aj_A}$ is:

$$G^{i_Aj_A} = \frac{1}{\hat{t}_A\hat{u}_A} \begin{pmatrix} 1 & -\hat{\nu}_A \\ -\hat{\nu}_A & (\hat{u}_A^2 + \hat{\nu}_A^2) \end{pmatrix} . \tag{5.24}$$

We will always use this parametrization for the massless fields coming from the metric.

This allows us to determine the Weyl rescaling parameter, which is different in $D = 10$ and in $D = 11$.

- In $D = 10$, the Einstein-Hilbert term in the $d = 4$ reduced action (5.13) is

$$\frac{1}{2\kappa_4^2} \int d^4x \sqrt{-\widetilde{G}_4}\,\sqrt{G_6}\,\left(e^{-2\Phi}\hat{s}^{-1}\right)\,\widetilde{R}_4 , \tag{5.25}$$

then, to go to the Einstein frame, we must require

$$\sqrt{G_6}\,e^{-2\Phi}\hat{s}^{-1} = 1 . \tag{5.26}$$

The square root of the metric moduli determinant is, according to (5.23),

$$\sqrt{G_6} = \hat{t}_1\hat{t}_2\hat{t}_3 , \tag{5.27}$$

hence

$$\hat{s} = e^{-2\Phi}\hat{t}_1\hat{t}_2\hat{t}_3 \quad \Leftrightarrow \quad e^{-2\Phi} = \frac{\hat{s}}{\hat{t}_1\hat{t}_2\hat{t}_3}\,, \tag{5.28}$$

so that

$$\begin{aligned}
&\sqrt{-\widetilde{G}_4}\Big[-\frac{3}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) + \\
&+4\left(\partial_\mu\left(\Phi-\frac{1}{4}\log G_6\right)\right)\left(\partial^\mu\left(\Phi-\frac{1}{4}\log G_6\right)\right)\Big] = \\
= \quad & -\frac{1}{2}\sqrt{-\widetilde{G}_4}\,\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s})\,.
\end{aligned} \tag{5.29}$$

- In $D=11$, the Einstein-Hilbert term in the reduced action (5.13) is

$$\frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\hat{t}_1\hat{t}_2\hat{t}_3\sqrt{\hat{v}}\,\hat{s}^{-1}\,\widetilde{R}_4\,. \tag{5.30}$$

Since the square root of the determinant of the internal metric is, as can be seen from (5.23),

$$\sqrt{G_7} = \hat{t}_1\hat{t}_2\hat{t}_3\,\sqrt{\hat{v}} \tag{5.31}$$

and there is no dilaton ($\Phi = 0$), to go to the Einstein frame we require

$$\hat{t}_1\hat{t}_2\hat{t}_3\sqrt{\hat{v}}\,\hat{s}^{-1} = 1 \quad \Longleftrightarrow \quad \sqrt{\hat{v}} = \frac{\hat{s}}{\hat{t}_1\hat{t}_2\hat{t}_3}\,, \tag{5.32}$$

so that

$$\begin{aligned}
&\sqrt{-\widetilde{G}_4}\Big[-\frac{3}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) + \frac{1}{4}\,(\partial_\mu\log G_7)\,(\partial^\mu\log G_7)\Big] = \\
= \quad & \sqrt{-\widetilde{G}_4}\Big[-\frac{1}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s})\Big]\,,
\end{aligned} \tag{5.33}$$

which has the same form as (5.29).

In both cases, the reduced action (5.13) specializes into

$$\begin{aligned}
S \quad = \quad & \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\sqrt{G_k}\,\left(e^{-2\Phi}\hat{s}^{-1}\right)\,\Big[\widetilde{R}_4 - \frac{1}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) + \\
&-\frac{1}{4}G^{mn}G^{pq}\,\left((\partial_\mu G_{mp})(\partial^\mu G_{nq}) + (\partial_\mu B_{mp})(\partial^\mu B_{nq})\right) + \\
&-\frac{1}{12}\,\hat{s}^2\,H_{\mu\nu\rho}H^{\mu\nu\rho} + \\
&-\frac{1}{2(p!)}\hat{s}^2 e^{2\Phi}G^{m_4n_4}\cdots G^{m_{p+1}n_{p+1}}(F_{\mu\nu\rho m_4\cdots m_{p+1}})\,(F^{\mu\nu\rho}{}_{n_4\cdots n_{p+1}}) + \\
&-\frac{1}{2}e^{2\Phi}\,\frac{1}{p!}G^{m_2n_2}\cdots G^{m_{p+1}n_{p+1}}(\partial_\mu A_{m_2\cdots m_{p+1}})\,(\partial^\mu A_{n_2\cdots n_{p+1}})\Big]\,.
\end{aligned} \tag{5.34}$$

which will be our starting point for four-dimensional compactifications of supergravity theories.

5.2.1 $\mathcal{N}=1$ supergravity (without Yang-Mills sector)

Consider the action (4.36). In this case we obtain a $N=1$ theory in $d=4$ just compactifying on $\mathbb{T}^6/(\mathbb{Z}_2\times\mathbb{Z}_2)$, because the orbifold action $\mathbb{Z}_2\times\mathbb{Z}_2$ reduces the supersymmetries in four dimensions from 4 to 1, with respect to toroidal compactification.

Besides the four-dimensional metric $\widetilde{G}_{\mu\nu}$, the only bosonic fields of the effective four-dimensional theory surviving the orbifold projection are ($A=1,2,3$):

Fields in $\mathcal{N}=1$ supergravity		
Dilaton	:	Φ
Metric moduli	:	$\hat{t}_A$, $\hat{u}_A$, $\hat{\nu}_A$
NSNS two-form $B_{(2)}$	:	B_{56}, B_{78}, B_{910}, $B_{\mu\nu}\leftrightarrow\sigma$

where we took notice of the fact that the field $B_{\mu\nu}$ is dual to a pseudoscalar σ in four dimensions.

Considering only the terms which contain the massless fields listed above, the supergravity action (4.36) compactified on $\mathbb{T}^6/(\mathbb{Z}_2\times\mathbb{Z}_2)$ assumes the form

$$\begin{aligned} S &= \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4-\frac{1}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s})+ \\ &\quad -\frac{1}{4}G^{mn}G^{pq}\,\left((\partial_\mu G_{mp})(\partial^\mu G_{nq})+(\partial_\mu B_{mp})(\partial^\mu B_{nq})\right)-\frac{1}{12}\,\hat{s}^2 H_{\mu\nu\rho}H^{\mu\nu\rho}\Big]\,. \end{aligned} \tag{5.35}$$

We now compute the explicit form of the reduced action (5.35) in terms of the massless fields.

- The terms involving only the internal metric components give, after a little algebra,

$$\begin{aligned} &-\frac{1}{4}\sqrt{-\widetilde{G}_4}\,G^{mn}G^{pq}\,(\partial_\mu G_{mp})(\partial^\mu G_{nq}) = \\ &= -\frac{1}{2}\sqrt{-\widetilde{G}_4}\,\sum_{A=1}^{3}\left[\frac{(\partial_\mu\hat{t}_A)(\partial^\mu\hat{t}_A)}{\hat{t}_A^2}+\frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A)+(\partial_\mu\hat{\nu}_A)(\partial^\mu\hat{\nu}_A)}{\hat{u}_A^2}\right]\,. \end{aligned} \tag{5.36}$$

- Those arising from the internal components of the 2-form are

$$\begin{aligned} &-\frac{1}{4}\sqrt{-\widetilde{G}_4}\,G^{mn}G^{pq}\,(\partial_\mu B_{mp})(\partial^\mu B_{nq}) = \\ &= -\frac{1}{2}\sqrt{-\widetilde{G}_4}\,\left[\frac{(\partial_\mu B_{56})(\partial^\mu B_{56})}{\hat{t}_1^2}+\frac{(\partial_\mu B_{78})(\partial^\mu B_{78})}{\hat{t}_2^2}+\frac{(\partial_\mu B_{910})(\partial^\mu B_{910})}{\hat{t}_3^2}\right]\,. \end{aligned}$$

This is the only place where the fields B_{56}, B_{78} and B_{910} appear. Moreover, they appear only in terms with two derivatives, i.e. kinetic terms. Those written above are the kinetic terms of three *real* scalar fields in a $N=1$ supergravity theory. We know from (2.43) that they should be accompanied by the kinetic terms of three other real scalar fields, to reconstruct altogether the standard supergravity form of the kinetic terms for three complex scalar fields belonging to three chiral multiplets. The field-dependent factor $\frac{1}{\hat{t}_A^2}$ must therefore appear also in the kinetic term of the "partner" scalar field. For this reason, we define

$$\tau_1 \equiv B_{56}\,, \quad \tau_2 \equiv B_{78}\,, \quad \tau_3 \equiv B_{910}\,, \tag{5.37}$$

and we find that the "partner" scalar field of τ_A is $\hat{t}_A$, $A=1,2,3$.

Duality in $d=4$ implies that the 2-form $B_{\mu\nu}$ is dual to a 0-form σ, according to

$$\int d^4x \sqrt{-\widetilde{G}_4}\,\left[-\frac{1}{12}\hat{s}^2 H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \leftrightarrow -\frac{1}{2}\int d^4x \sqrt{-\widetilde{G}_4}\,\left[\hat{s}^{-2}\,(\partial_\mu\sigma)(\partial^\mu\sigma)\right]\,, \tag{5.38}$$

where we remembered that the Hodge star contains the completely antisymmetric symbol in four dimensions, which provides for a factor $\hat{s}^{-4}$. Again, this is the only place where the scalar field σ appears, under derivative, so this is an axion. We find that the only other term involving $\hat{s}$ is of the form $\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s})$.

We have a total of 14 real scalar fields, which should correspond to seven complex scalar fields. Collecting all the previous results, for the reduced action (5.35) we obtain the expression

$$\begin{aligned} S &= \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4 - \frac{1}{2}\frac{(\partial_\mu\hat{s})(\partial^\mu\hat{s})+(\partial_\mu\sigma)(\partial^\mu\sigma)}{\hat{s}^2} + \\ &\quad - \frac{1}{2}\sum_{A=1}^{3}\frac{(\partial_\mu\hat{t}_A)(\partial^\mu\hat{t}_A)+(\partial_\mu\tau_A)(\partial^\mu\tau_A)}{\hat{t}_A^2} + \\ &\quad - \frac{1}{2}\sum_{A=1}^{3}\frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A)+(\partial_\mu\hat{\nu}_A)(\partial^\mu\hat{\nu}_A)}{\hat{u}_A^2}\Big]\,, \end{aligned} \tag{5.39}$$

which shows that the 14 real scalar fields can be cast as the kinetic terms for 7 complex scalar fields, according to (2.43). For uniformity with the following examples, we make the trivial redefinitions:

$$s \equiv \hat{s}\,, \quad t_A \equiv \hat{t}_A\,, \quad u_A \equiv \hat{u}_A\,, \quad \nu_A \equiv \hat{\nu}_A\,, \tag{5.40}$$

for $A=1,2,3$.

Then the seven complex scalar fields of the reduced theory are:

$$\begin{aligned} S &= s + i\sigma\,, \\ T_A &= t_A + i\tau_A\,, \\ U_A &= u_A + i\nu_A\,, \end{aligned} \tag{5.41}$$

and the supergravity action is characterized by a Kähler potential

$$K = -\log(S+\overline{S}) - \sum_{A=1}^{3}\log(T_A+\overline{T}_A) - \sum_{B=1}^{3}\log(U_B+\overline{U}_B)\,. \tag{5.42}$$

We call the complex scalars *main* moduli, their real parts *geometrical* moduli and their imaginary parts *axions*. The geometrical moduli are related to the size and the shape of the torus: in particular, the t_A moduli describe the volume of the torus, as can be seen from (5.27), and we choose them to be positive definite. The u_B moduli describe the ellipticity of each two-torus in the decomposition of the six-torus. The name axions is justified by the fact that, because of the underlying local invariances in the higher-dimensional theory, the corresponding fields appear only under derivative, hence they possess an axionic shift symmetry, e.g. $\sigma \to \sigma + const$.

5.2.2 IIA with O6 orientifolds

Consider now type IIA supergravity, described by the action (4.21). In this case, the compactification on the orbifold $\mathbb{T}^6/(\mathbb{Z}_2\times\mathbb{Z}_2)$ gives a $N=2$ four-dimensional effective supergravity. To obtain a $N=1$, $d=4$ theory, we introduce an orientifold projection $\mathcal{R}$. This is the first example of orientifold that we encounter, so we will study it in detail.

We begin by studying how the orientifold acts on the coordinates:

$$\mathcal{R}: \quad (x^5,x^6,x^7,x^8,x^9,x^{10}) \;\to\; (-x^5,+x^6,-x^7,+x^8,-x^9,+x^{10})\,. \tag{5.43}$$

The combined action of the orbifold and orientifold projection is then[3]:

Combined orbifold/orientifold actions							
	:	x^5	x^6	x^7	x^8	x^9	x^{10}
$\mathcal{R}$	:	$-$	$+$	$-$	$+$	$-$	$+$
$\mathbb{Z}_2\mathcal{R}$	:	$+$	$-$	$+$	$-$	$-$	$+$
$\mathbb{Z}_2'\mathcal{R}$	:	$-$	$+$	$+$	$-$	$+$	$-$
$\mathbb{Z}_2\mathbb{Z}_2'\mathcal{R}$	:	$+$	$-$	$-$	$+$	$+$	$-$

Each of the previous combinations leaves seven coordinates invariant, the four external space-time coordinates (which have not been displayed since the orientifold action is trivial on them) and three internal coordinates. This means that there are four invariant planes, each filling one time and six spatial dimensions, i.e. there are four O6-planes which span the internal coordinates

$$(6810)\,,\,(5710)\,,\,(679)\,,\,(589)\,.$$

The orientifold action on the fields, which can be motivated by string theory arguments, and can be checked to define, together with its action on the coordinates, a symmetry of the higher-dimensional supergravity theory, is:

[3] We use a short-hand notation in which we only write the sign of the internal components after the transformations.

Orientifold action on the fields		
Φ	$\rightarrow$	$+\Phi$
B	$\rightarrow$	$-B$
G	$\rightarrow$	$+G$
$C_{(1)}$	$\rightarrow$	$-C_{(1)}$
$C_{(3)}$	$\rightarrow$	$+C_{(3)}$

The only fields of the theory compatible with both the orbifold and the orientifold actions are ($A = 1, 2, 3$):

Fields in type IIA with O6	
Dilaton	Φ
Metric moduli	$\hat{t}_A$, $\hat{u}_A$
$B_{(2)}$	B_{56}, B_{78}, B_{910}
$C_{(3)}$	C_{5710}, C_{589}, C_{679}, C_{6810}

In this case the off-diagonal terms of the metric are projected out and the parametrization (5.21) reduces to

$$G_{i_A j_A} = \begin{pmatrix} \hat{t}_A \hat{u}_A & 0 \\ 0 & \frac{\hat{t}_A}{\hat{u}_A} \end{pmatrix} \quad , \qquad A = 1, 2, 3. \tag{5.44}$$

Now we can proceed with the dimensional reduction of the bosonic action. From the general formula (5.34), and keeping only the fields relevant for the present case, we get

$$\begin{aligned} S \;=\; & \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-\widetilde{G}_4} \Big[\widetilde{R}_4 - \frac{1}{2} \hat{s}^{-2} (\partial_\mu \hat{s})(\partial^\mu \hat{s}) + \\ & - \frac{1}{4} G^{mn} G^{pq} \left((\partial_\mu G_{mp})(\partial^\mu G_{nq}) + (\partial_\mu B_{mp})(\partial^\mu B_{nq}) \right) + \\ & - \frac{1}{12} e^{2\Phi} G^{mn} G^{pq} G^{rs} (\partial_\mu C_{mpr}) (\partial^\mu C_{nqs}) \Big] . \end{aligned} \tag{5.45}$$

We express this action in terms of the fields compatible with the orbifold and the orientifold.

- The terms in the action involving only the internal metric components give

$$\begin{aligned} & -\frac{1}{4} \sqrt{-\widetilde{G}_4}\, G^{mn} G^{pq} (\partial_\mu G_{mp})(\partial^\mu G_{nq}) = \\ = \; & -\frac{1}{2} \sqrt{-\widetilde{G}_4} \sum_{A=1}^{3} \left[\frac{(\partial_\mu \hat{u}_A)(\partial^\mu \hat{u}_A)}{\hat{u}_A^2} + \frac{(\partial_\mu \hat{t}_A)(\partial^\mu \hat{t}_A)}{\hat{t}_A^2} \right] . \end{aligned}$$

- Those involving the internal components of the 2-form are

$$
\begin{aligned}
& -\frac{1}{4}\sqrt{-\widetilde{G}_4}\, e^{2\Phi}\, G^{mn}G^{pq}\,(\partial_\mu B_{mp})(\partial^\mu B_{nq}) = \\
= \quad & -\frac{1}{2}\sqrt{-\widetilde{G}_4}\left[\frac{1}{\hat{t}_1^2}\,(\partial_\mu B_{56})(\partial^\mu B_{56}) + \frac{1}{\hat{t}_2^2}\,(\partial_\mu B_{78})(\partial^\mu B_{78}) + \right. \\
& \left. + \frac{1}{\hat{t}_3^2}\,(\partial_\mu B_{910})(\partial^\mu B_{910})\right] .
\end{aligned}
$$

- Finally, the terms involving the internal components of the 3-form are

$$
\begin{aligned}
& -\frac{1}{12}\sqrt{-\widetilde{G}_4}\, e^{2\Phi}\, G^{mn}G^{pq}G^{rs}\,(\partial_\mu C_{mpr})(\partial^\mu C_{nqs}) = \\
= \quad & -\frac{1}{2}\sqrt{-\widetilde{G}_4}\left(\frac{e^{2\Phi}}{\hat{t}_1\hat{t}_2\hat{t}_3}\right)\left[\frac{\hat{u}_3}{\hat{u}_1\hat{u}_2}\,(\partial_\mu C_{5710})(\partial^\mu C_{5710}) + \right. \\
& + \frac{\hat{u}_2}{\hat{u}_1\hat{u}_3}\,(\partial_\mu C_{589})(\partial^\mu C_{589}) + \\
& \left. + \frac{\hat{u}_1}{\hat{u}_2\hat{u}_3}\,(\partial_\mu C_{679})(\partial^\mu C_{679}) + \hat{u}_1\hat{u}_2\hat{u}_3\,(\partial_\mu C_{6810})(\partial^\mu C_{6810})\right]
\end{aligned}
$$

Collecting the results, we obtain for the reduced action the following expression:

$$
\begin{aligned}
S \quad = \quad & \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\left[\widetilde{R}_4 - \frac{1}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) + \right. \\
& - \frac{1}{2}\sum_{A=1}^{3}\left[\frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A)}{\hat{u}_A^2} + \frac{(\partial_\mu\hat{t}_A)(\partial^\mu\hat{t}_A)}{\hat{t}_A^2}\right] + \\
& - \frac{1}{2}\left[\frac{(\partial_\mu B_{56})(\partial^\mu B_{56})}{\hat{t}_1^2} + \frac{(\partial_\mu B_{78})(\partial^\mu B_{78})}{\hat{t}_2^2} + \frac{(\partial_\mu B_{910})(\partial^\mu B_{910})}{\hat{t}_3^2}\right] + \\
& - \frac{1}{2}\left(\frac{e^{2\Phi}}{\hat{t}_1\hat{t}_2\hat{t}_3}\right)\left[\frac{\hat{u}_3}{\hat{u}_1\hat{u}_2}\,(\partial_\mu C_{5710})(\partial^\mu C_{5710}) + \frac{\hat{u}_2}{\hat{u}_1\hat{u}_3}\,(\partial_\mu C_{589})(\partial^\mu C_{589}) + \right. \\
& \left.\left. + \frac{\hat{u}_1}{\hat{u}_2\hat{u}_3}\,(\partial_\mu C_{679})(\partial^\mu C_{679}) + \hat{u}_1\hat{u}_2\hat{u}_3\,(\partial_\mu C_{6810})(\partial^\mu C_{6810})\right]\right] .
\end{aligned} \tag{5.46}
$$

In this case the fields B_{56}, B_{78}, B_{910} and C_{5710}, C_{589}, C_{679}, C_{6810} appear only in terms containing two derivatives, which must be part of the kinetic term for a complex scalar field in supergravity $N = 1$ theories. Therefore they must be axions, that we denote by

$$
\tau_1 \equiv B_{56}\,, \quad \tau_2 \equiv B_{78}\,, \quad \tau_3 \equiv B_{910}\,, \tag{5.47}
$$

$$
\nu_1 \equiv -C_{679}\,, \quad \nu_2 \equiv -C_{589}\,, \quad \nu_3 \equiv -C_{5710}\,, \quad \sigma \equiv C_{6810}\,. \tag{5.48}
$$

Each of the functions multiplying the terms in τ_A depends on a single scalar field, which appears in the second line as it should be in a $N = 1$, $d = 4$ supergravity theory: the fields $\hat{t}_A$ are yet in the usual form.

The field-dependent factors multiplying the terms containing the fields ν_A and σ, instead, depend on more than one field. If we are in a $N=1$ supergravity theory, we then must be able to make proper redefinitions of the fields, as to find these new fields in the correct form.

Remembering that

$$\frac{e^{2\Phi}}{\hat{t}_1\hat{t}_2\hat{t}_3} = \hat{s}^{-1}\,, \tag{5.49}$$

we make the following non-linear redefinitions of the fields:

$$\frac{1}{u_A^2} \equiv \frac{\hat{u}_A^2}{\hat{s}\hat{u}_1\hat{u}_2\hat{u}_3}\,, \qquad A=1,2,3, \tag{5.50}$$

$$\frac{1}{s^2} \equiv \frac{\hat{u}_1\hat{u}_2\hat{u}_3}{\hat{s}}\,. \tag{5.51}$$

Inverting, we obtain

$$\hat{u}_A = \sqrt{\frac{u_1u_2u_3}{su_A^2}}\,, \qquad \hat{s} = \sqrt{su_1u_2u_3}\,. \tag{5.52}$$

Substituting these expression in the relevant terms of (5.46), we have:

- terms related to the dilaton

$$\begin{aligned}
&-\frac{1}{2}\sqrt{-\widetilde{G}_4}\,\hat{s}^{-2}(\partial_\mu\hat{s})(\partial^\mu\hat{s}) = \\
=\;& -\frac{1}{8}\sqrt{-\widetilde{G}_4}\left[\frac{1}{s^2}(\partial_\mu s)(\partial^\mu s) + \sum_{A=1}^{3}\frac{1}{u_A^2}(\partial_\mu u_A)(\partial^\mu u_A) + \right. \\
&\left. + 2\sum_{A=1}^{3}\frac{(\partial_\mu s)(\partial^\mu u_A)}{su_A} + 2\sum_{A<B}\frac{(\partial_\mu u_A)(\partial^\mu u_B)}{u_Au_B}\right],
\end{aligned}$$

- terms related to the u_B metric moduli

$$\begin{aligned}
&-\frac{1}{2}\sqrt{-\widetilde{G}_4}\sum_{A=1}^{3}\frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A)}{\hat{u}_1^2} = \\
=\;& -\frac{3}{8}\sqrt{-\widetilde{G}_4}\sum_{A=1}^{3}\frac{(\partial_\mu u_A)(\partial^\mu u_A)}{u_A^2} - \frac{3}{8}\sqrt{-\widetilde{G}_4}\,\frac{(\partial_\mu s)(\partial^\mu s)}{s^2} + \\
& + \frac{1}{4}\sqrt{-\widetilde{G}_4}\sum_{A<B}\frac{(\partial_\mu u_A)(\partial^\mu u_B)}{u_Au_B} + \frac{1}{4}\sqrt{-\widetilde{G}_4}\sum_{A=1}^{3}\frac{(\partial_\mu s)(\partial^\mu u_A)}{su_A}\,.
\end{aligned}$$

The reduced action (5.46) becomes then

$$\begin{aligned} S &= \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-\widetilde{G}_4}\Big[\widetilde{R}_4 - \frac{1}{2}\frac{(\partial_\mu s)(\partial^\mu s)+(\partial_\mu\sigma)(\partial^\mu\sigma)}{s^2} + \\ &\quad -\frac{1}{2}\sum_{A=1}^{3}\left[\frac{(\partial_\mu u_A)(\partial^\mu u_A)+(\partial_\mu\nu_A)(\partial^\mu\nu_A)}{u_A^2}\right] + \\ &\quad -\frac{1}{2}\sum_{A=1}^{3}\left[\frac{(\partial_\mu\hat{t}_A)(\partial^\mu\hat{t}_A)+(\partial_\mu\tau_A)(\partial^\mu\tau_A)}{\hat{t}_A^2}\right]\Big]. \end{aligned} \tag{5.53}$$

This has the usual form for the scalar kinetic term of a $N=1$, $d=4$ supergravity action. We make the last trivial redefinitions:

$$t_A \equiv \hat{t}_A\,, \qquad A=1,2,3\,. \tag{5.54}$$

We found that the reduced action describes, besides the supergravity multiplet, seven complex scalar fields:

$$S = s+i\sigma\,,\quad T_A = t_A + i\tau_A\,,\quad U_A = u_A + i\nu_A \tag{5.55}$$

and from (5.53) we obtain a Kähler potential

$$K = -\log(S+\overline{S}) - \sum_{A=1}^{3}\log(T_A+\overline{T}_A) - \sum_{B=1}^{3}\log(U_B+\overline{U}_B)\,. \tag{5.56}$$

The scheme will be the same in the following examples, the only differences being the redefinitions of the fields and the higher-dimensional origin of the 14 scalar degrees of freedom of the effective theory.

5.2.3 IIB with O3/O7 orientifolds

Consider type IIB supergravity, whose action is given by (4.30). As in the IIA case, we want to break the residual $N=2$ local supersymmetry which comes from the orbifold compactification down to a $N=1$ supersymmetry. In this case we introduce an orientifold projection which acts on the coordinates as

$$\mathcal{R}:\quad (x^5,x^6,x^7,x^8,x^9,x^{10}) \to (-x^5,-x^6,-x^7,-x^8,-x^9,-x^{10})\,. \tag{5.57}$$

In this case, we reflect all the internal dimensions. The combined action of the orbifold and orientifold projection is:

Combined orbifold/orientifold actions							
	:	x^5	x^6	x^7	x^8	x^9	x^{10}
$\mathcal{R}$	:	$-$	$-$	$-$	$-$	$-$	$-$
$\mathbb{Z}_2\mathcal{R}$	:	$+$	$+$	$+$	$+$	$-$	$-$
$\mathbb{Z}_2'\mathcal{R}$	:	$-$	$-$	$+$	$+$	$+$	$+$
$\mathbb{Z}_2\mathbb{Z}_2'\mathcal{R}$	:	$+$	$+$	$-$	$-$	$+$	$+$

We decided to respect the three two-tori analytical complex structure, since every combination of the generators acts in the same way on the corresponding two-planes.

Each of the previous combinations leaves either three or seven coordinates invariant. This corresponds to the existence of one invariant O3-plane (our four-dimensional space-time) and the three invariant O7-planes with internal coordinates:

$$(5678)\,,\ (78910)\,,\ (56910)\,.$$

The orientifold action on the fields is now

Orientifold action on the fields		
Φ	$\rightarrow$	$+\Phi$
B	$\rightarrow$	$-B$
G	$\rightarrow$	$+G$
$C_{(0)}$	$\rightarrow$	$+C_{(0)}$
$C_{(2)}$	$\rightarrow$	$-C_{(2)}$
$C_{(4)}$	$\rightarrow$	$+C_{(4)}$

The only fields of the theory compatible with both the orbifold and the orientifold actions are ($A = 1, 2, 3$):

Fields in type IIB with O3/O7	
Dilaton	Φ
Metric moduli	$\hat{t}_A$, $\hat{u}_A$, $\hat{\nu}_A$
RR 0-form	$C_{(0)}$
$C_{(4)}$	C_{5678}, C_{56910}, C_{78910}

The reduced action, deduced from the general formula (5.34), becomes then

$$\begin{aligned} S \;=\; & \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4 - \frac{1}{2}\hat{s}^{-2}\,(\partial_\mu \hat{s})(\partial^\mu \hat{s}) + \\ & - \frac{1}{4}G^{mn}G^{pq}\,(\partial_\mu G_{mp})(\partial^\mu G_{nq}) - \frac{1}{2}\,e^{2\Phi}\,(\partial_\mu C_{(0)})(\partial^\mu C_{(0)}) + \\ & - \frac{1}{4}\,e^{2\Phi}\,\frac{1}{4!}G^{m_1n_1}G^{m_2n_2}G^{m_3n_3}G^{m_4n_4}(\partial_\mu C_{m_1m_2m_3m_4})(\partial^\mu C_{n_1n_2n_3n_4})\Big]\,. \end{aligned} \tag{5.58}$$

Now we can compute the terms involving the fields surviving the orbifold and orientifold projections.

- The terms in the action involving only the internal components of the metric give

$$\begin{aligned} & -\frac{1}{4}\sqrt{-\widetilde{G}_4}\,G^{mn}G^{pq}\,(\partial_\mu G_{mp})(\partial^\mu G_{nq}) = \\ = \;& -\frac{1}{2}\sqrt{-\widetilde{G}_4}\,\sum_{A=1}^{3}\left[\frac{(\partial_\mu \hat{t}_A)(\partial^\mu \hat{t}_A)}{\hat{t}_A^2} + \frac{(\partial_\mu \hat{u}_A)(\partial^\mu \hat{u}_A) + (\partial_\mu \hat{\nu}_A)(\partial^\mu \hat{\nu}_A)}{\hat{u}_A^2}\right]\,. \end{aligned}$$

- The terms involving the internal components of the 4-form are

$$
\begin{aligned}
& -\frac{e^{2\Phi}}{4\cdot 4!}\sqrt{-\widetilde{G}_4}\,G^{m_1n_1}G^{m_2n_2}G^{m_3n_3}G^{m_4n_4}(\partial_\mu C_{m_1m_2m_3m_4})(\partial^\mu C_{n_1n_2n_3n_4}) = \\
= \;& -\frac{1}{4}\sqrt{-\widetilde{G}_4}\,e^{2\Phi}\left[\frac{1}{\hat{t}_1^2\hat{t}_2^2}\,(\partial_\mu C_{5678})(\partial^\mu C_{5678}) + \frac{1}{\hat{t}_1^2\hat{t}_3^2}\,(\partial_\mu C_{56910})(\partial^\mu C_{56910}) + \right. \\
& \left. + \frac{1}{\hat{t}_2^2\hat{t}_3^2}\,(\partial_\mu C_{78910})(\partial^\mu C_{78910})\right]. \qquad (5.59)
\end{aligned}
$$

We must remember that the 4-form is self-dual in the original theory. Thus we should include the dual fields

C_{5678}	$\leftrightarrow$	$C_{\mu\nu 910}$
C_{56910}	$\leftrightarrow$	$C_{\mu\nu 78}$
C_{78910}	$\leftrightarrow$	$C_{\mu\nu 56}$

but this would give exactly the same results as in (5.59), because of the self-duality constraint. This accounts for a factor 2 for these terms in the reduced action.

Collecting the previous results, eq. (5.58) takes the form:

$$
\begin{aligned}
S \;=\; & \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\left[\widetilde{R}_4 - \frac{1}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) + \right. \\
& -\frac{1}{2}\sum_{A=1}^{3}\left[\frac{(\partial_\mu\hat{t}_A)(\partial^\mu\hat{t}_A)}{\hat{t}_A^2} + \frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A) + (\partial_\mu\hat{\nu}_A)(\partial^\mu\hat{\nu}_A)}{\hat{u}_A^2}\right] + \\
& -\frac{1}{2}\left(\frac{e^{2\Phi}}{\hat{t}_1\hat{t}_2\hat{t}_3}\right)\left[\frac{\hat{t}_3}{\hat{t}_1\hat{t}_2}\,(\partial_\mu C_{5678})(\partial^\mu C_{5678}) + \frac{\hat{t}_2}{\hat{t}_1\hat{t}_3}\,(\partial_\mu C_{56910})(\partial^\mu C_{56910}) + \right. \\
& \left.\left. + \frac{\hat{t}_1}{\hat{t}_2\hat{t}_3}\,(\partial_\mu C_{78910})(\partial^\mu C_{78910}) + \hat{t}_1\hat{t}_2\hat{t}_3\,(\partial_\mu C_0)(\partial^\mu C_0)\right]\right]. \qquad (5.60)
\end{aligned}
$$

Again, we have the structure for the kinetic terms encountered in the IIA case. The axions, which appear only under derivative, are identified by

$$
\tau_1 \equiv C_{78910}\,,\;\; \tau_2 \equiv C_{56910}\,,\;\; \tau_3 \equiv C_{5678}\,, \quad \nu_A \equiv \hat{\nu}_A\,, \quad \sigma \equiv -C_{(0)}\,, \quad A = 1,2,3\,. \qquad (5.61)
$$

As in case IIA, we try to put the action in the usual form for the scalar kinetic terms of a $N = 1$, $d = 4$ supergravity theory. Recalling (5.28), we define

$$
\frac{1}{t_A^2} \equiv \frac{\hat{t}_A^2}{\hat{s}\hat{t}_1\hat{t}_2\hat{t}_3}\,, \qquad A = 1,2,3, \qquad (5.62)
$$

$$
\frac{1}{s^2} \equiv \frac{\hat{t}_1\hat{t}_2\hat{t}_3}{\hat{s}}\,. \qquad (5.63)
$$

Inverting:

$$\hat{t}_A = \sqrt{\frac{t_1 t_2 t_3}{s t_A^2}}\,, \quad \hat{s} = \sqrt{s t_1 t_2 t_3}\,. \tag{5.64}$$

When we substitute these expressions in the relevant terms of the action (5.60), we obtain:

- terms related to the dilaton

$$\begin{aligned} & -\frac{1}{2}\sqrt{-\widetilde{G}_4}\,\hat{s}^{-2}(\partial_\mu \hat{s})(\partial^\mu \hat{s}) = \\ = \;& \sqrt{-\widetilde{G}_4}\Big[-\frac{1}{8}\frac{1}{s^2}(\partial_\mu s)(\partial^\mu s) - \frac{1}{8}\sum_{A=1}^{3}\frac{1}{t_A^2}(\partial_\mu t_A)(\partial^\mu t_A) + \\ & -\frac{1}{4}\sum_{A=1}^{3}\frac{(\partial_\mu s)(\partial^\mu t_A)}{s t_A} - \frac{1}{4}\sum_{A<B}\frac{(\partial_\mu t_A)(\partial^\mu t_B)}{t_A t_B}\Big]\,, \end{aligned} \tag{5.65}$$

- terms related to the $\hat{t}_A$ geometric moduli

$$\begin{aligned} & -\frac{1}{2}\sqrt{-\widetilde{G}_4}\sum_{A=1}^{3}\frac{(\partial_\mu \hat{t}_A)(\partial^\mu \hat{t}_A)}{\hat{t}_A^2} = \\ = \;& \sqrt{-\widetilde{G}_4}\Big[-\frac{3}{8}\sum_{A=1}^{3}\frac{(\partial_\mu t_A)(\partial^\mu t_A)}{t_A^2} - \frac{3}{8}\frac{(\partial_\mu s)(\partial^\mu s)}{s^2} + \\ & +\frac{1}{4}\sum_{A<B}\frac{(\partial_\mu t_A)(\partial^\mu t_B)}{t_A t_B} + \frac{1}{4}\sum_{A=1}^{3}\frac{(\partial_\mu s)(\partial^\mu t_A)}{s t_A}\Big]\,. \end{aligned} \tag{5.66}$$

Finally, substituting (5.65) and (5.66) in the action (5.60), we get:

$$\begin{aligned} S \;=\; & \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\Big[\widetilde{R}_4 - \frac{1}{2}\frac{(\partial_\mu \hat{s})(\partial^\mu \hat{s}) + (\partial_\mu \sigma)(\partial^\mu \sigma)}{s^2} + \\ & -\frac{1}{2}\sum_{A=1}^{3}\Big[\frac{(\partial_\mu \hat{u}_A)(\partial^\mu \hat{u}_A) + (\partial_\mu \hat{\nu}_A)(\partial^\mu \hat{\nu}_A)}{\hat{u}_A^2} + \\ & + \frac{(\partial_\mu t_A)(\partial^\mu t_A) + (\partial_\mu \tau_A)(\partial^\mu \tau_A)}{t_A^2}\Big]\Big]\,. \end{aligned} \tag{5.67}$$

Finally, we identify

$$u_A \equiv \hat{u}_A\,, \quad s \equiv \hat{s}\,. \tag{5.68}$$

If we define seven complex scalar fields as

$$S = s + i\sigma\,, \quad T_A = t_A + i\tau_A\,, \quad U_A = u_A + i\nu_A\,, \tag{5.69}$$

then the reduced action (5.67) represents their kinetic terms, in a $N = 1$, $d = 4$ supergravity theory, with Kähler potential

$$K = -\log(S + \overline{S}) - \sum_{A=1}^{3} \log(T_A + \overline{T}_A) - \sum_{B=1}^{3} \log(U_B + \overline{U}_B)\,. \tag{5.70}$$

5.2.4 IIB with O5/O9 orientifolds

Consider type IIB supergravity, whose action is given by (4.30), and compactify it on the orbifold $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. As in the two previous cases, we want to break the residual $N = 2$ local supersymmetry which comes from the orbifold compactification down to a $N = 1$ supersymmetry. In this case we introduce an orientifold projection which acts on the coordinates as

$$\mathcal{R}: \quad (x^5, x^6, x^7, x^8, x^9, x^{10}) \;\rightarrow\; (+x^5, +x^6, +x^7, +x^8, +x^9, +x^{10})\,. \tag{5.71}$$

In this case, the orientifold does not change the sign of any of the ten-dimensional coordinates. The combined action of the orbifold and orientifold projection is:

Combined orbifold/orientifold actions							
	:	x^5	x^6	x^7	x^8	x^9	x^{10}
$\mathcal{R}$	:	$+$	$+$	$+$	$+$	$+$	$+$
$\mathbb{Z}_2\mathcal{R}$	:	$-$	$-$	$-$	$-$	$+$	$+$
$\mathbb{Z}_2'\mathcal{R}$	:	$+$	$+$	$-$	$-$	$-$	$-$
$\mathbb{Z}_2\mathbb{Z}_2'\mathcal{R}$	:	$-$	$-$	$+$	$+$	$-$	$-$

As in the previous case, we decided to respect the three two-tori complex structure, since every combination of the generators acts in the same way on the two coordinates which correspond to the same two-torus.

Each of the previous combinations leaves six or ten space-time coordinates invariant. This corresponds to the existence of one invariant O9-plane and three invariant O5-planes. Omitting as usual the four uncompactified space-time coordinates, they are identified by:

$$(5678910)\,,\ (910)\,,\ (56)\,,\ (78)\,.$$

The orientifold action on the fields of the theory is

Orientifold action on the fields		
Φ	$\rightarrow$	$+\Phi$
B	$\rightarrow$	$-B$
G	$\rightarrow$	$+G$
$C_{(0)}$	$\rightarrow$	$-C_{(0)}$
$C_{(2)}$	$\rightarrow$	$+C_{(2)}$
$C_{(4)}$	$\rightarrow$	$-C_{(4)}$

The only fields of the theory invariant under both the orbifold and the orientifold actions are ($A = 1, 2, 3$):

Fields in type IIB with O5/O9	
Dilaton	Φ
Metric moduli	$\hat{t}_A$, $\hat{u}_A$, $\hat{\nu}_A$
$C_{(2)}$	C_{56}, C_{78}, C_{910}, $C_{\mu\nu} \leftrightarrow \sigma$

Including only the fields compatible with the orbifold and orientifold conditions, the reduced action, deduced from the general formula (5.34), becomes

$$\begin{aligned} S &= \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4 - \frac{1}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) - \frac{1}{4}G^{mn}G^{pq}\,(\partial_\mu G_{mp})(\partial^\mu G_{nq}) + \\ &\quad - \frac{1}{4}\,e^{2\Phi}\,G^{mn}G^{pq}(\partial_\mu C_{mp})(\partial^\mu C_{nq}) - \frac{1}{12}\,e^{2\Phi}\,\hat{s}^2\,F_{\mu\nu\lambda}F^{\mu\nu\lambda}\Big]\,. \end{aligned} \tag{5.72}$$

We can express the reduced action in terms of the fields compatible with the orbifold and the orientifold.

- The term in the action involving only the internal components of the metric gives

$$\begin{aligned} &-\frac{1}{4}\sqrt{-\widetilde{G}_4}\,G^{mn}G^{pq}\,(\partial_\mu G_{mp})(\partial^\mu G_{nq}) = \\ &= -\frac{1}{2}\sqrt{-\widetilde{G}_4}\,\sum_{A=1}^{3}\left[\frac{(\partial_\mu\hat{t}_A)(\partial^\mu\hat{t}_A)}{\hat{t}_A^2} + \frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A) + (\partial_\mu\hat{\nu}_A)(\partial^\mu\hat{\nu}_A)}{\hat{u}_A^2}\right]\,. \end{aligned}$$

- The term involving the internal components of the 2-form is

$$\begin{aligned} &-\frac{1}{4}\sqrt{-\widetilde{G}_4}\,e^{2\Phi}\,G^{mn}G^{pq}(\partial_\mu C_{mp})(\partial^\mu C_{nq}) = \\ &= -\frac{1}{2}\sqrt{-\widetilde{G}_4}\,e^{2\Phi}\,\Big[\frac{(\partial_\mu C_{56})(\partial^\mu C_{56})}{\hat{t}_1^2} + \frac{(\partial_\mu C_{78})(\partial^\mu C_{78})}{\hat{t}_2^2} + \\ &\quad + \frac{(\partial_\mu C_{910})(\partial^\mu C_{910})}{\hat{t}_3^2}\Big] \end{aligned}$$

- Finally, we have a duality relation between the 2-form $C_{\mu\nu}$ and a scalar field, which we denote by σ:

$$\int d^4x\,\sqrt{-\widetilde{G}_4}\,\left[-\frac{1}{12}\hat{s}^2\,e^{2\Phi}\,F_{\mu\nu\lambda}F^{\mu\nu\lambda}\right] \leftrightarrow -\frac{1}{2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\left[\hat{s}^{-2}\,e^{-2\Phi}\,(\partial_\mu\sigma)(\partial^\mu\sigma)\right]\,.$$

The action (5.72) then takes the form:

$$\begin{aligned} S &= \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4 - \frac{1}{2}\frac{(\partial_\mu\hat{s})(\partial^\mu\hat{s})}{\hat{s}^2} - \frac{1}{2}e^{-2\Phi}\frac{(\partial_\mu\sigma)(\partial^\mu\sigma)}{\hat{s}^2} + \\ &\quad - \frac{1}{2}\sum_{A=1}^{3}\left[\frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A) + (\partial_\mu\hat{\nu}_A)(\partial^\mu\hat{\nu}_A)}{\hat{u}_A^2}\right] - \frac{1}{2}\sum_{A=1}^{3}\frac{(\partial_\mu\hat{t}_A)(\partial^\mu\hat{t}_A)}{\hat{t}_A^2} + \\ &\quad - \frac{1}{2}\,e^{2\Phi}\,\left[\frac{(\partial_\mu C_{56})(\partial^\mu C_{56})}{\hat{t}_1^2} + \frac{(\partial_\mu C_{78})(\partial^\mu C_{78})}{\hat{t}_2^2} + \frac{(\partial_\mu C_{910})(\partial^\mu C_{910})}{\hat{t}_3^2}\right]\Big]\,. \end{aligned} \tag{5.73}$$

The axions are:

$$\tau_1 \equiv C_{56}\,, \quad \tau_2 \equiv C_{78}\,, \quad \tau_3 \equiv C_{910}\,, \quad \nu_A \equiv \hat{\nu}_A\,, \tag{5.74}$$

and σ.

We try to put the action (5.73) in the standard form for the scalar kinetic terms of a $N=1$, $d=4$ supergravity theory through the definitions:

$$\frac{1}{t_A^2} \equiv \frac{e^{2\Phi}}{\hat{t}_A^2} = \frac{\hat{t}_1\hat{t}_2\hat{t}_3}{\hat{s}\hat{t}_A^2}\,, \qquad A=1,2,3, \tag{5.75}$$

$$\frac{1}{s^2} \equiv \frac{1}{\hat{s}\hat{t}_1\hat{t}_2\hat{t}_3}\,. \tag{5.76}$$

Inverting, we have:

$$\hat{t}_A = \sqrt{\frac{st_A^2}{t_1t_2t_3}}\,, \quad \hat{s} = \sqrt{st_1t_2t_3}\,. \tag{5.77}$$

When we substitute these expressions in the relevant terms of the action (5.73), we obtain:

- terms related to the dilaton

$$\begin{aligned} &-\frac{1}{2}\sqrt{-\widetilde{G}_4}\,\hat{s}^{-2}(\partial_\mu\hat{s})(\partial^\mu\hat{s}) = \\ = &\sqrt{-\widetilde{G}_4}\Big[-\frac{1}{8}\frac{1}{s^2}(\partial_\mu s)(\partial^\mu s) - \frac{1}{8}\sum_{A=1}^{3}\frac{1}{t_A^2}(\partial_\mu t_A)(\partial^\mu t_A) + \\ &-\frac{1}{4}\sum_{A=1}^{3}\frac{(\partial_\mu s)(\partial^\mu t_A)}{st_A} - \frac{1}{4}\sum_{A<B}\frac{(\partial_\mu t_A)(\partial^\mu t_B)}{t_At_B}\Big]\,, \end{aligned} \tag{5.78}$$

- terms related to the $\hat{t}_A$ geometric moduli

$$\begin{aligned} &-\frac{1}{2}\sqrt{-\widetilde{G}_4}\sum_{A=1}^{3}\frac{(\partial_\mu\hat{t}_A)(\partial^\mu\hat{t}_A)}{\hat{t}_A^2} = \\ = &\sqrt{-\widetilde{G}_4}\Big[-\frac{3}{8}\sum_{A=1}^{3}\frac{(\partial_\mu t_A)(\partial^\mu t_A)}{t_A^2} - \frac{3}{8}\frac{(\partial_\mu s)(\partial^\mu s)}{s^2} + \\ &+\frac{1}{4}\sum_{A<B}\frac{(\partial_\mu t_A)(\partial^\mu t_B)}{t_At_B} + \frac{1}{4}\sum_{A=1}^{3}\frac{(\partial_\mu s)(\partial^\mu t_A)}{st_A}\Big]\,. \end{aligned} \tag{5.79}$$

Substituting (5.78) and (5.79) into (5.73), we arrive at:

$$\begin{aligned} S &= \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4 - \frac{1}{2}\frac{(\partial_\mu\hat{s})(\partial^\mu\hat{s})+(\partial_\mu\sigma)(\partial^\mu\sigma)}{s^2} + \\ &\quad -\frac{1}{2}\sum_{A=1}^{3}\Big[\frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A)+(\partial_\mu\nu_A)(\partial^\mu\nu_A)}{\hat{u}_A^2} + \\ &\quad + \frac{(\partial_\mu t_A)(\partial^\mu t_A)+(\partial_\mu\tau_A)(\partial^\mu\tau_A)}{t_A^2}\Big]\Big]\,. \end{aligned} \tag{5.80}$$

Finally, we identify

$$u_A \equiv \hat{u}_A\,, \quad s \equiv \hat{s}\,. \tag{5.81}$$

Define seven complex scalar fields as

$$S = s + i\sigma\,, \quad T_A = t_A + i\tau_A\,, \quad U_A = u_A + i\nu_A\,. \tag{5.82}$$

The reduced action (5.80) represents their kinetic terms, in a $N=1$, $d=4$ supergravity theory, with Kähler potential

$$K = -\log(S+\overline{S}) - \sum_{A=1}^{3}\log(T_A+\overline{T}_A) - \sum_{B=1}^{3}\log(U_B+\overline{U}_B)\,. \tag{5.83}$$

5.2.5 M-theory ($\rightarrow$ IIA)

The first compactification of $D=11$ supergravity (4.11) is related to type IIA supergravity [34], so we will call it M-theory ($\rightarrow$ IIA). We will also consider the case related to heterotic string theory, which will be referred to as M-theory ($\rightarrow$ heterotic).

Dimensional reduction on $\mathbb{T}^7/(\mathbb{Z}_2\times\mathbb{Z}_2)$ gives $N=2$ supergravity in $d=4$. To eliminate the extra supersymmetry, we define a third orbifold parity $\mathcal{R}$, which acts on the coordinates as

$$\mathcal{R}:\quad (x^5,x^6,x^7,x^8,x^9,x^{10},x^{11}) \rightarrow (-x^5,+x^6,-x^7,+x^8,-x^9,+x^{10},-x^{11})\,. \tag{5.84}$$

We can easily compute the combined action of all the discrete symmetries:

Combined orbifold actions								
	:	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}
$\mathcal{R}$	:	$-$	$+$	$-$	$+$	$-$	$+$	$-$
$\mathbb{Z}_2\mathcal{R}$	:	$+$	$-$	$+$	$-$	$-$	$+$	$-$
$\mathbb{Z}_2'\mathcal{R}$	:	$-$	$+$	$+$	$-$	$+$	$-$	$-$
$\mathbb{Z}_2\mathbb{Z}_2'\mathcal{R}$	:	$+$	$-$	$-$	$+$	$+$	$-$	$-$

The action of $\mathcal{R}$ on the fields is trivial:

$\mathcal{R}$ **action on the fields**		
G	$\to$	$+G$
$A_{(3)}$	$\to$	$+A_{(3)}$

The only fields of the theory invariant under the full discrete group generated by $\mathbb{Z}_2$, $\mathbb{Z}_2'$ and $\mathcal{R}$ are ($A = 1, 2, 3$):

Fields in M-theory ($\to$ IIA)	
Metric moduli	$\hat{t}_A$, $\hat{u}_A$
$A_{(3)}$	A_{5611}, A_{5710}, A_{589}, A_{679}, A_{6810}, A_{7811}, A_{91011}

The reduced action, deduced from the general formula (5.34), takes the form

$$\begin{aligned} S \;=\; & \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\Big[\widetilde{R}_4 - \frac{1}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) + \\ & -\frac{1}{4}G^{mn}G^{pq}\,(\partial_\mu G_{mp})(\partial^\mu G_{nq}) - \frac{1}{12}G^{mn}G^{pq}G^{rs}\,(\partial_\mu A_{mpr})(\partial^\mu A_{nqs})\Big]\,. \end{aligned} \tag{5.85}$$

We can compute the relevant terms.

- The terms in the action involving only the internal components of the metric give

$$\begin{aligned} & -\frac{1}{4}\sqrt{-\widetilde{G}_4}\,G^{mn}G^{pq}\,(\partial_\mu G_{mp})(\partial^\mu G_{nq}) = \\ = \;& -\frac{1}{2}\sqrt{-\widetilde{G}_4}\sum_{A=1}^{3}\left[\frac{(\partial_\mu\hat{t}_A)(\partial^\mu\hat{t}_A)}{\hat{t}_A^2} + \frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A)}{\hat{u}_A^2}\right] + \\ & -\frac{1}{4}\sqrt{-\widetilde{G}_4}\,\frac{(\partial_\mu\hat{v})(\partial^\mu\hat{v})}{\hat{v}^2}\,. \end{aligned}$$

- The terms involving the internal components of the 3-form are

$$\begin{aligned} & -\frac{1}{12}\sqrt{-\widetilde{G}_4}\,G^{mn}G^{pq}G^{rs}\,(\partial_\mu A_{mpr})(\partial^\mu A_{nqs}) = \\ = \;& -\frac{1}{2}\sqrt{-\widetilde{G}_4}\Big[\frac{1}{\hat{t}_1^2\hat{v}}\,(\partial_\mu A_{5611})(\partial^\mu A_{5611}) + \frac{1}{\hat{t}_2^2\hat{v}}\,(\partial_\mu A_{7811})(\partial^\mu A_{7811}) + \\ & + \frac{1}{\hat{t}_3^2\hat{v}}\,(\partial_\mu A_{91011})(\partial^\mu A_{91011}) + \\ & + \frac{1}{\hat{t}_1\hat{t}_2\hat{t}_3}\Big[\frac{\hat{u}_3}{\hat{u}_1\hat{u}_2}\,(\partial_\mu A_{5710})(\partial^\mu A_{5710}) + \frac{\hat{u}_2}{\hat{u}_1\hat{u}_3}\,(\partial_\mu A_{589})(\partial^\mu A_{589}) + \\ & + \frac{\hat{u}_1}{\hat{u}_2\hat{u}_3}\,(\partial_\mu A_{679})(\partial^\mu A_{679}) + \hat{u}_1\hat{u}_2\hat{u}_3\,(\partial_\mu A_{6810})(\partial^\mu A_{6810})\Big]\Big]\,. \end{aligned}$$

We can recognize the axions, which appear only under derivatives in the previous terms:

$$\tau_1 \equiv A_{5611}\,, \quad \tau_2 \equiv A_{7811}\,, \quad \tau_3 \equiv A_{91011}\,, \tag{5.86}$$

$$\nu_1 \equiv -A_{5710}\,, \quad \nu_2 \equiv -A_{589}\,, \quad \nu_3 \equiv -A_{679}\,, \quad \sigma \equiv A_{6810}\,. \tag{5.87}$$

Then the action (5.85) takes the form:

$$\begin{aligned} S \;=\;& \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Bigg[\widetilde{R}_4 - \frac{1}{2}\hat{s}^{-2}\,(\partial_\mu \hat{s})(\partial^\mu \hat{s}) - \frac{1}{4}\,\frac{(\partial_\mu \hat{v})(\partial^\mu \hat{v})}{\hat{v}^2} + \\ & - \frac{1}{2}\sum_{A=1}^{3}\left[\frac{(\partial_\mu \hat{t}_A)(\partial^\mu \hat{t}_A)}{\hat{t}_A^2} + \frac{(\partial_\mu \hat{u}_A)(\partial^\mu \hat{u}_A)}{\hat{u}_A^2}\right] - \frac{1}{2}\sum_{A=1}^{3}\frac{1}{\hat{t}_A^2\hat{v}}\,(\partial_\mu \tau_A)(\partial^\mu \tau_A) + \\ & - \frac{1}{2}\frac{1}{\hat{t}_1\hat{t}_2\hat{t}_3}\Bigg[\frac{\hat{u}_3}{\hat{u}_1\hat{u}_2}\,(\partial_\mu \nu_3)(\partial^\mu \nu_3) + \frac{\hat{u}_2}{\hat{u}_1\hat{u}_3}\,(\partial_\mu \nu_2)(\partial^\mu \nu_2) + \\ & + \frac{\hat{u}_1}{\hat{u}_2\hat{u}_3}\,(\partial_\mu \nu_1)(\partial^\mu \nu_1) + \hat{u}_1\hat{u}_2\hat{u}_3\,(\partial_\mu \sigma)(\partial^\mu \sigma)\Bigg]\Bigg]\,. \end{aligned} \tag{5.88}$$

Now we look for the redefinitions of the fields, that are needed to obtain the usual form for the kinetic terms of $N = 1$, $d = 4$ supergravity theory. We first introduce a new field r such that

$$r^{-1} \equiv \sqrt{\hat{v}}\hat{s}^{-1} \Longleftrightarrow \hat{s} = \sqrt{\hat{v}}r \tag{5.89}$$

and t_A such that

$$t_A^2 \equiv \hat{t}_A^2\hat{v} \Longleftrightarrow t_A = \sqrt{\hat{v}}\hat{t}_A\,. \tag{5.90}$$

Then

$$\hat{v} = r^{-\frac{2}{3}}\;(t_1t_2t_3)^{\frac{2}{3}}\;, \tag{5.91}$$

$$\hat{s} = r^{\frac{2}{3}}\;(t_1t_2t_3)^{\frac{1}{3}}\;. \tag{5.92}$$

Substituting in (5.88) we obtain

$$\begin{aligned} S \;=\;& \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Bigg[\widetilde{R}_4 - \frac{1}{2}r^{-2}\,(\partial_\mu r)(\partial^\mu r) + \\ & - \frac{1}{2}\sum_{A=1}^{3}\left[\frac{(\partial_\mu t_A)(\partial^\mu t_A)}{t_A^2} + \frac{(\partial_\mu \hat{u}_A)(\partial^\mu \hat{u}_A)}{\hat{u}_A^2}\right] - \frac{1}{2}\sum_{A=1}^{3}\frac{1}{\hat{t}_A^2\hat{v}}\,(\partial_\mu \tau_A)(\partial^\mu \tau_A) + \\ & - \frac{1}{2}\frac{1}{\hat{t}_1\hat{t}_2\hat{t}_3}\Bigg[\frac{\hat{u}_3}{\hat{u}_1\hat{u}_2}\,(\partial_\mu \nu_3)(\partial^\mu \nu_3) + \frac{\hat{u}_2}{\hat{u}_1\hat{u}_3}\,(\partial_\mu \nu_2)(\partial^\mu \nu_2) + \\ & + \frac{\hat{u}_1}{\hat{u}_2\hat{u}_3}\,(\partial_\mu \nu_1)(\partial^\mu \nu_1) + \hat{u}_1\hat{u}_2\hat{u}_3\,(\partial_\mu \sigma)(\partial^\mu \sigma)\Bigg]\Bigg]\,. \end{aligned} \tag{5.93}$$

Now we perform other redefinitions for the terms multiplying the ν_A and the σ fields:

$$\frac{1}{u_A^2} \equiv \frac{\hat{u}_A^2}{r\hat{u}_1\hat{u}_2\hat{u}_3}\,, \qquad A = 1, 2, 3, \tag{5.94}$$

$$\frac{1}{s^2} \equiv \frac{\hat{u}_1\hat{u}_2\hat{u}_3}{r}\,. \tag{5.95}$$

Inverting:

$$\hat{u}_A = \sqrt{\frac{u_1u_2u_3}{su_A^2}}\,, \quad r = \sqrt{su_1u_2u_3}\,. \tag{5.96}$$

Then (5.93) becomes

$$\begin{aligned} S \;=\; & \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4 - \frac{1}{2}\frac{(\partial_\mu s)(\partial^\mu s) + (\partial_\mu \sigma)(\partial^\mu \sigma)}{s^2} + \\ & -\frac{1}{2}\sum_{A=1}^{3}\left[\frac{(\partial_\mu u_A)(\partial^\mu u_A) + (\partial_\mu \hat{\nu}_A)(\partial^\mu \hat{\nu}_A)}{u_A^2}\right] + \\ & -\frac{1}{2}\frac{(\partial_\mu t_A)(\partial^\mu t_A) + (\partial_\mu \tau_A)(\partial^\mu \tau_A)}{t_A^2}\Big]\,. \end{aligned} \tag{5.97}$$

After the last trivial redefinitions ($A = 1, 2, 3$)

$$\nu_A \equiv \hat{\nu}_A\,, \tag{5.98}$$

we obtain the scalar part of the $N = 1$, $d = 4$ supergravity Lagrangian with seven complex scalar fields

$$S = s + i\sigma\,, \quad T_A = t_A + i\tau_A\,, \quad U_A = u_A + i\nu_A \tag{5.99}$$

and with Kähler potential

$$K = -\log(S + \overline{S}) - \sum_{A=1}^{3}\log(T_A + \overline{T}_A) - \sum_{B=1}^{3}\log(U_A + \overline{U}_A)\,. \tag{5.100}$$

5.2.6 M-theory ($\rightarrow$ heterotic)

The starting point is again the action for M-theory supergravity, given by (4.11), but we now want to obtain fields which correspond to those of the low-energy limit of the heterotic string theory, by introducing the orbifold $\mathbb{T}^7/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, combined with an additional projection associated with a parity operation $\mathcal{R}$. We take this additional parity $\mathcal{R}$ to act on the coordinates as

$$\mathcal{R}: \quad (x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}) \rightarrow (+x^5, +x^6, +x^7, +x^8, +x^9, +x^{10}, -x^{11})\,. \tag{5.101}$$

The symmetry generators combine according to:

Action on the coordinates								
	:	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}
$\mathcal{R}$	:	$+$	$+$	$+$	$+$	$+$	$+$	$-$
$\mathbb{Z}_2\mathcal{R}$	:	$-$	$-$	$-$	$-$	$+$	$+$	$-$
$\mathbb{Z}_2'\mathcal{R}$	:	$+$	$+$	$-$	$-$	$-$	$-$	$-$
$\mathbb{Z}_2\mathbb{Z}_2'\mathcal{R}$	:	$-$	$-$	$+$	$+$	$-$	$-$	$-$

Invariance of the Chern-Simons term under the $\mathcal{R}$ projection calls for a non-trivial action on the fields:

Action on the fields		
G	$\to$	$+G$
$A_{(3)}$	$\to$	$-A_{(3)}$

The only fields of the theory invariant under all projections are ($A = 1, 2, 3$):

Fields in M-theory ($\to$ heterotic)	
Metric moduli	$\hat{t}_A$, $\hat{u}_A$, $\hat{\nu}_A$
$A_{(3)}$	A_{5611}, A_{7811}, A_{91011}, $A_{\mu\nu 11} \leftrightarrow \sigma$

The reduced action, deduced from the general formula (5.34), takes the form

$$\begin{aligned} S \;=\; & \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4 - \frac{1}{2}\hat{s}^{-2}\,(\partial_\mu \hat{s})(\partial^\mu \hat{s}) + \\ & - \frac{1}{4}G^{mn}G^{pq}\,(\partial_\mu G_{mp})(\partial^\mu G_{nq}) + \\ & - \frac{1}{12}G^{mn}G^{pq}G^{rs}\,(\partial_\mu A_{mpr})(\partial^\mu A_{nqs}) - \frac{1}{12}\hat{s}^2 G^{mn}F_{m\mu\nu\rho}F_n{}^{\mu\nu\rho}\Big]\,. \end{aligned} \tag{5.102}$$

- The term in the action involving only the internal components of the metric gives

$$\begin{aligned} & -\frac{1}{4}\sqrt{-\widetilde{G}_4}\,G^{mn}G^{pq}\,(\partial_\mu G_{mp})(\partial^\mu G_{nq}) = \\ = \; & -\frac{1}{2}\sqrt{-\widetilde{G}_4}\,\sum_{A=1}^{3}\left[\frac{(\partial_\mu \hat{t}_A)(\partial^\mu \hat{t}_A)}{\hat{t}_A^2} + \frac{(\partial_\mu \hat{u}_A)(\partial^\mu \hat{u}_A) + (\partial_\mu \hat{\nu}_A)(\partial^\mu \hat{\nu}_A)}{\hat{u}_A^2}\right] + \\ & - \frac{1}{4}\sqrt{-\widetilde{G}_4}\,\frac{(\partial_\mu \hat{v})(\partial^\mu \hat{v})}{\hat{v}^2}\,. \end{aligned}$$

- The term involving the internal components of the 3-form is

$$\begin{aligned} & -\frac{1}{12}\sqrt{-\widetilde{G}_4}\,G^{mn}G^{pq}G^{rs}\,(\partial_\mu A_{mpr})(\partial^\mu A_{nqs}) = \\ = \; & -\frac{1}{2}\sqrt{-\widetilde{G}_4}\,\Big[\frac{1}{\hat{t}_1^2\hat{v}}\,(\partial_\mu A_{5611})(\partial^\mu A_{5611}) + \frac{1}{\hat{t}_2^2\hat{v}}\,(\partial_\mu A_{7811})(\partial^\mu A_{7811}) + \\ & + \frac{1}{\hat{t}_3^2\hat{v}}\,(\partial_\mu A_{91011})(\partial^\mu A_{91011})\Big]\,. \end{aligned} \tag{5.103}$$

- Finally, there is a duality relation between the 2-form $A_{\mu\nu 11}$ and a scalar field, which we denote by σ:

$$\int d^4x \sqrt{-\widetilde{G}_4}\, \left[-\frac{1}{12}\hat{s}^2 G^{mn} F_{m\mu\nu\rho} F_n{}^{\mu\nu\rho}\right] \leftrightarrow \int d^4x \sqrt{-\widetilde{G}_4}\, \left[-\frac{1}{2}\hat{s}^{-2}\,(\partial_\mu\sigma)(\partial^\mu\sigma)\right] .$$

We immediately recognize the axions in (5.103):

$$\tau_1 \equiv A_{5611}\,, \quad \tau_2 \equiv A_{7811}\,, \quad \tau_3 \equiv A_{91011}\,. \tag{5.104}$$

The action (5.102) then takes the form:

$$\begin{aligned} S &= \frac{1}{2\kappa_4^2}\int d^4x \sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4 - \frac{1}{2}\hat{s}^{-2}\,(\partial_\mu\hat{s})(\partial^\mu\hat{s}) - \frac{1}{4}\frac{(\partial_\mu\hat{v})(\partial^\mu\hat{v})}{\hat{v}^2} + \\ &\quad - \frac{1}{2}\sum_{A=1}^{3}\left[\frac{(\partial_\mu\hat{t}_A)(\partial^\mu\hat{t}_A)}{\hat{t}_A^2} + \frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A) + (\partial_\mu\hat{\nu}_A)(\partial^\mu\hat{\nu}_A)}{\hat{u}_A^2}\right] + \\ &\quad - \frac{1}{2}\sum_{A=1}^{3}\frac{1}{\hat{t}_A^2\hat{v}}\,(\partial_\mu\tau_A)(\partial^\mu\tau_A) - \frac{1}{2}s^{-2}\,(\partial_\mu\sigma)(\partial^\mu\sigma)\Big] . \end{aligned} \tag{5.105}$$

We now introduce a field r such that

$$r^{-1} \equiv \sqrt{\hat{v}}\hat{s}^{-1} \Longleftrightarrow \hat{s} = \sqrt{\hat{v}}r \tag{5.106}$$

and t_A such that

$$t_A^2 \equiv \hat{t}_A^2\hat{v} \Longleftrightarrow t_A = \sqrt{\hat{v}}\hat{t}_A\,. \tag{5.107}$$

Then

$$\hat{v} = r^{-\frac{2}{3}}\,(t_1t_2t_3)^{\frac{2}{3}}\,, \tag{5.108}$$

$$\hat{s} = r^{\frac{2}{3}}\,(t_1t_2t_3)^{\frac{1}{3}} \tag{5.109}$$

and substituting into (5.105), we obtain

$$\begin{aligned} S &= \frac{1}{2\kappa_4^2}\int d^4x \sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4 - \frac{1}{2}\frac{(\partial_\mu r)(\partial^\mu r) + (\partial_\mu\sigma)(\partial^\mu\sigma)}{r^2} + \\ &\quad - \frac{1}{2}\sum_{A=1}^{3}\left[\frac{(\partial_\mu\hat{u}_A)(\partial^\mu\hat{u}_A) + (\partial_\mu\hat{\nu}_A)(\partial^\mu\hat{\nu}_A)}{\hat{u}_A^2}\right] + \\ &\quad - \frac{1}{2}\frac{(\partial_\mu t_A)(\partial^\mu t_A) + (\partial_\mu\tau_A)(\partial^\mu\tau_A)}{t_A^2}\Big] . \end{aligned} \tag{5.110}$$

Finally we identify ($A = 1, 2, 3$):

$$u_A \equiv \hat{u}_A\,, \quad \nu_2 \equiv \hat{\nu}_A\,, \quad s \equiv r\,. \tag{5.111}$$

Then (5.110) is the kinetic part for scalar fields of the $N = 1$, $d = 4$ supergravity Lagrangian with

$$S = s + i\sigma\,, \quad T_A = t_A + i\tau_A\,, \quad U_A = \hat{u}_A + i\hat{\nu}_A \tag{5.112}$$

and with Kähler potential

$$K = -\log(S + \overline{S}) - \sum_{A=1}^{3} \log(T_A + \overline{T}_A) - \sum_{B=1}^{3} (U_B + \overline{U}_B)\,. \tag{5.113}$$

Chapter 6

Effective $N = 1$ supergravities from fluxes

In this Chapter we describe flux compactifications of supergravity theories from ten or eleven to four dimensions.

First, we give an overview of the salient features and general properties of such compactifications, focusing on the emergence of a scalar potential for the moduli fields and on the consistency requirements through the Scherk-Schwarz mechanism and the presence of p-form fluxes. Then we analyze some concrete examples taken from Chapter 5. Although we are going to present the general procedure that has to be applied to any of the orbifold compactifications discussed in the previous chapter, we are going to give a detailed derivation of the potential and analyze the vacua only for the type IIB models, both with O3/O7 and O5/O9 orientifolds.

We start with the case of type IIB supergravity with O3/O7 orientifolds, which exhibits an interesting potential and simple vacua, passing then to the case with O5/O9 orientifolds, especially because this model has been neglected in the literature. We will actually show non-supersymmetric Minkowski vacua which had not been identified previously.

6.1 Flux compactifications

In Chapter 5 we studied six examples of compactifications of supergravity theories from ten or eleven to four dimensions, on the orbifold $\mathbb{T}^k/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ for $k = 6, 7$ and when necessary with a further $\mathbb{Z}_2$ projection. In each case, we obtained the kinetic terms for seven complex scalar fields in a $N = 1$, $d = 4$ supergravity theory. These are massless scalar fields, which can assume arbitrary v.e.v.'s, since there is no scalar potential governing their dynamics. This situation is unsatisfactory and can have very dangerous physical consequences. First of all, there are very strong experimental limits on exotic long-range forces mediated by massless particles. Moreover, moduli describe the geometry of the compact space and the coupling constants in the effective theory, thus having their v.e.v.'s undetermined means a loss of predicitivity. For this reason, we study two simple ways to generate a potential in the effective theory.

6.1.1 Geometric fluxes

In Chapter 3 we performed Scherk-Schwarz generalized dimensional reductions of a free complex scalar field and of a free Dirac field. In each case, we could introduce a mass parameter in the reduced theory, though the original theory was massless.

There are two kinds of Scherk-Schwarz dimensional reductions [28, 29, 44]. In the first type, which we studied in Chapter 3, a theory with a *global* symmetry is reduced on a circle or torus. The symmetry can be used to impose generalized boundary conditions on the fields, which allow to introduce mass parameters that appear in the reduced theory. In the case of a global symmetry, the effect of the Scherk-Schwarz twist can be absorbed in a local redefinition of the fields, depending on the internal coordinates.

In the second type of dimensional reduction, the symmetry used to generate the Scherk-Schwarz twist is *local*. When this symmetry is an internal gauge symmetry, this possibility is equivalent to a Wilson line. When it is general coordinate invariance, it is called *twisted torus*, because of the clear geometrical interpretation of generalized dimensional reduction in this case. It is convenient to begin the discussion of this case with an example, following [29].

In Section 3.3 we studied ordinary dimensional reduction of pure gravity from five to four dimensions. The original theory is invariant under five-dimensional general coordinate transformations, while the reduced theory had four-dimensional general coordinate transformation invariance, plus a $U(1)$ invariance and a global scale invariance. Ordinary dimensional reduction can be described as that in which all fields are taken independent of the internal coordinates x^i, i.e. the derivatives ∂_i, $i = d+1, \ldots, D$, are taken to vanish when acting on the fields. This summarizes the consequence of a mode expansion on the compact space, with a truncation of the modes which, in the limit of vanishing size of the compact dimension, decouple from the light sector.

It is possible to perform ordinary dimensional reduction of pure gravity by using the vielbein formalism [27, 28, 29], obtaining the same result as in (3.28). Nevertheless, we are interested to generalized dimensional reduction. Since this result will be useful in our supergravity compactifications, we consider pure gravity in $D = d + k$ dimensions and reduce it to d dimensions. We normalize the action as

$$S = -\frac{1}{4\kappa^2} \int d^d x \int \frac{d^k x}{\rho(k)} \, e_D \, R_D \,, \tag{6.1}$$

where e_D is the determinant of the vielbein in D dimensions and $\rho(k)$ is the invariant volume of the internal space, which will be conveniently fixed later.

Generalized dimensional reduction is obtained by allowing the fields to have a non-trivial dependence on the internal coordinates. Since we want to obtain an effective theory in d dimensions, this dependence cannot be arbitrary. First, it should be possible to define a limit which gives ordinary dimensional reduction. This implies that the number of physical degrees of freedom does not change, with respect to ordinary dimensional reduction. Then, the dependence of the fields and transformation laws on the internal coordinates must have a form that can be factored out of the transformation laws, so that the reduced theory is effectively d-dimensional.

The second requirement is satisfied when the x^i dependence cancels because of a symmetry. In pure gravity, we can use the general coordinate transformation (GCT) invariance in D dimensions. Then, the parameter of GCT, $\xi^M(x^\nu, x^n)$, will depend on the internal coordinates in some particular way. The choice of the x^i dependence must be such that:

- GCT in D dimensions describe GCT in d dimensions and a gauge transformation for the k vector fields, not necessarily abelian;
- the gauge algebra of GCT closes as (all parameters depend on both the internal and the external coordinates) $[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{\xi_3}$, where $\xi_3^M = \xi_2^N \partial_N \xi_1^M - \xi_1^N \partial_N \xi_2^M$.

These requirements are satisfied when

$$\xi^\mu(x^\nu, x^n) = \xi^\mu(x^\nu)\,, \tag{6.2}$$

$$\xi^i(x^\nu, x^n) = \left[U^{-1}(x^n)\right]_j^{\ i} \xi^j(x^\nu)\,, \tag{6.3}$$

where U is a $k \times k$ non-singular matrix. The results of ordinary dimensional reduction are recovered when U approaches unity.

The reduced theory is x^i-independent provided that the matrices $U^i_{\ j}$ satisfy the constraint that the coefficients

$$\omega_{ij}^{\ \ k} = \left[U^{-1}\right]_i^{\ i'} \left[U^{-1}\right]_j^{\ j'} \left(\partial_{j'} U^k_{\ i'} - \partial_{i'} U^k_{\ j'}\right) \tag{6.4}$$

are *constants*. The internal coordinates x^i may be thought as a system of coordinates on the manifold of a Lie group G having k generators. The infinitesimal generators of the group can be represented in terms of differential operators as

$$L_i(x^n) = \left[U^{-1}\right]_i^{\ j} \partial_j\,. \tag{6.5}$$

Then, if the L_i satisfy

$$[L_i, L_j] = \omega_{ij}^{\ \ k} L_k\,, \tag{6.6}$$

the $\omega_{ij}^{\ \ k}$ are constant, since they are structure constants for a Lie group. Moreover, since the matrices U are used to describe a change of basis, they are non-singular.

We have given a dependence on the internal coordinates to the parameters ξ^M in a way that the reduced theory be x^i-independent. For the same reason, we need to find appropriate compactification Ansatze for the fields in the theory, so that the reduced action is independent on the x^i. A consistent way to do it is by multiplying every lower internal (world) index by a $U(x^n)$ factor and every upper internal (world) index by a factor $U^{-1}(x^n)$.

Coming back to our example, the D-dimensional vielbein can be parametrized in a triangular form as

$$e_M{}^A = \left(\begin{array}{c|c} \delta^{-\frac{1}{d-2}}\, e_\mu{}^\alpha & 2\kappa\, A^i_\mu \phi_i{}^a \\ \hline 0 & \phi_i{}^a \end{array} \right) , \tag{6.7}$$

where $e_\mu{}^\alpha$ is the d-dimensional vielbein, the A^i_μ are k vector fields, the ϕ^a_i are k^2 scalar fields and $\delta = \det \phi_i{}^a$. This corresponds to a metric

$$g_{MN} = \left(\begin{array}{c|c} \delta^{-\frac{2}{d-2}} g_{\mu\nu} + 4\kappa^2\, A^m_\mu A_{\nu m} & 2\kappa\, A_{\mu j} \\ \hline 2\kappa A_{\nu i} & g_{ij} \end{array} \right) , \tag{6.8}$$

where

$$g_{ij} = \phi_i{}^a\, \delta_{ab}\, \phi_j{}^b \tag{6.9}$$

is a positive-definite metric. The dependence of the fields on the internal coordinates takes the form:

$$\begin{aligned} e_\mu{}^\alpha(x^\nu, x^n) &= e_\mu{}^\alpha(x^\nu) , \\ A^i_\mu(x^\nu, x^n) &= \left[U^{-1}(x^n)\right]_j{}^i\, A^j_\mu(x^\nu) , \\ \phi_i{}^a(x^\nu, x^n) &= U^j{}_i(x^n)\, \phi_j{}^a(x^\nu) . \end{aligned} \tag{6.10}$$

The counting of the degrees of freedom has not changed.

The transformation rules of the fields under GCT of parameter $\xi^m(x^\nu, x^n)$, defined as in (6.3), do not depend on the internal coordinates, but only on the constants $\omega_{ij}{}^k$:

$$\begin{aligned} \delta e_\mu{}^\alpha(x^\nu) &= 0 , \\ \delta A^i_\mu(x^\nu) &= \frac{1}{2\kappa}\partial_\mu \xi^i(x^\nu) + \omega_{jk}{}^i\, \xi^j(x^\nu)\, A^k_\mu(x^\nu) , \\ \delta \phi_i{}^a(x^\nu) &= \omega_{ij}{}^k\, \xi^j(x^\nu)\, \phi_k{}^a(x^\nu) . \end{aligned} \tag{6.11}$$

Thus the vector fields A^i_μ are gauge fields, for the gauge group G with structure constants $\omega_{ij}{}^k$. In (6.7), δ is the determinant of $\phi_i{}^a(x^\nu)$, so it does not dependent on the x^i. Then the determinant of the vielbein is

$$e_D = U(x^n)\, \delta^{-\frac{2}{d-2}}\, e_d , \tag{6.12}$$

where $U(x^n) \equiv \det U^i{}_j(x^n)$. This suggests to choose the measure $\rho(k)$ in (6.1) as

$$\rho(k) = \int_{\mathcal{K}} d^k x\, U(x^n) , \tag{6.13}$$

where $\mathcal{K}$ denotes the internal compact space.

The spin connections with all tangent space indices, which will be denoted by $\hat{\omega}_{\bullet,\circ\circ}$ to remember that the first index is a Lorentz index after contraction with the vielbein, assume the form

$$
\begin{aligned}
\hat{\omega}_{\alpha,\beta\gamma} &= \delta^{\frac{1}{d-2}} \left[\omega_{\alpha,\beta\gamma} + \frac{1}{d-2} \left(\eta_{\alpha\beta} e_{\gamma}{}^{\lambda} - \eta_{\alpha\gamma} e_{\beta}{}^{\lambda}\right) \partial_{\lambda} \ln \delta\right] , \\
\hat{\omega}_{\alpha,\beta a} &= \kappa\, \delta^{\frac{2}{d-2}}\, F^{i}_{\alpha\beta}\, \phi_{ia} , \\
\hat{\omega}_{\alpha,ab} &= \frac{1}{2}\, \delta^{\frac{1}{d-2}}\, \phi_{a}{}^{i}\, e_{\alpha}{}^{\mu}\, \mathcal{D}_{\mu}\phi_{ib} - (a \leftrightarrow b) , \\
\hat{\omega}_{c,\alpha\beta} &= -\kappa\, \delta^{\frac{2}{d-2}}\, F^{i}_{\alpha\beta}\, \phi_{ic} , \\
\hat{\omega}_{c,\alpha a} &= \frac{1}{2} \delta^{\frac{1}{d-2}}\, \phi_{a}{}^{i}\, \phi_{c}{}^{j} e_{\alpha}{}^{\lambda}\, \mathcal{D}_{\lambda} g_{ij} , \\
\hat{\omega}_{c,ab} &= \frac{1}{2} \omega_{ij}{}^{k} \left(\phi_{a}{}^{i}\phi_{b}{}^{j}\phi_{ck} + \phi_{a}{}^{i}\phi_{c}{}^{j}\phi_{bk} - \phi_{b}{}^{i}\phi_{c}{}^{j}\phi_{ak}\right) ,
\end{aligned}
\tag{6.14}
$$

with the covariant derivatives

$$
\begin{aligned}
F^{i}_{\mu\nu} &\equiv \partial_{\mu} A^{i}_{\nu} - \partial_{\nu} A^{i}_{\mu} - 2\kappa\, \omega_{jk}{}^{i} A^{j}_{\mu} A^{k}_{\nu} , \\
\mathcal{D}_{\mu}\phi_{i}{}^{a} &= \partial_{\mu}\phi_{i}{}^{a} - 2\kappa\, \omega_{ij}{}^{k} A^{j}_{\mu} \phi_{k}{}^{a} .
\end{aligned}
\tag{6.15}
$$

In these expressions, the fields depend only on the external coordinates x^{ν}, not on the internal coordinates x^{n}, therefore the Ricci scalar does not depend on the internal coordinates. As a consequence, the factor $U(x^{n})$ in (6.12) exactly cancels the measure (6.13) and the reduced action is indeed x^{n}-independent.

The last of the (6.14) shows the effect of the Scherk-Schwarz generalized dimensional reduction on a theory of gravitation. The ordinary case, which can be recovered by putting $\omega_{ij}{}^{k} = 0$, has a vanishing spin connection with all internal components $\hat{\omega}_{c,ab}$. Thus there is a correspondence between the structure constants $\omega_{ij}{}^{k}$ and the internal components of the spin connection $\hat{\omega}_{c,ab}$, which suggested the name *geometric fluxes* for the former.

From the spin connection, it is possible to calculate the Ricci scalar. The resulting reduced action, to be compared with the ordinary case (3.28), is

$$
\begin{aligned}
S = \int d^{d}x\, e_{d} \Big[&- \frac{1}{4\kappa^{2}} R_{d} - \frac{1}{4} \delta^{\frac{2}{d-2}} F^{\mu\nu i} F^{j}_{\mu\nu}\, g_{ij} + \\
&- \frac{1}{16\kappa^{2}} \left(\mathcal{D}_{\lambda} g_{ij}\right) \left(\mathcal{D}^{\lambda} g^{ij}\right) + \frac{1}{4\kappa^{2}(d-2)} \left(\partial_{\rho} \ln \delta\right) \left(\partial^{\rho} \ln \delta\right) - V_{E} \Big] ,
\end{aligned}
\tag{6.16}
$$

where V_{E} is the potential generated by generalized dimensional reduction of the Einstein term. This potential takes the form

$$
V_{E} = \frac{1}{8\kappa^{2}}\, \delta^{-\frac{2}{d-2}}\, \omega_{ij}{}^{k} \left(2\, \omega_{kl}{}^{i} g^{jl} + \omega_{i'j'}{}^{k'} g_{kk'} g^{ii'} g^{jj'}\right) . \tag{6.17}
$$

Since the geometric fluxes are structure constants for a Lie group, they cannot be arbitrary. The first constraint is the Jacobi identity, which follows from nilpotency of the exterior differential. Consider the 1-forms η^{i}, defined as

$$
\eta^{i} \equiv \left[U^{-1}(x^{n})\right]_{j}{}^{i}\, dx^{j} . \tag{6.18}
$$

The η^i define a basis, called *twisted basis* because the $[U(x^n)]_j{}^i$ represents the deformation (twist) of the toroidal case and we recover the case of a torus when these matrices reduce to identity.

Let Z_i be the vectors dual to η^i, such that $Z_i(\eta^j) = \delta_i^j$. Then

$$Z_i = [U(x^n)]^j{}_i \, \partial_j \, , \tag{6.19}$$

and satisfy

$$[Z_i, Z_j] = \omega_{ij}{}^k \, Z_k \, . \tag{6.20}$$

With these definitions, it is possible to show (note 2 in [38]) that

$$d\eta^i + \frac{1}{2} \omega_{ij}{}^k \, \eta^i \wedge \eta^j = 0 \, . \tag{6.21}$$

The Jacobi identity is then a consequence of $d^2\eta^i = 0$, which reads

$$\omega_{i[j}{}^k \, \omega_{mn]}{}^i = 0 \, . \tag{6.22}$$

Geometric fluxes must also satisfy another constraint. The volume form on the internal space, $V_k = \frac{1}{k!} \epsilon_{i_1 \cdots i_k} \, \eta^{i_1} \wedge \ldots \wedge \eta^{i_k} = U \, d^k x$, must be invariant under the action of the Z_i, which is equivalent to the requirement of invariance under GCT with parameter $\xi^i(x^\nu, x^n)$. This gives a further constraint [29]

$$\omega_{ij}{}^i = 0 \, . \tag{6.23}$$

Last remark: the potential V_E is entirely determined by the geometry of the compact space $\mathcal{K}$ and by the structure constants of the Lie group, and it can be in general unbounded from below.

This geometric and algebraic construction has been widely studied in the literature, but more details go far beyond our purposes.

6.1.2 p-form fluxes

Given a p-form potential $A_{(p)}$ in D dimensions, the corresponding action, expressed in terms of the field strength $F_{(p+1)} = dA_{(p)}$, is

$$-\frac{1}{2} \int d^D x \, \sqrt{-G_D} \, \left| F_{(p+1)} \right|^2 = -\frac{1}{2} \int F_{(p+1)} \wedge * F_{(p+1)} \, , \tag{6.24}$$

where we use the convention (4.12).

This is a generalization of Maxwell's action for the electromagnetic field, but there is an important remark. In Maxwell's theory, there is a 1-form potential which couples to point-like particles, but in supergravity theories the fields have other sources: the NSNS 2-form, $B_{(2)}$, couples to the two-dimensional string world sheet, while the RR p-forms, $A_{(p)}$, are sourced by D$(p-1)$-branes, extended objects in p space-time dimensions.

In compactifications of field theories from D to d dimensions, we require to obtain a background with a maximally symmetric d-dimensional space-time (in our case Minkowski space), but without any assumption on the geometry of the internal space, as long as it is compact. This allows to change the background values of the internal components of gauge invariant fields, without affecting the space-time symmetry in d dimensions. Non-trivial vacuum expectation values of the internal components of the p-form field strength are called *p-form fluxes*. The resulting physics can be interpreted as a generalized Zeeman effect, in which the fluxes are parallel to the internal space and remove (part of) the mass degeneracy of the moduli, according to their coupling to the fluxes themselves.

Although the discussion of the moduli stabilization will be performed at a purely classical level, there are important constraints on the fluxes which have to be imposed for quantum consistency. These constraints can be understood by drawing a parallel to quantum electrodynamics (QED). Quantum electrodynamics contains magnetic monopoles, particles which source magnetic flux: taken a closed surface Σ surrounding a monopole, the flux of the magnetic field on it is non vanishing,

$$g = \int_{\Sigma} F_{(2)} \neq 0\,. \tag{6.25}$$

This implies the Dirac quantization condition

$$\frac{1}{2\pi}\, e \cdot g \in \mathbb{Z}\,, \tag{6.26}$$

where e is the elementary electric charge. The analogy with p-form fluxes continues, since they must also satisfy a quantization condition analogous to (6.26). Consider the homology group $H_{p+1}(\mathcal{K})$ of the manifold on which we compactify and suppose that it is non-trivial. Let Σ a non-trivial element of $H_{p+1}(\mathcal{K})$. Then we can consider a configuration with non-zero flux. The flux of the field strength

$$\int_{\Sigma} F_{(p+1)} \tag{6.27}$$

must be an integer, in suitable units. Moreover, we can turn on an independent flux for each basis element of $H_{p+1}(\mathcal{K})$, which give $\dim H_{p+1}(\mathcal{K}) = b_{p+1}$ Dirac quantization conditions.

In general, it is not possible to turn on p-form and geometric fluxes arbitrarily. This can be viewed as a consequence of the Bianchi identities (BI) for the $(p+1)$ field strengths in the NSNS and RR sectors [45], however it is explained in string theories as a consequence of tadpole conditions, which are the analogue of Gauss's theorem: the total charge on the compactification manifold, including all sources, must vanish. In the following we will not consider sources for the fluxes, since this would require the introduction of D-branes and goes beyond our purposes.

For the sake of simplicity, we discuss the NSNS sector, which contains the field strength $H_{(3)}$. The BI read

$$dH = 0\,, \tag{6.28}$$

with general solution

$$H = dB + \overline{H}\,, \qquad \overline{H} = const\,. \tag{6.29}$$

Given a form X, the effects of non-trivial geometric fluxes ω can be described by the rule

$$dX \to dX + \omega\, X\,, \tag{6.30}$$

where explicitly

$$\omega\, X = \omega_{[ij}{}^{k}\, X_{k|mn]}\,. \tag{6.31}$$

When we introduce geometric fluxes, the BI change into

$$dH + \omega\, H = 0\,, \tag{6.32}$$

The general solution is

$$H = dB + \omega\, B + \overline{H}\,, \quad \overline{H} = const\,, \tag{6.33}$$

provided that

$$\omega\, \overline{H} = 0\,. \tag{6.34}$$

These are the tadpole conditions.

6.2 Examples: classification of invariant fluxes

We want now to discuss the effect of fluxes on the toroidal compactifications studied in Chapter 5. Since we introduced some (orbifold and orientifold) projections, we must classify those that are invariant under these projections and therefore can appear in the potential.

We denote by ω the geometric fluxes, while the set H_{mnr} represents those arising from $H_{(3)} = dB_{(2)}$. Finally, the fluxes $F_{(p)}$ come from the field strengths $F_{(p)} = dA_{(p-1)}$.

- $\mathcal{N} = 1$ **supergravity (without Yang-Mills sector)**.
  In this case the allowed fluxes, in the absence of the Yang-Mills sector, come from generalized Scherk-Schwarz reductions as twenty-four geometric fluxes:

$$\omega_{nr}{}^{m} : \begin{array}{ccc} \omega_{79}{}^{5} & \omega_{95}{}^{7} & \omega_{57}{}^{9} \\ \omega_{710}{}^{5} & \omega_{105}{}^{7} & \omega_{57}{}^{10} \\ \omega_{89}{}^{5} & \omega_{95}{}^{8} & \omega_{58}{}^{9} \\ \omega_{810}{}^{5} & \omega_{105}{}^{8} & \omega_{58}{}^{10} \\ \omega_{79}{}^{6} & \omega_{96}{}^{7} & \omega_{67}{}^{9} \\ \omega_{710}{}^{6} & \omega_{106}{}^{7} & \omega_{67}{}^{10} \\ \omega_{89}{}^{6} & \omega_{96}{}^{8} & \omega_{68}{}^{9} \\ \omega_{810}{}^{6} & \omega_{106}{}^{8} & \omega_{68}{}^{10} \end{array} \tag{6.35}$$

  and from the NSNS sector as eight 3-form fluxes from H_{ijk}:

$$H_{mnr} : H_{579},\ H_{5710},\ H_{589},\ H_{5810},\ H_{679},\ H_{6710},\ H_{689},\ H_{6810}. \tag{6.36}$$

- **IIA with O6 orientifold**.
  There are twelve allowed geometric fluxes:

$$\omega_{nr}{}^{m}: \begin{array}{ccc} \omega_{710}{}^{5} & \omega_{105}{}^{7} & \omega_{57}{}^{10} \\ \omega_{89}{}^{5} & \omega_{95}{}^{8} & \omega_{58}{}^{9} \\ \omega_{79}{}^{6} & \omega_{96}{}^{7} & \omega_{67}{}^{9} \\ \omega_{810}{}^{6} & \omega_{106}{}^{8} & \omega_{68}{}^{10} \end{array} . \tag{6.37}$$

  There are also four fluxes from the NSNS sector:

$$H_{mnr} : H_{579},\, H_{5810},\, H_{6710},\, H_{689}. \tag{6.38}$$

  In the RR sector, there are the flux associated to the mass parameter

$$F_{(0)} : M\,, \tag{6.39}$$

  three $F_{(2)}$ fluxes,

$$F_{(2)} : F_{56},\, F_{78},\, F_{910}, \tag{6.40}$$

  three $F_{(4)}$ fluxes,

$$F_{(4)} : F_{5678},\, F_{56910},\, F_{78910}, \tag{6.41}$$

  and one $F_{(6)}$ flux, which is originated by a duality relation with $F_{\mu\nu\rho\sigma}$,

$$F_{(6)} : F_{5678910}. \tag{6.42}$$

- **IIB with O3/O7 orientifold**.
  In IIB with O3/O7 orientifold no geometric fluxes are allowed. There are eight 3-form fluxes from the NSNS sector,

$$H_{mnr} : H_{579},\, H_{5710},\, H_{589},\, H_{5810},\, H_{679},\, H_{6710},\, H_{689},\, H_{6810}, \tag{6.43}$$

  while, from the RR sector, the $F_{(1)}$ and the $F_{(5)}$ fluxes are not consistent with the orbifold and orientifold projections. The only RR allowed fluxes come from $F_{(3)}$ and are

$$F_{(3)} : F_{579},\, F_{5710},\, F_{589},\, F_{5810},\, F_{679},\, F_{6710},\, F_{689},\, F_{6810}. \tag{6.44}$$

- **IIB with O5/O9 orientifold**.
  When in theory IIB is introduced the O5/O9 orientifold, the following twenty-four geometric fluxes are allowed

$$\omega_{nr}{}^{m}: \begin{array}{ccc} \omega_{79}{}^{5} & \omega_{95}{}^{7} & \omega_{57}{}^{9} \\ \omega_{710}{}^{5} & \omega_{105}{}^{7} & \omega_{57}{}^{10} \\ \omega_{89}{}^{5} & \omega_{95}{}^{8} & \omega_{58}{}^{9} \\ \omega_{810}{}^{5} & \omega_{105}{}^{8} & \omega_{58}{}^{10} \\ \omega_{79}{}^{6} & \omega_{96}{}^{7} & \omega_{67}{}^{9} \\ \omega_{710}{}^{6} & \omega_{106}{}^{7} & \omega_{67}{}^{10} \\ \omega_{89}{}^{6} & \omega_{96}{}^{8} & \omega_{68}{}^{9} \\ \omega_{810}{}^{6} & \omega_{106}{}^{8} & \omega_{68}{}^{10} \end{array} \tag{6.45}$$

  but the NSNS 3-form fluxes H_{mnr} are all incompatible with the orbifold and orientifold projections. In the RR sector, the $F_{(1)}$ and $F_{(5)}$ are not allowed, so there are only eight $F_{(3)}$ possible fluxes,

$$F_{(3)} : F_{579},\, F_{5710},\, F_{589},\, F_{5810},\, F_{679},\, F_{6710},\, F_{689},\, F_{6810}. \tag{6.46}$$

- **M-theory (→ IIA)**.
  In this case there are twenty-one allowed geometric fluxes,

$$\omega_{nr}{}^{m}: \begin{array}{ccc} \omega_{611}{}^{5} & \omega_{115}{}^{6} & \omega_{56}{}^{11} \\ \omega_{710}{}^{5} & \omega_{105}{}^{7} & \omega_{57}{}^{10} \\ \omega_{89}{}^{5} & \omega_{95}{}^{8} & \omega_{58}{}^{9} \\ \omega_{79}{}^{6} & \omega_{96}{}^{7} & \omega_{67}{}^{9} \\ \omega_{810}{}^{6} & \omega_{106}{}^{8} & \omega_{68}{}^{10} \\ \omega_{811}{}^{7} & \omega_{117}{}^{8} & \omega_{78}{}^{11} \\ \omega_{1011}{}^{9} & \omega_{119}{}^{10} & \omega_{910}{}^{11} \end{array} \tag{6.47}$$

  seven fluxes coming from $F_{(4)}$,

$$F_{(4)} : F_{5678},\, F_{56910},\, F_{57911},\, F_{581011},\, F_{671011},\, F_{68911},\, F_{78910}\,, \tag{6.48}$$

  and the flux $F_{(7)}$, which comes from a duality relation with $F_{\mu\nu\rho\sigma}$,

$$F_{(7)} : F_{567891011}\,. \tag{6.49}$$

- **M-theory (→ heterotic).**
  There are twenty-four geometric fluxes compatible with the orbifold projections,

$$
\omega_{nr}{}^{m}: \quad \begin{array}{ccc}
\omega_{79}{}^{5} & \omega_{95}{}^{7} & \omega_{57}{}^{9} \\
\omega_{710}{}^{5} & \omega_{105}{}^{7} & \omega_{57}{}^{10} \\
\omega_{89}{}^{5} & \omega_{95}{}^{8} & \omega_{58}{}^{9} \\
\omega_{810}{}^{5} & \omega_{105}{}^{8} & \omega_{58}{}^{10} \\
\omega_{79}{}^{6} & \omega_{96}{}^{7} & \omega_{67}{}^{9} \\
\omega_{710}{}^{6} & \omega_{106}{}^{7} & \omega_{67}{}^{10} \\
\omega_{89}{}^{6} & \omega_{96}{}^{8} & \omega_{68}{}^{9} \\
\omega_{810}{}^{6} & \omega_{106}{}^{8} & \omega_{68}{}^{10}
\end{array} \tag{6.50}
$$

  and eight fluxes coming from the 3-form $F_{(4)}$,

$$
F_{(4)} : F_{57911},\ F_{571011},\ F_{58911},\ F_{581011},\ F_{67911},\ F_{671011},\ F_{68911},\ F_{681011}. \tag{6.51}
$$

6.3 Type IIB superpotential and vacua

In this section we compute the effective superpotential of the type IIB compactifications studied in Chapter 5, which arise when considering non-trivial fluxes.

When there are no fluxes, the Kähler potential is given by (5.70):

$$
K = -\log(S+\overline{S}) - \sum_{A=1}^{3}\log(T_A+\overline{T}_A) - \sum_{B=1}^{3}\log(U_B+\overline{U}_B)\,. \tag{6.52}
$$

It is convenient to denote the seven main moduli (5.69) by

$$
z^i \equiv (S, T_1, T_2, T_3, U_1, U_2, U_3)\ , \qquad i = 1, \ldots, 7. \tag{6.53}
$$

The scalar potential is given, in general, by

$$
\begin{aligned}
V(z^i) &= \sqrt{-\widetilde{G}_4}\, e^K \left[K^{i\bar{\jmath}}(D_iW)(D_{\bar{\jmath}}\overline{W}) - 3|W|^2\right] = \\
&= \sqrt{-\widetilde{G}_4}\, e^K \left[K^{i\bar{\jmath}}(W_i + K_iW)(\overline{W}_{\bar{\jmath}} + \overline{K}_{\bar{\jmath}}\overline{W}) - 3|W|^2\right]\,.
\end{aligned} \tag{6.54}
$$

From the explicit form of the Ansatz for the Kähler potential (6.52), we deduce:

$$
K_i = -\frac{1}{z^i+\overline{z}^{\bar{\imath}}} = \begin{pmatrix} -(S+\overline{S})^{-1} \\ -(T_A+\overline{T}_A)^{-1} \\ -(U_B+\overline{U}_B)^{-1} \end{pmatrix} \tag{6.55}
$$

and the Kähler metric is

$$
K_{\bar{\jmath}i} = \frac{\delta_{\bar{\jmath}i}}{(z^i+\overline{z}^{\bar{\imath}})^2} = diag\left((S+\overline{S})^{-2}, (T_A+\overline{T}_A)^{-2}, (U_B+\overline{U}_B)^{-2}\right)\,. \tag{6.56}
$$

Then the scalar potential becomes:

$$
\begin{aligned}
V(z^i) &= \sqrt{-\widetilde{G}_4}\, e^K \left[\sum_{i=1}^{7} \left| (z^i + \overline{z^{\imath}}) W_i - W \right|^2 - 3\, |W|^2 \right] = \\
&= \sqrt{-\widetilde{G}_4}\, e^K \Big[|(S+\overline{S}) W_S - W|^2 + \sum_{A=1}^{3} |(T_A + \overline{T}_A) W_{T_A} - W|^2 + \\
&\quad + \sum_{B=1}^{3} |(U_B + \overline{U}_B) W_{U_B} - W|^2 - 3|W|^2 \Big] . \qquad (6.57)
\end{aligned}
$$

6.3.1 IIB with O3/O7 orientifolds

In the type IIB orbifold compactification with O3/O7 orientifolds there are only two types of allowed fluxes, those arising from H_{mnr}, the field strength of $B_{(2)}$, in the NSNS sector and those arising from F_{mnr}, the field strength of $A_{(2)}$, in the RR sector. The other fluxes, in particular the geometric ones, are absent, because they are incompatible with the presence of the orbifold and of the orientifold projections.

After dimensional reduction, the part of the action containing the fluxes coming from the three-forms is, in the four-dimensional Einstein frame,

$$
S_3 = -\frac{1}{4\kappa_4^2} \int d^4x \sqrt{-\widetilde{G}_4}\, \frac{1}{3!}\, \hat{s}^{-1}\, G^{mn} G^{pq} G^{rs} \left[H_{mpr} H_{nqs} + e^{2\Phi} \widetilde{F}_{mpr} \widetilde{F}_{nqs} \right] , \qquad (6.58)
$$

where, according to (4.34) and to (5.61), we have

$$
\widetilde{F}_{mpr} = F_{mpr} - \sigma\, H_{mpr} \, . \qquad (6.59)
$$

We recall the redefinitions of the fields made in Sec. 5.2.3:

$$
e^{-2\Phi} = \frac{\hat{s}}{\hat{t}_1 \hat{t}_2 \hat{t}_3} \, , \qquad (6.60)
$$

$$
\frac{1}{t_A^2} \equiv \frac{\hat{t}_A^2}{\hat{s} \hat{t}_1 \hat{t}_2 \hat{t}_3} \, , \quad A = 1, 2, 3, \qquad (6.61)
$$

$$
\frac{1}{s^2} \equiv \frac{\hat{t}_1 \hat{t}_2 \hat{t}_3}{\hat{s}} \, . \qquad (6.62)
$$

Using these redefinitions, the reduced action for the fluxes reads

$$
S_3 = -\frac{1}{4\kappa_4^2} \int d^4x \sqrt{-\widetilde{G}_4}\, \frac{1}{3!}\, \hat{s}^{-1}\, G^{mn} G^{pq} G^{rs} \left[H_{mpr} H_{nqs} + s^{-2} \widetilde{F}_{mpr} \widetilde{F}_{nqs} \right] . \qquad (6.63)
$$

We start by turning on the following fluxes:

$$
F_{579} \, , \, F_{5710} \, , \, F_{6810} \, , \, F_{6710} \, , \, H_{6710} \, . \qquad (6.64)
$$

Noting that

$$\frac{\hat{s}^{-1}}{\hat{t}_1\hat{t}_2\hat{t}_3\hat{u}_1\hat{u}_2\hat{u}_3} = \frac{s}{t_1 t_2 t_3 u_1 u_2 u_3}\,, \tag{6.65}$$

we can write S_3 as a function of the real moduli (geometric moduli and axions):

$$\begin{aligned} S_3 \;=\; & -\frac{1}{\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,(4st_1t_2t_3u_1u_2u_3)^{-1}\;\cdot \\ & \cdot\Big[\,(F_{579})^2 + (u_3^2+\nu_3^2)\,(F_{5710})^2 + \\ & + (u_1^2+\nu_1^2)(u_2^2+\nu_2^2)(u_3^2+\nu_3^2)\;(F_{6810})^2 + \\ & + (u_1^2+\nu_1^2)(u_3^2+\nu_3^2)\,(F_{6710})^2 + (s^2+\sigma^2)\,(u_1^2+\nu_1^2)\,(u_3^2+\nu_3^2)(H_{6710})^2 + \\ & - 2\,\nu_3F_{579}F_{5710} - 2\,\nu_1\nu_2\nu_3F_{579}F_{6810} + \\ & + 2\,\nu_1\nu_3F_{579}F_{6710} - 2\,\sigma\nu_1\nu_3F_{579}H_{6710} + \\ & + 2\,\nu_1\nu_2(u_3^2+\nu_3^2)F_{5710}F_{6810} + \\ & - 2\,\nu_1(u_3^2+\nu_3^2)F_{5710}F_{6710} + 2\,\sigma\nu_1(u_3^2+\nu_3^2)F_{5710}H_{6710} + \\ & - 2\,\nu_2(u_1^2+\nu_1^2)(u_3^2+\nu_3^2)F_{6810}F_{6710} + \\ & + 2\,\sigma\nu_2(u_1^2+\nu_1^2)(u_3^2+\nu_3^2)F_{6810}H_{6710} + \\ & - 2\,\sigma(u_1^2+\nu_1^2)(u_3^2+\nu_3^2)F_{6710}H_{6710}\Big]\,. \end{aligned} \tag{6.66}$$

In the square brackets, the only field-independent term is $(F_{579})^2$. If this scalar potential, which has the form (6.57), comes from a superpotential W, we can use this flux to recognize the form of the terms in this superpotential. In facts, we can obtain a field-independent term (apart from the overall factor e^K) only from terms in $|W|^2$ or from the part proportional to W in D_iW. Therefore, the terms proportional to F_{579} must give the combinations of complex scalar fields and fluxes which appear in the superpotential. The superpotential has then the form

$$W = a_0\,F_{579} + a_1\,U_3F_{5710} + a_2\,U_1U_2U_3F_{6810} + a_3\,U_1U_3F_{6710} + a_4\,SU_1U_3H_{6710}\,, \tag{6.67}$$

where the a_n are complex parameters to be determined. Substituting in the general expression (6.57), we obtain the scalar potential (6.66) coming from dimensional reduction for the values

$$\begin{aligned} a_0 &= 2\sqrt{2}\,i\,e^{i\alpha}\,, \\ a_1 &= -2\sqrt{2}\,e^{i\alpha}\,, \\ a_2 &= 2\sqrt{2}\,e^{i\alpha}\,, \\ a_3 &= -2\sqrt{2}\,i\,e^{i\alpha}\,, \\ a_4 &= 2\sqrt{2}\,e^{i\alpha}\,, \end{aligned} \tag{6.68}$$

where $\alpha \in \mathbb{R}$ is a real parameter which remains undetermined. Therefore the superpotential is

$$W = 2\sqrt{2}\,e^{i\alpha}\,[i\,F_{579} - U_3F_{5710} + U_1U_2U_3F_{6810} - i\,U_1U_3F_{6710} + SU_1U_3H_{6710}]\,. \tag{6.69}$$

The superpotential has a constant term, which is proportional to F_{579} as expected, while it is independent of the T_A main moduli. The field S appears only with a flux of the NSNS sector, H_{6710}, while the U_B main moduli appear related to the fluxes coming from both the NSNS and the RR sector.

As a consequence, the scalar potential (6.57) becomes

$$V_{O3/O7} \;=\; \sqrt{-\widetilde{G}_4}\, e^K \left[\left|(S+\overline{S})W_S - W\right|^2 + \sum_{B=1}^{3} \left|(U_B+\overline{U}_B)W_{U_B} - W\right|^2 \right] , \quad (6.70)$$

which is positive semi-definite. Exploiting the fact that W is a polynomial of degree at most one in each of the seven fields z^i, it is useful to notice the identity:

$$\left|(z^i+\overline{z}^{\overline{\imath}})W_i - W\right|^2 = \left|W(z^i \to -\overline{z}^{\overline{\imath}})\right|^2 . \quad (6.71)$$

We are interested in the vacua of the theory. It is useful to calculate the Kähler covariant derivatives $D_iW = W_i + K_iW$ $(A = 1, 2, 3)$:

$$\begin{aligned}
D_SW &= -2\sqrt{2}\, e^{i\alpha}(S+\overline{S})^{-1} \Big[iF_{579} - U_3F_{5710} + U_1U_2U_3F_{6810} + \\
&\qquad - iU_1U_3F_{6710} - \overline{S}U_1U_3H_{6710} \Big] , \\
D_{T_A}W &= -2\sqrt{2}\, e^{i\alpha}(T_A+\overline{T}_A)^{-1} \Big[iF_{579} - U_3F_{5710} + U_1U_2U_3F_{6810} + \\
&\qquad - iU_1U_3F_{6710} + SU_1U_3H_{6710} \Big] , \\
D_{U_1}W &= -2\sqrt{2}\, e^{i\alpha}(U_1+\overline{U}_1)^{-1} \Big[iF_{579} - U_3F_{5710} - \overline{U}_1U_2U_3F_{6810} + \\
&\qquad + i\overline{U}_1U_3F_{6710} - S\overline{U}_1U_3H_{6710} \Big] , \\
D_{U_2}W &= -2\sqrt{2}\, e^{i\alpha}(U_2+\overline{U}_2)^{-1} \Big[iF_{579} - U_3F_{5710} - U_1\overline{U}_2U_3F_{6810} + \\
&\qquad - iU_1U_3F_{6710} + SU_1U_3H_{6710} \Big] , \\
D_{U_3}W &= -2\sqrt{2}\, e^{i\alpha}(U_3+\overline{U}_3)^{-1} \Big[iF_{579} + \overline{U}_3F_{5710} - U_1U_2\overline{U}_3F_{6810} + \\
&\qquad + iU_1\overline{U}_3F_{6710} - SU_1\overline{U}_3H_{6710} \Big] .
\end{aligned} \quad (6.72)$$

Supersymmetric vacua are those which satisfy the conditions $D_iW = 0$, $i = 1, \ldots, 7$ (v.e.v. is intended). Since the superpotential does not depend on the T_A, it is interesting to inspect, for the moment, the consequences of the four equations

$$D_SW = 0 \,, \quad D_{U_B}W = 0 \,, \qquad B = 1, 2, 3. \quad (6.73)$$

When these four conditions are satisfied, the scalar potential (6.70) vanishes. Since it is non-negative, the solutions are vacua of the theory. Vice versa, the scalar potential is a sum of non-negative terms, thus the vacua of the theory are those which satisfy the conditions (6.73), because minimization with respect to the scalars t_A requires that V vanishes at the vacuum. Therefore, the equations (6.73) identify the vacua of the theory.

Taking the real and the imaginary parts of the four equations (6.73), we arrive at the conditions:

$$\left\{\begin{array}{lclcl}
(u_1\nu_3+\nu_1u_3)\,\lambda & + & (u_1u_3-\nu_1\nu_3)\,(u_2F_{6810}-sH_{6710})-u_3F_{5710} & = & 0\,,\\
(u_1\nu_3-\nu_1u_3)\,\lambda & + & (u_1u_3+\nu_1\nu_3)\,(u_2F_{6810}+sH_{6710})+u_3F_{5710} & = & 0\,,\\
(u_1\nu_3+\nu_1u_3)\,\lambda & - & (u_1u_3-\nu_1\nu_3)\,(u_2F_{6810}-sH_{6710})-u_3F_{5710} & = & 0\,,\\
(u_1\nu_3-\nu_1u_3)\,\lambda & - & (u_1u_3+\nu_1\nu_3)\,(u_2F_{6810}+sH_{6710})+u_3F_{5710} & = & 0\,,\\
(u_1u_3-\nu_1\nu_3)\,\lambda & - & (u_1\nu_3+\nu_1u_3)\,(u_2F_{6810}-sH_{6710})+\nu_3F_{5710}-F_{579} & = & 0\,,\\
(u_1u_3+\nu_1\nu_3)\,\lambda & - & (u_1\nu_3-\nu_1u_3)\,(u_2F_{6810}+sH_{6710})-\nu_3F_{5710}+F_{579} & = & 0\,,\\
(u_1u_3-\nu_1\nu_3)\,\lambda & + & (u_1\nu_3+\nu_1u_3)\,(u_2F_{6810}-sH_{6710})+\nu_3F_{5710}-F_{579} & = & 0\,,\\
(u_1u_3+\nu_1\nu_3)\,\lambda & + & (u_1\nu_3-\nu_1u_3)\,(u_2F_{6810}+sH_{6710})-\nu_3F_{5710}+F_{579} & = & 0\,,
\end{array}\right.$$

where we defined

$$\lambda \equiv F_{6710}-\nu_2\,F_{6810}-\sigma\,H_{6710}\,. \tag{6.74}$$

Since the geometric moduli are positive definite (and in particular non-vanishing), there exists a solution to this system of equations only if all the fluxes are zero:

$$F_{579}=F_{5710}=F_{6810}=F_{6710}=H_{6710}=0\,. \tag{6.75}$$

This shows that vacua in type IIB supergravity with O3/O7 orientifold, without sources for the fluxes, are Minkowski supersymmetric vacua, which are allowed only for trivial values of the fluxes.

6.3.2 IIB with O5/O9 orientifolds

In IIB compactifications with O5/O9 orientifolds, twenty-four geometric fluxes are allowed by the orbifold and orientifold projections.

The potential which comes from Scherk-Schwarz generalized reduction of the Einstein term is, with our normalizations and conventions [48, 45],

$$V_E=\frac{1}{8}\sqrt{-\widetilde{G}_4}\,\hat{s}^{-1}\,\left[2\,\omega_{jk}{}^{i}\,\omega_{il}{}^{j}G^{kl}+\omega_{jk}{}^{i}\,\omega_{mn}{}^{l}G_{il}G^{jm}G^{kn}\right]\,, \tag{6.76}$$

while the scalar potential coming from the generalized $F_{(3)}$ flux is

$$V_3=\frac{1}{4}\sqrt{-\widetilde{G}_4}\,\hat{s}^{-1}\,\left[\frac{1}{3!}G^{mn}G^{pq}G^{rs}\,\left(\omega_{mp}{}^{t}\,C_{tr}\,\omega_{nq}{}^{v}\,C_{vs}-\omega_{mp}{}^{t}\,C_{tr}\,\omega_{ns}{}^{v}\,C_{vq}\right)\right] \tag{6.77}$$

We recall the redefinitions of the fields, calculated in Chapter 5, which put the action in the standard form for $N=1$ supergravity:

$$e^{2\Phi}=\hat{s}^{-1}\hat{t}_1\hat{t}_2\hat{t}_3\,, \tag{6.78}$$

$$t_A\equiv\sqrt{\frac{\hat{s}\hat{t}_A^2}{\hat{t}_1\hat{t}_2\hat{t}_3}}\,,\qquad A=1,2,3, \tag{6.79}$$

$$s \equiv \sqrt{\hat{s}\hat{t}_1\hat{t}_2\hat{t}_3}\,, \tag{6.80}$$

$$\tau_1 \equiv C_{56}\,, \quad \tau_2 \equiv C_{78}\,, \quad \tau_3 \equiv C_{910}\,. \tag{6.81}$$

Inverting, we obtain

$$\hat{t}_A = \sqrt{\frac{s t_A^2}{t_1 t_2 t_3}}\,, \qquad \hat{s} = \sqrt{s t_1 t_2 t_3}\,. \tag{6.82}$$

We begin by turning on the following fluxes

$$\omega_{57}{}^{9}\,, \quad \omega_{59}{}^{7}\,, \quad \omega_{610}{}^{8}\,, \quad \omega_{68}{}^{10}\,. \tag{6.83}$$

Expanding the expression of V_E+V_3 in components and using the redefinitions of the fields introduced before, the scalar potential obtained from dimensional reduction becomes

$$\begin{aligned}
\frac{V_E+V_3}{\sqrt{-\widetilde{G}_4}} \;=\; & 2^5\, e^K \Big[(t_2^2+\tau_2^2)\ \big[(\omega_{59}{}^7)^2\,(u_2^2+\nu_2^2) + (\omega_{610}{}^8)^2\,(u_1^2+\nu_1^2)\,(u_3^2+\nu_3^2)\big] + \\
& + (t_3^2+\tau_3^2)\ \big[(\omega_{57}{}^9)^2\,(u_3^2+\nu_3^2) + (\omega_{68}{}^{10})^2\,(u_1^2+\nu_1^2)\,(u_2^2+\nu_2^2)\big] + \\
& + 2\,\nu_1\nu_2\nu_3\ \big[\omega_{59}{}^7\,\omega_{610}{}^8(t_2^2+\tau_2^2) + \omega_{57}{}^9\,\omega_{68}{}^{10}(t_3^2+\tau_3^2)\big] + \\
& + 2\ (t_2t_3u_2u_3 - \tau_2\tau_3\nu_2\nu_3)\ \big[\omega_{59}{}^7\,\omega_{57}{}^9 + \omega_{610}{}^8\,\omega_{68}{}^{10}\,(u_1^2+\nu_1^2)\big] + \\
& - 2\,\nu_1\tau_2\tau_3\,\big[\omega_{59}{}^7\,\omega_{68}{}^{10}(u_2^2+\nu_2^2) + \omega_{610}{}^8\,\omega_{57}{}^9(u_3^2+\nu_3^2)\big]\Big]\,.
\end{aligned} \tag{6.84}$$

In the bracket multiplying the overall factor e^K, there are not field-independent terms, thus we must use another trick to understand the form of the superpotential. The terms quadratic in the fluxes suggest the following structure for the superpotential:

$$\begin{aligned}
\omega_{57}{}^9 &\leftrightarrow T_3U_3\,, \\
\omega_{59}{}^7 &\leftrightarrow T_2U_2\,, \\
\omega_{68}{}^{10} &\leftrightarrow T_3U_1U_2\,, \\
\omega_{610}{}^8 &\leftrightarrow T_2U_1U_3\,.
\end{aligned} \tag{6.85}$$

This allows us to put forward the following Ansatz for the superpotential:

$$W = a_1\,\omega_{59}{}^7\,T_2U_2 + a_2\,\omega_{610}{}^8\,T_2U_1U_3 + a_3\,\omega_{57}{}^9\,T_3U_3 + a_4\,\omega_{68}{}^{10}\,T_3U_1U_2\,. \tag{6.86}$$

The superpotential does not depend on S or on T_1, hence the general form of the scalar potential (6.57) becomes:

$$\begin{aligned}
V(z^i) \;=\; & \sqrt{-\widetilde{G}_4}\, e^K \Big[\sum_{A=2}^{3} |(T_A+\overline{T}_A)W_{T_A} - W|^2 + \\
& + \sum_{B=1}^{3} |(U_B+\overline{U}_B)W_{U_B} - W|^2 - |W|^2\Big]\,.
\end{aligned} \tag{6.87}$$

In general, one does not expect this to be positive semi-definite, however we now prove this to be the case. The scalar potential (6.84) is obtained for the following vales of the parameters:

$$\begin{aligned} a_1 &= 2\sqrt{2}\, e^{i\alpha}\,, \\ a_2 &= -2\sqrt{2}\, i\, e^{i\alpha}\,, \\ a_3 &= -2\sqrt{2}\, e^{i\alpha}\,, \\ a_4 &= 2\sqrt{2}\, i\, e^{i\alpha}\,, \end{aligned} \tag{6.88}$$

where $\alpha \in \mathbb{R}$ is a real parameter. The superpotential is then

$$W_{O5/O9} = 2\sqrt{2}\, e^{i\alpha}\left[{\omega_{59}}^{7}\, T_2 U_2 - i\, {\omega_{610}}^{8}\, T_2 U_1 U_3 - {\omega_{57}}^{9}\, T_3 U_3 + i\, {\omega_{68}}^{10}\, T_3 U_1 U_2\right]\,. \tag{6.89}$$

An important observation is that the scalar potential (6.84) can be expressed as a sum of non-negative terms:

$$\begin{aligned} V_{O5/O9} &= 2^5 \sqrt{-\widetilde{G}_4}\, e^K \Big[({\omega_{59}}^{7}\, t_2 u_2 + {\omega_{57}}^{9}\, t_3 u_3)^2 + \\ &+ (u_1^2 + \nu_1^2)\,({\omega_{610}}^{8}\, t_2 u_3 + {\omega_{68}}^{10}\, t_3 u_2)^2 + \\ &+ u_1^2\,({\omega_{610}}^{8}\, \tau_2 \nu_3 - {\omega_{68}}^{10}\, \tau_3 \nu_2)^2 + \\ &+ \left({\omega_{59}}^{7}\, \tau_2 \nu_2 - {\omega_{57}}^{9}\, \tau_3 \nu_3 + \nu_1\left({\omega_{610}}^{8}\, \tau_2 \nu_3 - {\omega_{68}}^{10}\, \tau_3 \nu_2\right)\right)^2 + \\ &+ u_1^2\, \left[({\omega_{610}}^{8})^2\,(t_2^2 \nu_3^2 + \tau_2^2 u_3^2) + ({\omega_{68}}^{10})^2\,(t_3^2 \nu_2^2 + \tau_3^2 u_2^2)\right] + \\ &+ t_2^2\,({\omega_{59}}^{7} \nu_2 + {\omega_{610}}^{8}\, \nu_1 \nu_3)^2 + t_3^2\,({\omega_{57}}^{9} \nu_3 + {\omega_{68}}^{10}\, \nu_1 \nu_2)^2 + \\ &+ u_2^2\,({\omega_{59}}^{7} \tau_2 - {\omega_{68}}^{10}\, \nu_1 \tau_3)^2 + u_3^2\,({\omega_{57}}^{9} \tau_3 - {\omega_{610}}^{8}\, \nu_1 \tau_2)^2\Big]\,. \end{aligned} \tag{6.90}$$

Though the scalar potential (6.87) has not a definite sign, we have shown that it is a sum of square terms, thus being semi-positive definite.

We can begin the study of vacua in this situation. Again, it is useful to calculate the Kähler covariant derivatives $D_i W = W_i + K_i W$:

$$\begin{aligned} D_S W &= -(S + \overline{S})^{-1}\, W\,, \\ D_{T_1} W &= -(T_1 + \overline{T}_1)^{-1}\, W\,, \\ D_{T_2} W &= -2\sqrt{2}\, e^{i\alpha}\,(T_2 + \overline{T}_2)^{-1}\Big[- {\omega_{59}}^{7}\, \overline{T}_2 U_2 + i\, {\omega_{610}}^{8}\, \overline{T}_2 U_1 U_3 + \\ &\quad - {\omega_{57}}^{9}\, T_3 U_3 + i\, {\omega_{68}}^{10}\, T_3 U_1 U_2\Big] \\ D_{T_3} W &= -2\sqrt{2}\, e^{i\alpha}\,(T_3 + \overline{T}_3)^{-1}\Big[{\omega_{59}}^{7}\, T_2 U_2 - i\, {\omega_{610}}^{8}\, T_2 U_1 U_3 + \\ &\quad + {\omega_{57}}^{9}\, \overline{T}_3 U_3 - i\, {\omega_{68}}^{10}\, \overline{T}_3 U_1 U_2\Big] \\ D_{U_1} W &= -2\sqrt{2}\, e^{i\alpha}\,(U_1 + \overline{U}_1)^{-1}\Big[{\omega_{59}}^{7}\, T_2 U_2 + i\, {\omega_{610}}^{8}\, T_2 \overline{U}_1 U_3 + \\ &\quad - {\omega_{57}}^{9}\, T_3 U_3 - i\, {\omega_{68}}^{10}\, T_3 \overline{U}_1 U_2\Big]\,, \end{aligned}$$

$$\begin{aligned}
D_{U_2}W &= -2\sqrt{2}\,e^{i\alpha}\,(U_2+\overline{U}_2)^{-1}\Big[-\omega_{59}{}^{7}\,T_2\overline{U}_2 - i\,\omega_{610}{}^{8}\,T_2U_1U_3 + \\
&\quad - \omega_{57}{}^{9}\,T_3U_3 - i\,\omega_{68}{}^{10}\,T_3U_1\overline{U}_2\Big]\,, \\
D_{U_3}W &= -2\sqrt{2}\,e^{i\alpha}\,(U_3+\overline{U}_3)^{-1}\Big[\omega_{59}{}^{7}\,T_2U_2 + i\,\omega_{610}{}^{8}\,T_2U_1\overline{U}_3 + \\
&\quad + \omega_{57}{}^{9}\,T_3\overline{U}_3 + i\,\omega_{68}{}^{10}\,T_3U_1U_2\Big]\,.
\end{aligned}$$

Supersymmetric vacua satisfy

$$W=0\,,\quad D_{T_2}W = D_{T_3}W = D_{U_1}W = D_{U_2}W = D_{U_3}W = 0\,. \tag{6.91}$$

These conditions are equivalent to

$$\begin{aligned}
\omega_{59}{}^{7}\,T_2U_2 - i\,\omega_{610}{}^{8}\,T_2U_1U_3 - \omega_{57}{}^{9}\,T_3U_3 + i\,\omega_{68}{}^{10}\,T_3U_1U_2 &= 0\,, \\
\omega_{59}{}^{7}\,U_2 - i\,\omega_{610}{}^{8}\,U_1U_3 &= 0 \\
\omega_{59}{}^{7}\,T_2 + i\,\omega_{68}{}^{10}\,T_3U_1 &= 0 \\
\omega_{610}{}^{8}\,T_2U_3 - \omega_{68}{}^{10}\,T_3U_2 &= 0 \\
\omega_{57}{}^{9}\,U_3 - i\,\omega_{68}{}^{10}\,U_1U_2 &= 0 \\
\omega_{57}{}^{9}\,T_3 + i\,\omega_{610}{}^{8}\,T_2U_1 &= 0
\end{aligned} \tag{6.92}$$

which can be satisfied only for trivial values of the geometric fluxes

$$\omega_{59}{}^{7} = \omega_{610}{}^{8} = \omega_{57}{}^{9} = \omega_{68}{}^{10} = 0\,. \tag{6.93}$$

Thus, among the supersymmetric vacua, we have only the trivial Minkowski ones, as in type O3/O7 compactification.

The only dependence on s and t_1 in the superpotential is in the overall factor e^K, thus generic vacua must be Minkowski vacua. Since (6.90) is a sum of non-negative terms, each term must be zero at the vacuum. This is achieved when the following system of equations holds:

$$\omega_{59}{}^{7}\,t_2u_2 + \omega_{57}{}^{9}\,t_3u_3 = 0\,, \tag{6.94}$$
$$\omega_{610}{}^{8}\,t_2u_3 + \omega_{68}{}^{10}\,t_3u_2 = 0\,, \tag{6.95}$$
$$\omega_{610}{}^{8}\,\tau_2\nu_3 - \omega_{68}{}^{10}\,\tau_3\nu_2 = 0\,, \tag{6.96}$$
$$\omega_{59}{}^{7}\,\tau_2\nu_2 - \omega_{57}{}^{9}\,\tau_3\nu_3 = 0\,, \tag{6.97}$$
$$(\omega_{610}{}^{8})^2\,(t_2^2\nu_3^2 + \tau_2^2u_3^2) = 0\,, \tag{6.98}$$
$$(\omega_{68}{}^{10})^2\,(t_3^2\nu_2^2 + \tau_3^2u_2^2) = 0\,, \tag{6.99}$$
$$\omega_{59}{}^{7}\nu_2 + \omega_{610}{}^{8}\,\nu_1\nu_3 = 0\,, \tag{6.100}$$
$$\omega_{57}{}^{9}\nu_3 + \omega_{68}{}^{10}\,\nu_1\nu_2 = 0\,, \tag{6.101}$$
$$\omega_{59}{}^{7}\tau_2 - \omega_{68}{}^{10}\,\nu_1\tau_3 = 0\,, \tag{6.102}$$
$$\omega_{57}{}^{9}\tau_3 - \omega_{610}{}^{8}\,\nu_1\tau_2 = 0\,. \tag{6.103}$$

We can use equations (6.98) and (6.99) to classify the solutions of these equations.

There are four cases:

1. **Case** ${\omega_{610}}^{8} = {\omega_{68}}^{10} = 0$.
Equations (6.94)-(6.103) reduce to

$$\begin{cases} {\omega_{59}}^{7}\, t_2 u_2 + {\omega_{57}}^{9}\, t_3 u_3 & = & 0\,, \\ {\omega_{59}}^{7}\, \tau_2 \nu_2 - {\omega_{57}}^{9}\, \tau_3 \nu_3 & = & 0\,, \\ {\omega_{59}}^{7} \nu_2 & = & 0\,, \\ {\omega_{57}}^{9} \nu_3 & = & 0\,, \\ {\omega_{59}}^{7} \tau_2 & = & 0\,, \\ {\omega_{57}}^{9} \tau_3 & = & 0\,. \end{cases} \tag{6.104}$$

The geometric moduli are strictly positive, thus there are two possibilities:

- ${\omega_{59}}^{7} = {\omega_{57}}^{9} = 0$, which is the trivial supersymmetric Minkowski vacuum;
- if ${\omega_{59}}^{7} \neq 0$, ${\omega_{57}}^{9}/{\omega_{59}}^{7} < 0$,

$$\begin{cases} t_2 & = & -\frac{{\omega_{57}}^{9}}{{\omega_{59}}^{7}}\, t_3 \frac{u_3}{u_2}\,, \\ \nu_2 & = & 0\,, \\ \nu_3 & = & 0\,, \\ \tau_2 & = & 0\,, \\ \tau_3 & = & 0\,. \end{cases} \tag{6.105}$$

2. **Case** ${\omega_{610}}^{8} = 0$, $\nu_2 = 0$, $\tau_3 = 0$.
Equations (6.94)-(6.103) become

$$\begin{cases} {\omega_{59}}^{7}\, t_2 u_2 + {\omega_{57}}^{9}\, t_3 u_3 & = & 0\,, \\ {\omega_{68}}^{10}\, t_3 u_2 & = & 0\,, \\ {\omega_{57}}^{9} \nu_3 & = & 0\,, \\ {\omega_{59}}^{7} \tau_2 & = & 0\,, \end{cases} \tag{6.106}$$

There are two possible solutions:

- if ${\omega_{59}}^{7} = 0$ there is the trivial solution for

$${\omega_{59}}^{7} = {\omega_{57}}^{9} = {\omega_{68}}^{10} = 0\,; \tag{6.107}$$

- if ${\omega_{59}}^{7} \neq 0$, there is a solution only for ${\omega_{68}}^{10} = 0$, ${\omega_{57}}^{9} \neq 0$, ${\omega_{57}}^{9}/{\omega_{59}}^{7} < 0$, which is given by

$$\begin{cases} t_2 & = & -\frac{{\omega_{57}}^{9}}{{\omega_{59}}^{7}}\, t_3 \frac{u_3}{u_2}\,, \\ \nu_3 & = & 0\,, \\ \tau_2 & = & 0\,, \end{cases} \tag{6.108}$$

3. **Case** ${\omega_{68}}^{10} = 0$, $\nu_3 = 0$, $\tau_2 = 0$.
   Equations (6.94)-(6.103) read

$$\left\{\begin{array}{lcl} {\omega_{59}}^{7}\, t_2 u_2 + {\omega_{57}}^{9}\, t_3 u_3 & = & 0\,, \\ {\omega_{610}}^{8}\, t_2 u_3 & = & 0\,, \\ {\omega_{59}}^{7} \nu_2 & = & 0\,, \\ {\omega_{57}}^{9} \tau_3 & = & 0\,. \end{array}\right. \tag{6.109}$$

   Again, there are two possible solutions:

   - if ${\omega_{59}}^{7} = 0$, there is the trivial supersymmetric solution for
$$ {\omega_{57}}^{9} = {\omega_{610}}^{8} = 0\,; \tag{6.110}$$
   - if ${\omega_{59}}^{7} \neq 0$, there is a solution only for ${\omega_{610}}^{8} = 0$ and ${\omega_{57}}^{9} \neq 0$, ${\omega_{57}}^{9}/{\omega_{59}}^{7} < 0$:
$$\left\{\begin{array}{lcl} t_2 & = & -\frac{{\omega_{57}}^{9}}{{\omega_{59}}^{7}}\, t_3 \frac{u_3}{u_2}\,, \\ \nu_2 & = & 0\,, \\ \tau_3 & = & 0\,. \end{array}\right. \tag{6.111}$$

4. **Case** $\nu_2 = 0$, $\nu_3 = 0$, $\tau_2 = 0$, $\tau_3 = 0$.
   Equations (6.94)-(6.103) give

$$\left\{\begin{array}{lcl} {\omega_{59}}^{7}\, t_2 u_2 + {\omega_{57}}^{9}\, t_3 u_3 & = & 0\,, \\ {\omega_{610}}^{8}\, t_2 u_3 + {\omega_{68}}^{10}\, t_3 u_2 & = & 0\,. \end{array}\right. \tag{6.112}$$

   If ${\omega_{610}}^{8} \neq 0$ (otherwise we recover the case 2.), ${\omega_{68}}^{10} \neq 0$ and $\frac{{\omega_{68}}^{10}}{{\omega_{610}}^{8}} < 0$, these equations are equivalent to

$$ t_2 = -\frac{{\omega_{68}}^{10}}{{\omega_{610}}^{8}}\, t_3 \frac{u_2}{u_3}\,, \tag{6.113}$$
$$ {\omega_{57}}^{9}\, u_3^2 = \frac{{\omega_{59}}^{7}\, {\omega_{68}}^{10}}{{\omega_{610}}^{8}}\, u_2^2\,. \tag{6.114}$$

   If ${\omega_{57}}^{9} = 0$, we have solutions only when ${\omega_{59}}^{7} = {\omega_{68}}^{10} = 0$. The interesting case is then ${\omega_{57}}^{9} \neq 0$ with the further constraint $\frac{{\omega_{57}}^{9}}{{\omega_{59}}^{7}} < 0$, which gives:

$$\begin{aligned} t_2 &= -\sqrt{\frac{{\omega_{57}}^{9}}{{\omega_{59}}^{7}}\frac{{\omega_{68}}^{10}}{{\omega_{610}}^{8}}}\, t_3\,, \\ u_2 &= \sqrt{\frac{{\omega_{57}}^{9}\, {\omega_{610}}^{8}}{{\omega_{59}}^{7}\, {\omega_{68}}^{10}}}\, u_3\,. \end{aligned} \tag{6.115}$$

6.4 Discussion

In flux compactifications of type IIB supergravity with O3/O7 orientifolds and without sources, we showed that vacua are allowed only for trivial values of the fluxes. This is a consequence of the runaway form of the scalar potential. It can be explained as a consequence of a no-go theorem, first formulated in [39] and recently perfected by Maldacena and Nunez [40], which forbids, under general hypotheses, non-singular warped compactifications to Minkowski or de Sitter space.

In section 6.3.1 we studied the potential and superpotential when only the fluxes

$$F_{579}\,,\; F_{5710}\,,\; F_{6810}\,,\; F_{6710}\,,\; H_{6710}\,, \tag{6.116}$$

were turned on. From that, we can guess the general form of the superpotential in type IIB supergravity compactifications with O3/O7 orientifolds:

$$\begin{aligned} W_{O3/O7} \;=\;& 2\sqrt{2}\,e^{i\alpha}\Big[i\,F_{579} - (U_3F_{5710} + U_2F_{589} + U_1F_{679}) + \\ & - i\,(U_1U_2F_{689} + U_1U_3F_{6710} + U_2U_3F_{5810}) + U_1U_2U_3F_{6810} + \\ & -\; SH_{579} - iS(U_3H_{5710} + U_2H_{589} + U_1H_{679}) + \\ & +\; S(U_1U_2H_{689} + U_1U_3H_{6710} + U_2U_3H_{5810}) + iSU_1U_2U_3H_{6810}\Big]\,. \end{aligned} \tag{6.117}$$

Given the dependence of this superpotential only on the S and the U_B main moduli, the scalar potential is still positive semi-definite, with Minkowski vacua. In general, the corresponding potential is no-scale. Moreover, in the absence of sources it is also a runaway potential, thus having vacua only for trivial values of the fluxes and regaining the special case studied before.

In type IIB supergravity compactification with O5/O9 orientifolds there are non-trivial vacua. Since there are not sources, we cannot have non-trivial p-form fluxes, but we can consider geometric fluxes, which arise from the Scherk-Schwarz twist. In this case, the Maldacena-Nunez no-go theorem is evaded, since the Riemann tensor vanishes identically on the compactification manifold.

This explains the presence of non-trivial vacua, which fall in two classes:

i) the first class is obtained for the following values of the fluxes

$$\begin{cases} {\omega_{68}}^{10} \;=\; 0\,, \\ {\omega_{610}}^{8} \;=\; 0\,, \end{cases} \tag{6.118}$$

with the further conditions ${\omega_{57}}^{9} \neq 0$, ${\omega_{59}}^{7} \neq 0$ and

$$\frac{{\omega_{57}}^{9}}{{\omega_{59}}^{7}} < 0\,, \tag{6.119}$$

with the v.e.v.'s

$$\begin{cases} \langle t_2 \rangle &= -\frac{\omega_{57}{}^9}{\omega_{59}{}^7} \left\langle \frac{t_3\, u_3}{u_2} \right\rangle \,, \\ \langle \nu_2 \rangle &= 0\,, \\ \langle \nu_3 \rangle &= 0\,, \\ \langle \tau_2 \rangle &= 0\,, \\ \langle \tau_3 \rangle &= 0\,. \end{cases} \tag{6.120}$$

ii) the second class is more general, since all the geometric fluxes considered must be non zero, $\omega_{68}{}^{10} \neq 0$, $\omega_{610}{}^{8} \neq 0$, $\omega_{57}{}^{9} \neq 0$ and $\omega_{59}{}^{7} \neq 0$. Moreover, geometric fluxes must satisfy:

$$\frac{\omega_{59}{}^7}{\omega_{57}{}^9} < 0\,, \qquad \frac{\omega_{68}{}^{10}}{\omega_{610}{}^8} < 0 \tag{6.121}$$

with the v.e.v.'s

$$\begin{cases} \langle t_2 \rangle &= -\sqrt{\frac{\omega_{57}{}^9}{\omega_{59}{}^7}\frac{\omega_{68}{}^{10}}{\omega_{610}{}^8}}\; \langle t_3 \rangle \,, \\ \langle u_2 \rangle &= \sqrt{\frac{\omega_{57}{}^9\, \omega_{610}{}^8}{\omega_{59}{}^7\, \omega_{68}{}^{10}}}\; \langle u_3 \rangle \,, \\ \langle \nu_2 \rangle &= 0\,, \\ \langle \nu_3 \rangle &= 0\,, \\ \langle \tau_2 \rangle &= 0\,, \\ \langle \tau_3 \rangle &= 0\,. \end{cases} \tag{6.122}$$

We can see that these vacua break spontaneously $N = 1$ supersymmetry on a classically flat background. Moreover, some of the moduli are stabilized and acquire non-vanishing masses. However, a number of flat directions remain, and also the gravitino mass slides along these flat directions.

To conclude, we can also infer the general form of the superpotential in type IIB compactifications on the O5/O9 orientifold:

$$\begin{aligned} W_{O5/O9} \;=\; & 2\sqrt{2}\, e^{i\alpha} \Big[-i\,(\omega_{89}{}^5\, T_1 + \omega_{105}{}^7\, T_2 + \omega_{67}{}^9\, T_3) + \\ & - (\omega_{79}{}^5\, T_1 U_1 + \omega_{95}{}^7\, T_2 U_2 + \omega_{57}{}^9\, T_3 U_3) + \\ & + i\,(\omega_{810}{}^6\, T_1 U_2 U_3 + \omega_{106}{}^8\, T_2 U_1 U_3 + \omega_{68}{}^{10}\, T_3 U_1 U_2) + \\ & + U_1 U_2 U_3\,(\omega_{710}{}^6\, T_1 + \omega_{96}{}^8\, T_2 + \omega_{58}{}^{10}\, T_3) + \\ & + (\omega_{89}{}^6\, T_1 U_2 + \omega_{106}{}^7\, T_1 U_3 + \omega_{68}{}^9\, T_2 U_1 + \\ & + \omega_{105}{}^8\, T_2 U_3 + \omega_{67}{}^{10}\, T_3 U_1 + \omega_{610}{}^5\, T_3 U_2) + \\ & - i\,(\omega_{58}{}^9\, T_3 U_3 U_1 + \omega_{79}{}^6\, T_3 U_3 U_2 + \omega_{710}{}^5\, T_1 U_1 U_2 + \\ & + \omega_{95}{}^8\, T_1 U_1 U_3 + \omega_{57}{}^{10}\, T_2 U_2 U_1 + \omega_{96}{}^7\, T_2 U_2 U_3) + \\ & - \; F_{579} - i(U_3 F_{5710} + U_2 F_{589} + U_1 F_{679}) + \end{aligned}$$

$$
\begin{aligned}
&+\ (U_1U_2F_{689}+U_1U_3F_{6710}+U_2U_3F_{5810})+iU_1U_2U_3F_{6810}+\\
&-i\,SH_{579}+S(U_3H_{5710}+U_2H_{589}+U_1H_{679})+\\
&+i\,S(U_1U_2H_{689}+U_1U_3H_{6710}+U_2U_3H_{5810})-SU_1U_2U_3H_{6810}\Big]
\end{aligned} \tag{6.123}
$$

We should keep in mind, however, that the coefficients are not arbitrary, but must satisfy constraints corresponding to Bianchi identities or tadpole cancellation conditions. These constraints can be relaxed in the presence of localized sources such as branes.

Spectrum in type IIB supergravity with O5/O9 orientifolds.
We found two non-supersymmetric vacua in type IIB supergravity with O5/O9 orientifolds. Although we expect that turning on just four geometric fluxes cannot stabilize all the moduli, given the form of the scalar potential, we can directly compute the spectrum of the theory in each of the two vacua.

The theory consists of seven complex scalar fields, that we split in their real and imaginary part according to (5.82). The mass matrix is obtained from the Hessian matrix of the scalar potential. However, given the form of the Kähler potential and remembering that we have Minkowski vacua, it is easy to show that we can directly substitute

$$
e^K \;\rightarrow\; \left\langle e^K \right\rangle = 2^{-7}\,\left\langle (st_1t_2t_3u_1u_2u_3)^{-1}\right\rangle\,. \tag{6.124}
$$

To compute the spectrum of the scalar fields, we expand the fields around their v.e.v.'s. We must also take into account for the non-canonical normalization of the kinetic terms of the scalar fields:

$$
\begin{aligned}
S_{kin} \;=\; & \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4-\frac{1}{2}\frac{(\partial_\mu s)(\partial^\mu s)+(\partial_\mu\sigma)(\partial^\mu\sigma)}{s^2}+\\
& -\frac{1}{2}\sum_{A=1}^{3}\Big[\frac{(\partial_\mu u_A)(\partial^\mu u_A)+(\partial_\mu\nu_A)(\partial^\mu\nu_A)}{u_A^2}+\\
& +\frac{(\partial_\mu t_A)(\partial^\mu t_A)+(\partial_\mu\tau_A)(\partial^\mu\tau_A)}{t_A^2}\Big]\Big]\,.
\end{aligned} \tag{6.125}
$$

For this reason we define fourteen new real scalar fields through:

$$
\begin{aligned}
s &\equiv \langle s\rangle\,(s'-1)\,, \qquad \sigma\equiv\langle s\rangle\,\sigma'-\langle\sigma\rangle\,,\\
t_A &\equiv \langle t_A\rangle\,(t'_A-1)\,, \qquad \tau_A\equiv\langle t_A\rangle\,\tau'_A-\langle\tau_A\rangle\,,\\
u_A &\equiv \langle u_A\rangle\,(u'_A-1)\,, \qquad \nu_A\equiv\langle u_A\rangle\,\nu'_A-\langle\nu_A\rangle\,.
\end{aligned} \tag{6.126}
$$

Then the scalar kinetic terms become canonically normalized:

$$
\begin{aligned}
S_{kin} \;=\; & \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-\widetilde{G}_4}\,\Big[\widetilde{R}_4-\frac{1}{2}(\partial_\mu s')(\partial^\mu s')-\frac{1}{2}(\partial_\mu\sigma')(\partial^\mu\sigma')+\\
& -\frac{1}{2}\sum_{A=1}^{3}\Big[(\partial_\mu u'_A)(\partial^\mu u'_A)+(\partial_\mu\nu'_A)(\partial^\mu\nu'_A)+\\
& +(\partial_\mu t'_A)(\partial^\mu t'_A)+(\partial_\mu\tau'_A)(\partial^\mu\tau'_A)\Big]\Big]\,.
\end{aligned} \tag{6.127}
$$

We can substitute the redefinitions (6.126) in the scalar potential (6.84) and evaluate this expression in the vacua, keeping only terms at most quadratic in the fields.

Since we are in Minkowski vacua, an important consistency check is that we must not find field-independent or linear terms. This will always turn to be the case.

We can now study the spectrum in the simplest of the two vacua previously found, the case i). The scalar potential, expanded around the vacuum, takes the form:

$$\begin{aligned} V^{i)}_{O5/O9} &= 2^5 \left\langle e^K \right\rangle \left(\omega_{57}{}^9\right)^2 \left\langle t_3^2\, u_3^2 \right\rangle \Big[4 \left(\frac{t_3' - t_2' + u_3' - u_2'}{2} \right)^2 + \\ &\quad + (\tau_2')^2 + (\tau_3')^2 + (\nu_2')^2 + (\nu_3')^2 + \ldots \Big] , \end{aligned} \tag{6.128}$$

where the ellipses represent interaction terms, which are irrelevant for our discussion. The gravitino mass is:

$$m^2_{3/2} = \left\langle e^K |W|^2 \right\rangle = -\frac{1}{4}\, \omega_{57}{}^9 \omega_{59}{}^7 \, \left\langle (s t_1 u_1)^{-1} \right\rangle . \tag{6.129}$$

Observe that the scalar potential does not contain terms which are constant or linear in the fields, as it should be in a Minkowski vacuum.

The spectrum of the theory around the vacuum is summarized in Table 6.1.

Field	**Squared mass**
$\left(\frac{t_3'-t_2'+u_3'-u_2'}{2}\right)$	$32\ m^2_{3/2}$
τ_2', τ_3', ν_2', ν_3'	$8\ m^2_{3/2}$
$\left(\frac{t_3'+t_2'+u_3'+u_2'}{2}\right)$	0
$\left(\frac{t_3'+t_2'-u_3'-u_2'}{2}\right)$	0
$\left(\frac{t_3'-t_2'-u_3'+u_2'}{2}\right)$	0
s', σ', t_1', τ_1', u_1', ν_1'	0

Table 6.1: Spectrum in class i) vacua.

We have five massive scalars fields: τ_2', τ_3', ν_2' and ν_3' with a degenerate mass. The combination $\left(\frac{t_3'-t_2'+u_3'-u_2'}{2}\right)$ has a mass four times greater than the massive axionic scalars. The massless states are then $\left(\frac{t_3'+t_2'+u_3'+u_2'}{2}\right)$, $\left(\frac{t_3'+t_2'-u_3'-u_2'}{2}\right)$, $\left(\frac{t_3'-t_2'-u_3'+u_2'}{2}\right)$, as well as s', σ', u_1', ν_1', t_1' and τ_1'.

This shows that in type i) vacuum five moduli are stabilized.

Analogously, it is possible to find the spectrum of the fields in the second vacuum, where we expect to stabilize six moduli, but we stop here.

Conclusions and outlook

In this thesis, we considered the effective $N = 1$, $d = 4$ supergravities obtained from some simple flux compactifications of the ten- and eleven-dimensional supergravities that are obtained, in a suitable field-theory limit, from superstring theories and M-theory, respectively.

We started with a pedagogical introduction to global supersymmetry, supergravity, compactifications of higher-dimensional theories, ten- and eleven-dimensional supergravities, contained in the first four chapters.

The first original step, performed in Chapter 5, was to take six examples of compactifications on the orbifold $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, with an additional $\mathbb{Z}_2$ projection when needed, leading to a $N = 1$, $d = 4$ effective theory. We focused on the dependence of the effective theory on the scalar fields coming from the closed string sector and invariant under the orbifold projection, which fall into seven chiral multiplets. We performed explicitly the dimensional reduction, we showed that the resulting theory can be expressed in the standard supergravity formalism only after some non-linear field redefinitions, and we determined its Kähler potential. In the absence of fluxes, there is no scalar potential for the moduli, which corresponds to a vanishing superpotential.

The second original part of our work, described in Chapter 6, consisted in the study of flux compactifications of type IIB supergravity, with O3/O7 and O5/O9 orientifolds. In each of the two cases, we obtained the potential and superpotential generated by the allowed p-form and geometric fluxes, and discussed the resulting vacua of the theory.

In our first example, the IIB theory with O3/O7 orientifold, we found that geometric fluxes are not consistent with the orbifold and orientifold projections. We thus studied the effects of a simple set of fluxes for the 3-form field strengths, both in the RR and in the NSNS sector. By dimensional reduction, we obtained the scalar potential. Then we deduced the corresponding superpotential, and found (in agreement with known general arguments) that it is a function only of the S and U (dilaton and complex structure) moduli, but does not depend on the T (Kähler) moduli. In such a situation, we expect that not all of the moduli can be classically stabilized, if the Kähler potential is well approximated by the one found by ordinary dimensional reduction without fluxes. Actually, in agreement with a general no-go theorem, we showed explicitly that this model admits stable classical vacua only for trivial values of the fluxes (in the absence of sources, as chosen in our example to avoid some technical complications). Hence, the chosen model admits only trivial supersymmetric Minkowski vacua, without stabilization of any modulus.

Our second example consisted in a type IIB compactification with O5/O9 orientifolds, and turned out to be more interesting, because it allows for non-trivial vacua. In this situation, the fluxes compatible with the orbifold and orientifold projections are the geometric ones and those of the 3-form field strength in the RR sector. As before, we concentrated on a simple set of fluxes that does not require the introduction of localized sources. In the chosen example, we found that the model admits supersymmetric Minkowski vacua only for vanishing fluxes. However, we found that for some possible choices of the geometric fluxes the model admits Minkowski vacua with spontaneously broken supersymmetry, where at least part of the moduli can be stabilized. We then computed the spectrum on the simplest of such classical vacua.

In our examples, we could not consider the most general systems of fluxes. A more complete study would have required, in fact, the consistent inclusion of additional terms in the action corresponding to localized sources, such as D-branes and O-planes, with additional technical difficulties. Moreover, we could have also introduced the so-called "non-geometric" fluxes, whose existence has been suggested on the basis of certain duality properties of the underlying string theories and their effective supergravities. The introduction of these elements would lead to a richer structure, with new possibilities for moduli stabilization and supersymmetry breaking.

The present study can thus be extended in many ways. First, as suggested above, we could include generic systems of D-branes and O-planes, and possibly non-geometric fluxes. Moreover, we neglected the "twisted" moduli that arise at the orbifold fixed points, considering only the untwisted moduli that are invariant under the orbifold projection: we could include both twisted and untwisted moduli in a more complete treatment. Finally, we could include in the effective theory the additional vector and chiral multiplets arising from the open strings modes, localized on branes or at brane intersections. A consistent inclusion of all the above-mentioned effects is clearly beyond the purpose of the present thesis and is left for future work.

Appendix A

Notation and conventions

When dealing with global supersymmetry in four space-time dimensions we will closely follow the conventions of [8], preferring the two-component notation. We will always use the summation convention over repeated indices, unless otherwise stated.

A.1 Four-component notation

Metric tensor for flat space-time:

$$\eta^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)\,. \tag{A.1}$$

Poincaré algebra:

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}\right)\,, \tag{A.2}$$

$$[P^{\mu}, M^{\nu\rho}] = i\left(\eta^{\mu\nu}P^{\rho} - \eta^{\mu\rho}P^{\nu}\right)\,, \tag{A.3}$$

$$[P^{\mu}, P^{\rho}] = 0\,. \tag{A.4}$$

Gamma matrices (in four space-time dimensions) satisfy:

$$\{\gamma^{\mu},\gamma^{\nu}\} \equiv \gamma^{\mu}\,\gamma^{\nu} + \gamma^{\nu}\,\gamma^{\mu} = 2\,\eta^{\mu\nu}\,, \qquad (\mu,\nu = 0,1,2,3)\,. \tag{A.5}$$

We will use the following representation for the γ matrices:

$$\gamma_{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ \bar{\sigma}_{\mu} & 0 \end{pmatrix}\,, \tag{A.6}$$

where $\sigma^{\mu} = (\mathbb{I}_2, \sigma^i)$, $\bar{\sigma}^{\mu} = (\mathbb{I}_2, -\sigma^i)$, $i = 1,2,3$, and σ^i are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{A.7}$$

and satisfy $(i,j,k = 1,2,3)$

$$\{\sigma_i,\sigma_j\} = 2\,\delta_{ij}\,, \qquad [\sigma_i,\sigma_j] = 2i\,\epsilon_{ijk}\,\sigma_k\,, \tag{A.8}$$

$$\sigma_i{}^2 = \mathbb{I}_2\,, \qquad \sigma_i{}^\dagger = \sigma_i\,. \tag{A.9}$$

Then:

$$\gamma^0 = \gamma_0\,, \qquad \gamma^i = -\gamma_i\,, \tag{A.10}$$

$$\left(\gamma^0\right)^2 = \mathbb{I}_4 = -\left(\gamma^i\right)^2\,, \qquad (i = 1,2,3)\,, \tag{A.11}$$

$$\left(\gamma^0\right)^\dagger = \gamma^0\,, \qquad \left(\gamma^i\right)^\dagger = -\gamma^i\,, \tag{A.12}$$

$$\gamma^0\gamma^\mu\gamma^0 = \left(\gamma^\mu\right)^\dagger\,. \tag{A.13}$$

Dirac conjugate spinors are defined by

$$\bar{\psi} \equiv \psi^\dagger\gamma^0\,.$$

We will choose:

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.14}$$

so that

$$\gamma_5{}^\dagger = \gamma_5\,, \quad \left(\gamma_5\right)^2 = \mathbb{I}_4\,, \qquad \{\gamma_5, \gamma_\mu\} = 0\,. \tag{A.15}$$

The helicity projectors are taken as

$$P_{L,R} \equiv \frac{\mathbb{I}_4 \mp \gamma_5}{2}\,. \tag{A.16}$$

With our choice of the γ matrices,

$$\gamma_5 = \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}\,, \quad P_L = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & 0 \end{pmatrix}\,, \quad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}\,, \tag{A.17}$$

and a Dirac spinor reads

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} \tag{A.18}$$

where $\chi_{L,R}$ are left-handed (right-handed) Weyl spinors.

Majorana spinors are defined by the condition

$$\lambda = \lambda_C = C\left(\bar{\lambda}\right)^T \tag{A.19}$$

where C is the charge-conjugation matrix such that:

$$\left(C\gamma^0\right)\gamma^{\mu *}\left(C\gamma^0\right)^{-1} = -\gamma^\mu\,. \tag{A.20}$$

In our representation, $C = -i\gamma^2\gamma^0$. With these conventions, the charge conjugate spinor of λ can be written

$$\lambda_C = C\bar{\lambda}^T = C\gamma^0\lambda^* = -i\gamma^2\lambda^*\,, \tag{A.21}$$

thus a Majorana spinor has the form

$$\lambda = \begin{pmatrix} \chi_L \\ i\sigma_2\chi_L{}^* \end{pmatrix}\,. \tag{A.22}$$

A.2 Two-component notation

In the chosen representation for the γ-matrices, a Dirac spinor ψ can be written

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{A.23}$$

where ψ_L is a left-handed Weyl spinor, which transforms under the Lorentz group $SO(1,3) \sim SL(2,\mathbb{C})$ in the $(1/2,0)$ representation, while ψ_R is a right-handed Weyl spinor, which transforms in the $(0,1/2)$ representation. We will use then the Van der Waerden notation, which indicates the transformation properties:

$$\begin{array}{ll} \psi_L = \psi_\alpha & \alpha = 1,2 \\ \psi_R = \bar{\psi}^{\dot{\alpha}} & \dot{\alpha} = 1,2 \end{array} . \tag{A.24}$$

The Lorentz group acts on these two-component Weyl spinors via matrices $M \in SL(2,\mathbb{C})$ (2×2 complex matrices of unit determinant); if $M \in SL(2,\mathbb{C})$, then M^*, $(M^T)^{-1}$, $(M^\dagger)^{-1} \in SL(2,\mathbb{C})$, so they also represent the action of the Lorentz group on two-component spinors.

Two-component spinors with upper or lower and dotted or undotted indices transform as follows under M:

$$\begin{array}{lllll} \psi'_\alpha &=& M_\alpha{}^\beta\, \psi_\beta & \psi'^\alpha &=& (M^{-1})_\beta{}^\alpha\, \psi^\beta \\ \bar{\psi}'_{\dot{\alpha}} &=& (M^*)_{\dot{\alpha}}{}^{\dot{\beta}}\, \bar{\psi}_{\dot{\beta}} & \bar{\psi}'^{\dot{\alpha}} &=& (M^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}}\, \bar{\psi}^{\dot{\beta}} \end{array} . \tag{A.25}$$

The definition means that $\psi_1^\alpha \psi_{2\alpha}$ and $\bar{\psi}_1^{\dot{\alpha}} \bar{\psi}_{2\dot{\alpha}}$ are Lorentz invariant:

$$\begin{array}{lllll} \psi'^\alpha_1 \psi'_{2\alpha} &=& \psi^\beta\, (M^{-1})_\beta{}^\alpha\, M_\alpha{}^\gamma\, \psi_{2\gamma} &=& \psi_1^\alpha \psi_{2\alpha} \\ \bar{\psi}'^{\dot{\alpha}}_1 \psi'_{2\dot{\alpha}} &=& \bar{\psi}^{\dot{\beta}}\, (M^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}}\, (M^*)_{\dot{\alpha}}{}^{\dot{\gamma}}\, \bar{\psi}_{2\gamma} &=& \bar{\psi}_1^{\dot{\alpha}} \bar{\psi}_{2\dot{\alpha}} \end{array} . \tag{A.26}$$

Let us define $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ such that

$$\epsilon_{\alpha\beta}\, \epsilon^{\beta\gamma} = \delta_\alpha^\gamma \,. \tag{A.27}$$

$\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ can be used to raise and lower spinor indices:

$$\psi^\alpha = \epsilon^{\alpha\beta}\, \psi_\beta \,, \qquad \psi_\alpha = \epsilon_{\alpha\beta}\, \psi^\beta \,. \tag{A.28}$$

Four-component Dirac spinors contain two Weyl spinors:

$$\psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \qquad \bar{\psi} = \begin{pmatrix} \chi^\alpha & \bar{\psi}_{\dot{\alpha}} \end{pmatrix} \tag{A.29}$$

so that $\bar{\psi}\psi = \chi^\alpha \psi_\alpha + \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$ and we choose the conventions

$$\begin{array}{lllll} \psi_1 \psi_2 &\equiv& \psi_1^\alpha \psi_{2\alpha} &=& \psi_2 \psi_1 \\ \bar{\psi}_1 \bar{\psi}_2 &\equiv& \bar{\psi}_{1\dot{\alpha}} \bar{\psi}_2^{\dot{\alpha}} &=& \bar{\psi}_2 \bar{\psi}_1 \end{array} \tag{A.30}$$

with

$$\bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\,\bar{\psi}^{\dot{\beta}}\,, \qquad \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\,\bar{\psi}_{\dot{\beta}}\,. \tag{A.31}$$

A Majorana spinor satisfies the condition

$$\bar{\chi}^{\dot{\alpha}} = (i\sigma^2\bar{\psi}^T)^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}(\chi_\beta)^*\,, \tag{A.32}$$

thus we choose $\epsilon^{\dot{\alpha}\dot{\beta}} = (i\sigma^2)^{\dot{\alpha}\dot{\beta}}$.
Many useful identities for spinors and σ matrices can be found in [1] and [2].

Appendix B

Spinors beyond $SO(1,3)$

Supergravity theories describe both bosons and fermions, in curved space and in generic dimension ($2 \leq d \leq 11$). This appendix reviews some mathematical tools which are used for the two extensions:

1. the vielbein formalism to describe fermions in curved space-time;

2. spinors in general dimensionality.

B.1 The vielbein formalism

Supergravity is a theory of gravitation, hence space-time is dynamical and, in general, curved. In supersymmetric theories we must describe fermionic, as well as bosonic, degrees of freedom, but spinors cannot be defined in curved space-time [4, 17, 18].

Space-time is a manifold, therefore at every point x^μ of d-dimensional space-time we can define a locally inertial frame of reference, on the tangent space at x^μ (Equivalence Principle). Let $X^\alpha(\overline{x})$, $\alpha = 1, \ldots, d$, be the coordinates[1] that define the inertial frame erected at $\overline{x}$. We define the *vielbein* to be the $d \times d$ matrix (not necessarily symmetric)

$$e_\mu{}^\alpha \equiv \left(\frac{\partial X^\alpha}{\partial x^\mu}(x) \right)_{|x=\overline{x}} . \tag{B.1}$$

The vielbein transforms as a Lorentz vector under local Lorentz transformations (LL) and as a world vector under general coordinate transformations (GCT). We take it to be orthonormalized as:

$$e_\mu{}^\alpha e_\nu{}^\beta g^{\mu\nu} = \eta^{\alpha\beta} . \tag{B.2}$$

The *inverse vielbein* $e_\alpha{}^\mu$ is defined as

$$e_\alpha{}^\mu e_\nu{}^\alpha = \delta^\mu_\nu , \quad e_\alpha{}^\mu e_\mu{}^\beta = \delta^\beta_\alpha . \tag{B.3}$$

[1] We use letters from the middle of the Greek alphabet to denote *world indices*, which describe the space-time manifold, and letters from the beginning of the Greek alphabet to denote *Lorentz indices*, which are used in the tangent space.

The contraction of a Lorentz (world) vector with the vielbein or its inverse is a world (Lorentz) vector. Therefore the vielbein has the fundamental property

$$g_{\mu\nu} = e_\mu{}^\alpha e_\nu{}^\beta \eta_{\alpha\beta} \,. \tag{B.4}$$

Under infinitesimal LL and GCT of parameters $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ and ξ^λ, respectively, it transforms as

$$\begin{aligned} \delta_{LL} e_\mu{}^\alpha &= \omega^\alpha{}_\beta e_\mu{}^\beta \,, \\ \delta_{GCT} e_\mu{}^\alpha &= (\partial_\mu \xi^\lambda) e_\lambda{}^\alpha + \xi^\lambda \partial_\lambda e_\mu{}^\alpha \,. \end{aligned} \tag{B.5}$$

Thus the vielbein has the the same number of physical degrees of freedom as the corresponding metric: the vielbein must be GCT invariant, thus the number of independent physical components is reduced by a LL transformation. We are left with

$$d^2 - \frac{1}{2} d(d-1) = \frac{1}{2} d(d+1)$$

physical degrees of freedom, which are the same number as those carried by the metric.

As in the metric formalism, we can introduce a *spin connection* $\widetilde{\omega}_\mu{}^{\alpha\beta}$, analogous to the metric connection $\widetilde{\Gamma}^\lambda_{\mu\nu}$, to define covariant derivatives.

- A derivative which is covariant only to respect with LL,

  $$\widetilde{D}_\mu e_\nu{}^\alpha \equiv \partial_\mu e_\nu{}^\alpha - \widetilde{\omega}_\mu{}^\alpha{}_\beta e_\nu{}^\beta \,. \tag{B.6}$$

  Given a spinor λ, which transforms under an infinitesimal LL transformation of parameter $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ as $\delta_{LL}\lambda = \frac{1}{2}\omega_{\alpha\beta}\sigma^{\alpha\beta}\lambda$, $\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha, \gamma^\beta]$, the Lorentz covariant derivative is given by

  $$\widetilde{D}_\mu \lambda = \partial_\mu \lambda - \frac{i}{4} \widetilde{\omega}_{\mu\alpha\beta} \sigma^{\alpha\beta} \lambda \,. \tag{B.7}$$

- A derivative covariant under GCT,

  $$\widetilde{\mathcal{D}}_\mu e_\nu{}^\alpha \equiv \partial_\mu e_\nu{}^\alpha + \widetilde{\omega}_\mu{}^\alpha{}_\beta e_\nu{}^\beta - \widetilde{\Gamma}^\lambda_{\mu\nu} e_\lambda{}^\alpha \,. \tag{B.8}$$

  For a spinor-vector field,

  $$\widetilde{\mathcal{D}}_\mu \Psi_\nu = \partial_\mu \Psi_\nu - \frac{i}{4} \widetilde{\omega}_{\mu\alpha\beta} \sigma^{\alpha\beta} \Psi_\nu - \widetilde{\Gamma}^\lambda_{\mu\nu} \Psi_\lambda \,. \tag{B.9}$$

Compatibility of the spin connection with the vielbein requires

$$\widetilde{\mathcal{D}}_\mu e_\nu{}^\alpha = 0 \,. \tag{B.10}$$

We denote by $\Gamma^\lambda_{\mu\nu}$ the symmetric part of the metric connection and by $C^\lambda_{\mu\nu}$ the torsion tensor:

$$\widetilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + C^\lambda_{\mu\nu}\,, \quad \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}\,. \tag{B.11}$$

Therefore $\Gamma^\lambda_{\mu\nu}$ is the Christoffel symbol of the second kind.

Analogously, we can find an expression for the torsionless part of the spin connection $\omega_{\mu\alpha\beta}$, decomposed as $\widetilde{\omega}_\mu{}^{\alpha\beta} = \omega_\mu{}^{\alpha\beta} + K_\mu{}^{\alpha\beta}$,

$$D_\mu e_\nu{}^\alpha - D_\nu e_\mu{}^\alpha = 0\,, \quad D_\mu e_\nu{}^\alpha = \partial_\mu e_\nu{}^\alpha - \omega_\mu{}^\alpha{}_\beta e_\nu{}^\beta\,, \tag{B.12}$$

in terms of the vielbein:

$$\omega_\mu{}^{\alpha\beta} = \frac{1}{2}\, e^{\alpha\lambda}\left(\partial_\mu e_\lambda{}^\beta - \partial_\lambda e_\mu{}^\beta\right) + \frac{1}{2}\, e^{\alpha\nu} e^{\beta\lambda} e_{\mu\gamma}\, \partial_\nu e_\lambda{}^\gamma - (\alpha \leftrightarrow \beta)\,, \tag{B.13}$$

such that

$$\omega_\mu{}^{\alpha\beta} = -\omega_\mu{}^{\beta\alpha}\,. \tag{B.14}$$

Under infinitesimal LL and GCT of parameter $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ and ξ^λ, respectively, the spin connection transforms as

$$\begin{aligned} \delta_{LL}\widetilde{\omega}_\mu{}^{\alpha\beta} &= \partial_\mu \omega^{\alpha\beta} - [\widetilde{\omega}_\mu, \omega]^{\alpha\beta} \equiv \widetilde{D}_\mu \omega^{\alpha\beta}\,, \\ \delta_{GCT}\widetilde{\omega}_\mu{}^{\alpha\beta} &= (\partial_\mu \xi^\lambda)\widetilde{\omega}_\lambda{}^{\alpha\beta} + \xi^\lambda \partial_\lambda \widetilde{\omega}_\mu{}^{\alpha\beta}\,. \end{aligned} \tag{B.15}$$

From (B.13) we obtain an expression for the torsionless metric connection in terms of the vielbein:

$$\Gamma^\lambda_{\mu\nu} = e_\alpha{}^\lambda R^\alpha_{\mu\nu}\,, \tag{B.16}$$

where $R^\alpha_{\mu\nu}$ is the antisymmetric part of the covariant derivative (B.8) with torsionless metric and spin connection:

$$R^\alpha_{\mu\nu} \equiv \mathcal{D}_\mu e_\nu{}^\alpha - \mathcal{D}_\nu e_\mu{}^\alpha = \partial_{[\mu} e_{\nu]}{}^\alpha - \omega_{[\mu}{}^\alpha{}_{|\beta|} e_{\nu]}{}^\beta\,. \tag{B.17}$$

The torsion of the metric connection can be written:

$$C^\lambda_{\mu\nu} \equiv \widetilde{\Gamma}^\lambda_{[\mu\nu]} = e_\alpha{}^\lambda R^\alpha_{\mu\nu}\,. \tag{B.18}$$

In an ordinary gauge theory, we would have introduced the spin connection as an independent gauge field, but since we want a formulation equivalent to the metric formulation of General Relativity, we can express the *torsionless* spin connection in terms of the vielbein, i.e. the spin connection does not represent new degrees of freedom. This is true *only as long as we do not consider torsion tensors.*

We can define the curvature as the torsion of the spin connection:

$$\widetilde{R}^{\alpha\beta}_{\mu\nu} \equiv \widetilde{\mathcal{D}}_\mu \widetilde{\omega}^{\alpha\beta}_\nu - \widetilde{\mathcal{D}}_\nu \widetilde{\omega}^{\alpha\beta}_\mu = \partial_{[\mu} \widetilde{\omega}_{\nu]}{}^{\alpha\beta} - [\widetilde{\omega}_\mu, \widetilde{\omega}_\nu]^{\alpha\beta}\,, \tag{B.19}$$

which is related to the Riemann tensor by

$$\widetilde{R}^{\mu}{}_{\nu\rho\sigma} = e_{\alpha}{}^{\mu} e_{\nu}{}^{\beta} \widetilde{R}^{\alpha\beta}_{\rho\sigma} \,. \tag{B.20}$$

Finally, we can define the analogue of the Ricci tensor:

$$\widetilde{R}^{\alpha}_{\mu} = e_{\beta}{}^{\nu} \widetilde{R}^{\alpha\beta}_{\mu\nu} \,. \tag{B.21}$$

As seen in Chapter 2, in supergravity theories there are some quartic terms in the spinor fields of the theory which can be absorbed in the torsion of the spin connection, $K_{\mu}{}^{\alpha\beta}$, with a form depending on the specific theory under consideration.

B.2 Spinors in higher dimensions

We would like to discuss [51, 52, 53] some properties of spinor representations of the group $SO(t,s)$, $d = t + s$, with an invariant metric

$$\eta_{AB} = diag\left(\underbrace{+1,\ldots,+1}_{t},\underbrace{-1,\ldots,-1}_{s}\right) \,. \tag{B.22}$$

The d-dimensional Minkowski space corresponds to the choice $t = 1$, $s = d - 1$.

We can define, in every dimensionality, the gamma matrices Γ^A which satisfy the Clifford algebra:

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} \,, \quad A, B = 1, \ldots, d. \tag{B.23}$$

These matrices are represented by $2^{\left[\frac{d}{2}\right]} \times 2^{\left[\frac{d}{2}\right]}$ matrices, where $[x]$ indicates the integer part of x.

The main differences in the description of spinors in higher dimensions depend on whether d is even or odd.

- d **even**. t of the gamma matrices can be taken as hermitian, while the other can be taken as anti-hermitian:

$$\begin{aligned} \left(\Gamma^A\right)^\dagger &= \Gamma^A \,, \quad A = 1, \ldots, t \,, \\ \left(\Gamma^A\right)^\dagger &= -\Gamma^A \,, \quad A = t+1, \ldots, d \,. \end{aligned} \tag{B.24}$$

An explicit representation[2] is given in terms of the Pauli matrices by

$$\Gamma^1 = \sigma^1 \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2 \,,$$

[2]This is different from that given in Appendix A.1, but this choice is harmless, since we will use it only throughout this section.

$$
\begin{array}{rcl}
\Gamma^2 & = & \sigma^2 \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2 \,, \\
\Gamma^3 & = & \sigma^3 \otimes \sigma^1 \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2 \,, \\
\Gamma^4 & = & \sigma^3 \otimes \sigma^2 \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2 \,, \\
\ldots & & \ldots \\
\Gamma^{2k+1} & = & \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{k} \otimes \sigma^1 \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2 \,, \\
\Gamma^{2k+2} & = & \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{k} \otimes \sigma^2 \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2 \,, \\
\ldots & & \ldots \\
\Gamma^{d-1} & = & \sigma^3 \otimes \cdots \otimes \sigma^3 \otimes \sigma^1 \,, \\
\Gamma^{d} & = & \sigma^3 \otimes \cdots \otimes \sigma^3 \otimes \sigma^2 \,.
\end{array}
$$

- d **odd**. Gamma matrices can be built starting from those in the even dimension $(d-1)$. For example, we can use the gamma matrices of $SO(t, s-1)$ and take the last matrix as

$$\Gamma^d = (-1)^{\frac{1}{4}(s-t)}\, i\, \Gamma^1 \Gamma^2 \cdots \Gamma^{d-1} \,. \tag{B.25}$$

In general dimensionality, it is not possible to define chirality or charge conjugation, which are used to define Weyl and Majorana spinors. As we will see, these properties will depend on $(s-t)$, as well as on d.

Chirality

The difference between d even and d odd is crucial.

- d **even**. In even dimensions we can define a matrix

$$\overline{\Gamma} \equiv (-1)^{\frac{1}{4}(s-t)}\, \Gamma^1 \cdots \Gamma^d \,, \tag{B.26}$$

which has the properties

$$\overline{\Gamma}^2 = 1 \,, \quad \{\overline{\Gamma}, \Gamma^A\} = 0 \,, \quad \forall\, A = 1, \ldots, d \,. \tag{B.27}$$

We can thus define two orthogonal projectors

$$P_\pm \equiv \frac{1}{2}\left(\mathbb{I} \pm \overline{\Gamma}\right) \,, \tag{B.28}$$

$$P_+ + P_- = \mathbb{I} \,, \quad P_+ P_- = 0 = P_- P_+ \,. \tag{B.29}$$

Weyl spinors are those which satisfy

$$P_\pm \psi_\pm = \pm\psi_\pm \,, \quad P_\pm \psi_\mp = 0 \,. \tag{B.30}$$

- d **odd**. In this case the $\overline{\Gamma}$ matrix is proportional to (B.25), therefore it is not possible to define chirality in odd dimensionality.

Charge conjugation

- d **even**. The matrices $\pm\left(\Gamma^A\right)^*$ form an equivalent representation of the Clifford algebra (B.23), then there exist two unitary matrices $B_\pm$ such that

$$
\begin{aligned}
\left(\Gamma^A\right)^* &= B_+\Gamma^A\left(B_+\right)^{-1}\,, \\
-\left(\Gamma^A\right)^* &= B_-\Gamma^A\left(B_-\right)^{-1}\,, \quad A=1,\dots,d\,.
\end{aligned}
\tag{B.31}
$$

It can be shown [53] that these matrices satisfy

$$B_\pm^* B_\pm = \epsilon_\pm(s,t)\mathbb{I}\,, \quad \epsilon_\pm(s,t) = \sqrt{2}\cos\left(\frac{\pi}{4}(s-t\pm 1)\right)\,. \tag{B.32}$$

The values $\epsilon_\pm$ are periodic of period 8 in $(s-t)$. We list the possible values in Table B.1.

$(s-t)$	1	2	3	4	5	6	7	8	9
ϵ_+	0	-1	-1	-1	0	$+1$	$+1$	$+1$	0
ϵ_-	$+1$	$+1$	0	-1	-1	-1	0	$+1$	$+1$

Table B.1: Values of the function $\epsilon_\pm(s,t)$ for $(s-t)=1,\dots,9$. $\epsilon_\pm$ are periodic of period 8 in $(s-t)$. For Minkowski space, $(s-t)=d-2$.

Charge conjugation can be defined by using $B_\pm$ as

$$\psi_C \equiv (B_+)^{-1}\psi^* \quad \text{or} \quad \psi_C \equiv (B_-)^{-1}\psi^*\,. \tag{B.33}$$

This is equivalent to the usual definition (A.19). In facts, if we define the matrix

$$A \equiv \Gamma^1\dots\Gamma^t\,, \tag{B.34}$$

the Dirac conjugate can be taken as

$$\overline{\psi} \equiv \psi^\dagger A\,, \tag{B.35}$$

so the charge conjugation matrices such that

$$\psi_C \equiv C_+\overline{\psi}^T \quad \text{or} \quad \psi_C \equiv C_-\overline{\psi}^T \tag{B.36}$$

are given by

$$C_{\pm} = (B_{\pm})^{-1} A^{-T} . \tag{B.37}$$

They satisfy:

$$\begin{aligned} \left(\Gamma^{A}\right)^{T} &= \pm(-1)^{t+1} (C_{\pm})^{-1} \Gamma^{A} C_{\pm} , \\ (C_{\pm})^{\dagger} C_{\pm} &= \mathbb{I} , \\ C_{\pm}^{T} &= (\pm 1)^{t} \, (-1)^{\frac{1}{2} t(t-1)} \epsilon_{\pm} C_{\pm} . \end{aligned}$$

The usual charge conjugation is obtained by using C_{-}: spinors satisfying

$$\psi_{C} = C_{-} \overline{\psi}^{T} \tag{B.38}$$

are called *Majorana spinors*, while those satisfying

$$\psi_{C} = C_{+} \overline{\psi}^{T} \tag{B.39}$$

are called *pseudo-Majorana spinors*.

The Majorana condition is consistent only if

$$(\psi_{C})_{C} = \psi , \tag{B.40}$$

which can be satisfied if and only if $B_{\pm}^{*} B_{\pm} = +1$, or

$$\epsilon_{\pm} = +1 . \tag{B.41}$$

- d **odd**. Charge conjugation is defined by using the same $B_{\pm}$ as in $(d-1)$ dimensions. They satisfy, in addition to (B.31) for $A = 1, \ldots, (d-1)$, the relations

$$(-1)^{\frac{1}{2}(s-t+1)} \left(\Gamma^{d}\right)^{*} = B_{\pm} \Gamma^{d} (B_{\pm})^{-1} . \tag{B.42}$$

When $(-1)^{\frac{1}{2}(s-t+1)} = \pm 1$, the signs generated by $B_{\pm}$ are the same for all of the d matrices, thus we can use $B_{\pm}$ to define charge conjugation.

In general dimensionality, we can look for the conditions which allow us to define (pseudo) Majorana-Weyl spinors, i.e. spinors which satisfy both of the (pseudo) Majorana and Weyl conditions. For this definition to be consistent, the spinor ψ and its charge conjugate ψ_{C} must have the same chirality. Recalling that Weyl spinors are defined only in even dimensions, we obtain

$$P_{\pm} (\psi_{\pm})_{C} = \pm(-1)^{\frac{1}{2}(s-t)} (\psi_{\pm})_{C} , \tag{B.43}$$

d	Weyl	Majorana	pseudo-Maj	Majorana-Weyl	pseudo-Maj-Weyl
2	Yes	Yes	Yes	Yes	Yes
3		Yes			
4	Yes	Yes			
5					
6	Yes				
7					
8	Yes		Yes		
9			Yes		
10	Yes	Yes	Yes	Yes	Yes
11		Yes			

Table B.2: Spinors in d-dimensional Minkowski space $t = 1$, $s = d - 1$. The table is periodic in d with period 8.

then we require $(-1)^{\frac{1}{2}(s-t)} = 1$, i.e. $(s - t) = 0 \pmod 4$. In particular, for Minkowski space $t = 1$, $s = d - 1$, this is true in $d = 2 \pmod 8$, since in $d = 6 \pmod 8$ the (pseudo) Majorana condition cannot be imposed.

We summarize in Table B.2 the possible types of spinors which can be defined in the Minkowski space.

When $\epsilon_\pm = -1$ we cannot impose the (pseudo) Majorana condition, so we must use Dirac or Weyl spinors (where available). Another possibility is to introduce *symplectic* (pseudo) Majorana spinors. Given an *even* number of spinors ψ^i, $i = 1, \ldots, 2n$, they are obtained by imposing the constraint

$$\psi^i = \Omega^{ij}(\psi^j)_C\,, \tag{B.44}$$

where $\Omega^{ij} = -\Omega^{ji}$ is a constant antisymmetric matrix. $2n$ symplectic (pseudo) Majorana spinors are equivalent to n Dirac spinors.

Using the vielbein formalism, we can describe fermions in curved space-time and in general dimensionality, by defining the analogue of gamma matrices in curved space-time as

$$\gamma^\mu \equiv e_\alpha{}^\mu \gamma^\alpha\,. \tag{B.45}$$

This is consistent, since

$$\{\gamma^\mu, \gamma^\nu\} = \{\gamma^\alpha, \gamma^\beta\} e_\alpha{}^\mu e_\beta{}^\nu = 2g^{\mu\nu}\,. \tag{B.46}$$

Now we have the mathematical tools necessary to describe supergravity in every dimensionality.

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